A Maple interface for computing variational symmetries in optimal control
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Abstract
A computer algebra package, for the automatic computation of variational symmetries in optimal control, was recently developed by the authors [2, 3]. We present a graphical user interface which permit to interact, in a point-and-click environment, with all the previous symbolical tools.

1. Introduction
The concept of variational symmetry entered into optimal control in the seventies of the twentieth century. Variational symmetries, which keep an optimal control problem invariant, are very useful in optimal control, but unfortunately their study is not easy, requiring lengthy and cumbersome calculations. Recently there has been an interest in the application of Computer Algebra Systems to the study of control systems, and collections of symbolical tools are being developed to help on the analysis and solution of complex problems. The first computer algebra package for computing the variational symmetries in the calculus of variations, and respective Noether’s first integral, was given by the authors in [2] then extended to the more general setting of optimal control [3] and, more recently, upgraded in [4] with the introduction of new capacities, several optimal parameters, and graphical routines. Here we provide a graphical user interface to our computer algebra package [4]. This application is named octool and was created with Maple tools (a graphical programming language of the Maple system).

2. Defining Variational Symmetries and Noether Conservation Laws in Optimal Control
The optimal control problem consists to minimize an integral functional,
\[ J(u, x) = \int L(x, \dot{x}, u) \, dt \]
subject to a control system described by a system of ODEs,
\[ x(t) = \Phi(t, x, u) \]
with appropriate boundary conditions on the values of \( x(0) \) and \( x(T) \). The Lagrangian \( L(x, \dot{x}, u) \) is a real function, assumed to be continuously differentiable in \( (t, x, \dot{x}, u) \in R^4 \) the independent variables; \( x \) the state vector of variables; \( u \in R^n \) the vector of controls, assumed to be piecewise continuous functions and \( \dot{x} \in R^{n+m} \) the velocity vector, assumed to be a continuously differentiable vector function. Recently, the authors developed analytic computational methods [3, 4] that permits to obtain symmetries and conservation laws for given optimal control problem (1)–(2) based on the extension of the famous Noether’s theorem to optimal control [6]. Let us consider a one-parameter group of \( C \) transformations \( \{ t, x, u \rightarrow (t, \tilde{x}(t), \tilde{u}(t)) \} \) generating the optimal control problem (1)–(2)–(3) of which reduces to the identity transformation when the parameters vanishes: \( h_0 \cdot \tilde{x} \cdot h_0^{-1} = x \), \( h_0 \cdot \tilde{u} \cdot h_0^{-1} = u \). Associated with a one-parameter group of transformations (3), we introduce the infinitesimal generators \( \begin{bmatrix} T(t, x, u) = \partial_t \phi(t, x, u) \\ U(t, x, u) = \partial_x \phi(t, x, u) \\ \psi(t, x, u) = \partial_u \phi(t, x, u) \end{bmatrix} \)
(4)
Emmy Noether was the first who established a relationship between the existence of invariance transformations of the problem and the existence of conservation laws. Since the work pioneered by Noether, several definitions of invariance have been introduced for the problems of the calculus of variations and for the problems of optimal control. All these definitions are given with respect to a one-parameter group of transformations (3). Although written in a different way, one gets, in terms of the generators \( \psi \), a necessary and sufficient condition of invariance that, essentially, coincide to all those definitions. For this reason, here we define in variance directly in terms of the generators (4).

Theorem 1 (Invariance) An optimal control problem is invariant under the \( C \) transformations (3) if and only if \( \Phi(t, x, u) = \phi(t, x, u) \) or equivalently, (4) is a symmetry of the problem up to \( C, \dot{X} \), and \( \Psi \), with \( \Phi \) the Hamiltonian:
\[ H(t, x, u, \dot{x}) = \phi(t, x, u) - \partial_t \phi(t, x, u) + \partial_x \phi(t, x, u) + \partial_u \phi(t, x, u) \]
(5)

3. The Maple interface octool
The Maple package [4] is very general and provides a myriad of optional parameters. In this work we provide a new graphical routine, called octool, which permits a user to take full power of (4) without having to learn the corresponding Maple commands and optional parameters. With our users can, in a point-and-click environment, interact with all the symbolical tools of our Maple package and deal with concrete problems of optimal control. More precisely, one has now an interface to the base main procedures of [4]: i. Symmetry, to obtain the variational symmetries (4) and gauge term \( c \) that satisfy the invariance condition (5); ii. Noether, to obtain the conservation laws (7); and iii. PMP (Pontryagin Maximum Principle), to try to obtain the Pontryagin extremals or, alternatively, to obtain the equations of the Hamiltonian system, stationary condition or just Hamiltonian.
We refer the reader to [3, 4] for the description and detailed specification of these Maple procedures.

In the case of optimal control problems with additional algebraic manipulations. The complete Maple package [4] without having to learn the corresponding Maple commands and optional parameters. With our users can, in a point-and-click environment, interact with all the symbolical tools of our Maple package and deal with concrete problems of optimal control. More precisely, one has now an interface to the base main procedures of [4]: i. Symmetry, to obtain the variational symmetries (4) and gauge term \( c \) that satisfy the invariance condition (5); ii. Noether, to obtain the conservation laws (7); and iii. PMP (Pontryagin Maximum Principle), to try to obtain the Pontryagin extremals or, alternatively, to obtain the equations of the Hamiltonian system, stationary condition or just Hamiltonian.

4. An illustrative example
Let us consider the following optimal control problem:
\[ \int_{t_0}^{t_1} \left( (x^1)^2 + (x^2)^2 \right) \, dt \]
where the control system serves as model for the kinematics of a car (Example 18, p. 750). We begin with the problem definition:

\[
\begin{align*}
\dot{x}^1 &= \psi_1(u), \\
\dot{x}^2 &= \psi_2(u), \\
\end{align*}
\]

Choosing, by a pop-up menu, the substitutions

\[
\begin{align*}
\psi_1(u) &= x^1, \\
\psi_2(u) &= x^2, \\
\end{align*}
\]

we obtain the Conservation Law
\[ \lambda = x^1 + x^2 \]

which corresponds to the symmetry group of planar (orientation preserving) isometries [5, Ex. 18, p. 750].

References