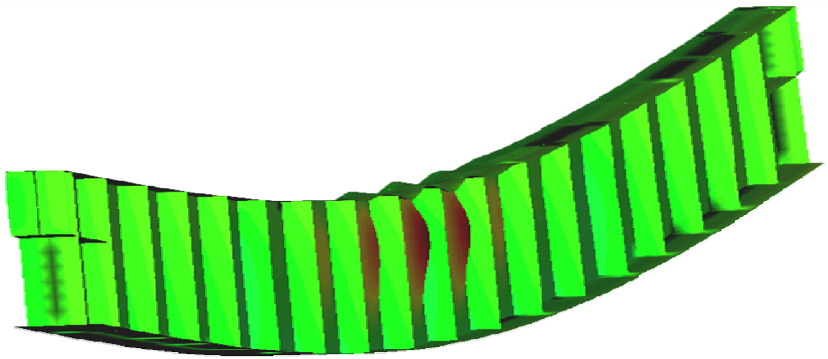
INSTITUTO POLITÉCNICO DE BRAGANÇA
Escola Superior de Tecnologia e Gestão

LECTURE (CU)

COMPUTATIONAL MECHANICS

(2º cycle, Master of Science Degree)



Paulo Piloto

Applied Mechanics Department

School of Technology and Management

Polytechnic Institute of Braganza

CURRICULAR UNIT - SYLLABUS

- Subject: Computational Mechanics.
- Course Speciality: Construction and Industrial Engineering.
- Main Scientific area: Structures and Solid Mechanics (MSE).
- Classes: 60 h/Semester: T (Theoretical) PL (Practice and Laboratory).
- Cycle: 2º (Master degree of Science).
- Year / Semester: 1º year/ 2nd Semester.
- Learning outcomes and competences:
  - Understand and apply Finite Element Method formulation.
  - To be aware of beam and bar finite element formulation.
  - Understand and apply two and three dimensional elasticity formulation.
  - Understand and apply plate and shell finite element formulation.
  - Understand and apply the finite element method and the numerical solutions.
  - Learn to use commercial finite element software.

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CURRICULAR UNIT - SYLLABUS

- Course contents (Extended version):
  - Chapter 1 - Stages of the FEM. Bar finite element:
    - Introduction, advantages and applications of the finite element method (FEM). Basic concepts in matrix analysis of structures. Types of analysis. Fundamental steps in the FEM. Phases of the method. Mathematical model formulation. Discrete mathematical models. Static and dynamic formulations. Stiffness matrix and element assembly. Continuous mathematical models. Variational formulation. Bar element formulation. Matrix formulation of the element equations. Isoparametric formulation and numerical integration.
  - Chapter 2 - Finite element formulation:
    - Standard flowchart of a finite element code. FEM general methodology. Shape functions. Interpolation of displacements. Displacement and strain fields. Stress field. Constitutive models. Solution of the FEM equations. FEM convergence requirements and error types. Optimal points for stress calculations.
  - Chapter 3 – Beam finite elements:
    - Euler-Bernoulli beam finite element. Timoshenko beam finite elements. Reduced integration and alternative solutions for the shear locking problem.
  - Chapter 4 - 2D and 3D formulations:
    - Two and three dimensional finite elements in elasticity. Finite elements formulation. Lagrangian and Serendipity elements. Numerical integration. Application of plate and shell finite elements: Kirchhoff and Reissner-Mindlin theories.
  - Chapter 5 - Computer Applications in Engineering.
    - Computational applications in structural (static, dynamic, instability), thermal and fluid flow problems, using a commercial finite element code.

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- Assessment:
  - Final season (EF) and Appeal season (ER):
    - Distributed assessment with 4 working projects to be presented at classes (oral presentations with power point slides, with written reports in word format) with 80% weight for final classification;
    - Final Exam with 20 % weight for final classification;
    - Labor students, with special statute, may require full Examination during final and appeal season, with 100 % weight for final classification.
  - Special season (EE):
    - Full Examination with 100 % weight for final classification.
- Language of classes: Portuguese and English
- Bibliography:

    - Moaveni S., Finite Element Analysis, Theory and Application with Ansys, 2nd Edition, Prentice Hall, 2003.
    - Onãte E., Cálculo de estructuras por el Método de Elementos Finitos, Centro Internacional de Métodos Numéricos en Ingeniería, Barcelona, 1995.
    - Zienkiewicz OC, Taylor RL., The finite element method. Vol.1: The basis. Oxford: Butterworth, 2000.
    - Zienkiewicz OC, Taylor RL., The finite element method. Vol.2: Solid mechanics. Oxford: Butterworth, 2000.
    - Fonseca, E.M.M, Sebenta de Mecânica Computacional, ESTIG-IPB, 2008.

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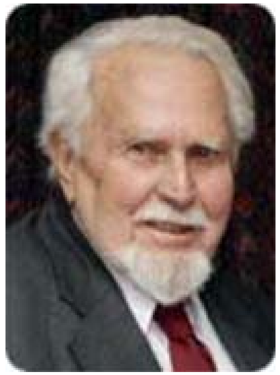
FINITE ELEMENT METHOD INTRODUCTION

- First reference to FEM, since 1940.
- Basically consists of an adaptation or modification of the approximation methods used in engineering and science, such as, Ritz method, 1909.
- Usually known as a mathematical method for solving PDE partial differential equations, such as, Poisson , Laplace, Navier-Stokes, and so on.
- Easy to program on computer codes:
  - Using high level programming languages, such as FORTRAN, C, and so on.
- Consider as an advance solution method in several designer codes
- Commercially available:
  - Ansys, Abaqus, Adina, and so on.
- First book to be published:
  - 1967, **Zienkiewicz** and Chung.
  - *''Professor* ***Zienkiewicz*** *is one of the originators of the finite element technique and has since the early 1960s dominated the finite element field internationally''*



- Special awards in computation mechanics (by the European Community on Computational Methods in Applied Sciences and Engineering (ECCOMAS)):
  - Prandtl and Euler Medals.
  - O. C. Zienkiewicz Award for Young Scientists in Computational Engineering Science.

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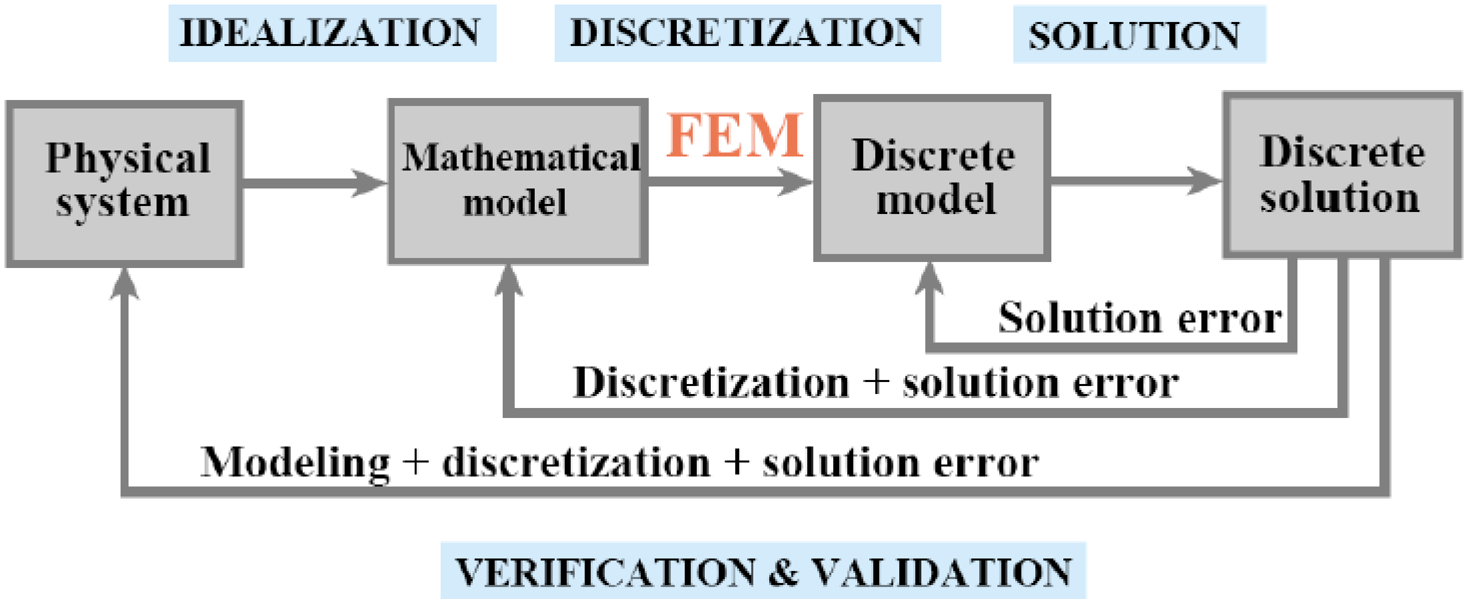
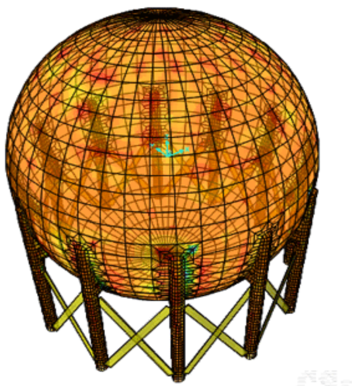
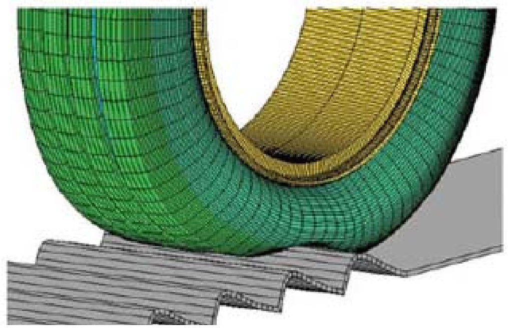
FINITE ELEMENT METHOD INTRODUCTION

- Fundamental steps in the FEM:

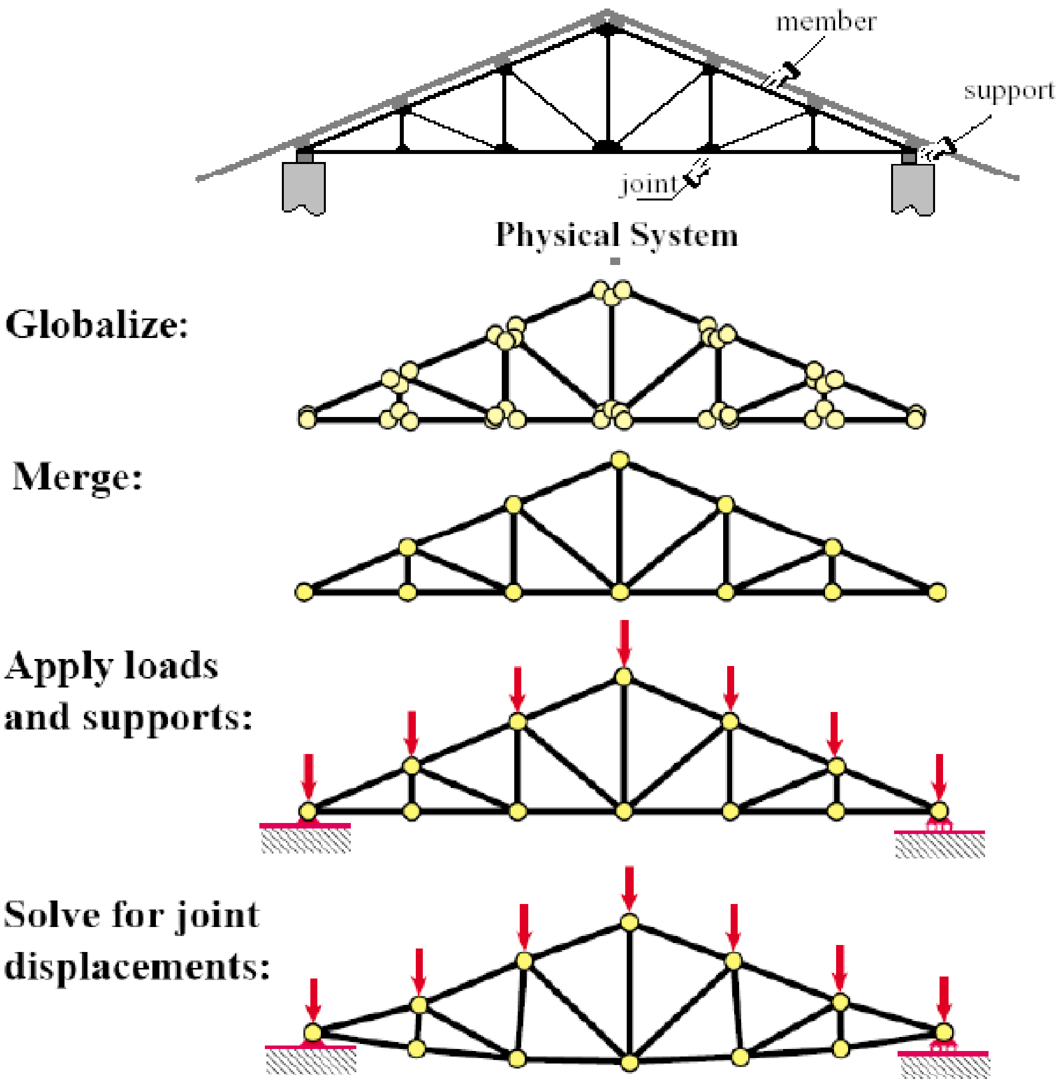






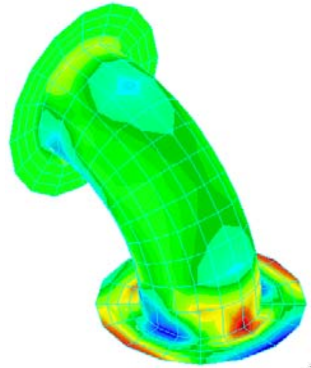
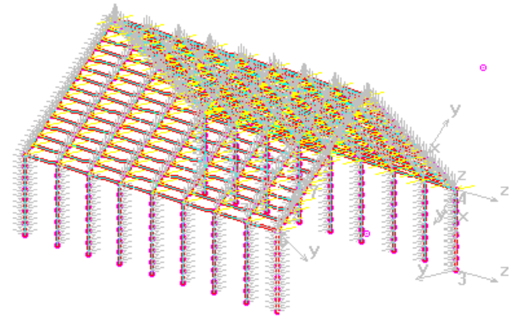
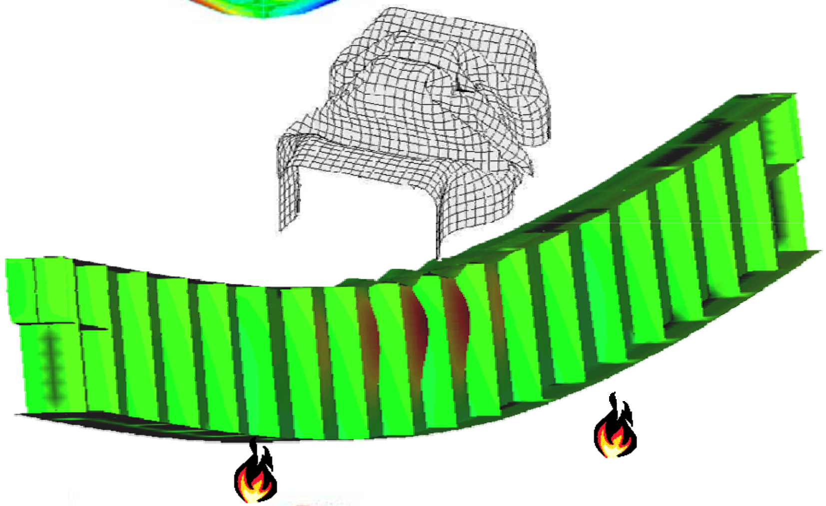
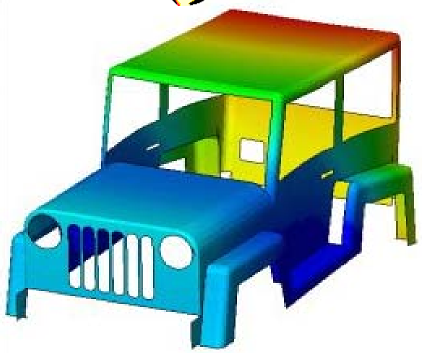
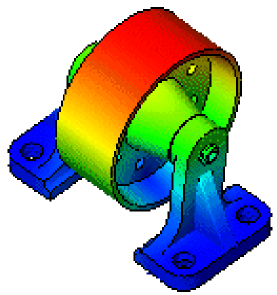
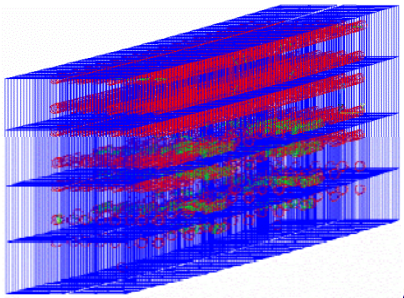
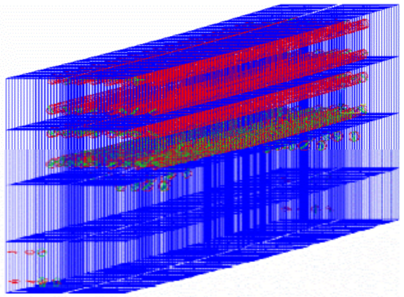
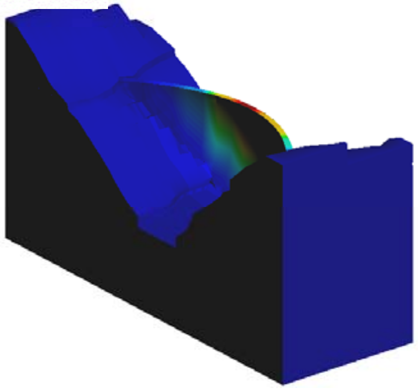
FINITE ELEMENT METHOD INTRODUCTION

- Basic steps:



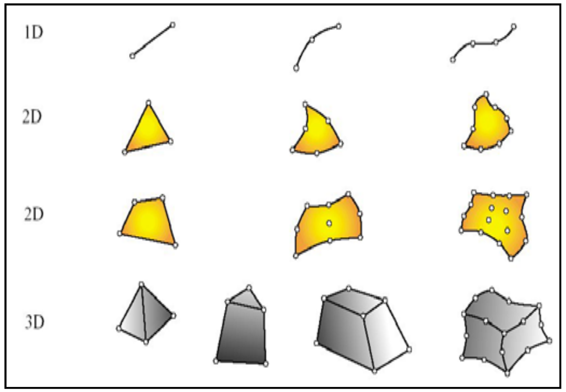
FINITE ELEMENT METHOD INTRODUCTION

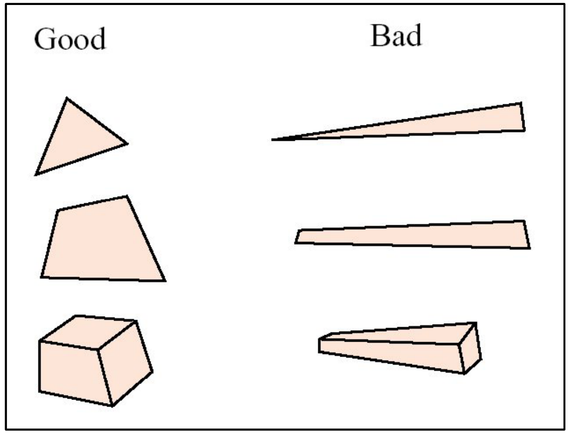
- Engineering applications:
  - Linear, material / geometry non-linear, structural analysis (static and dynamic);
  - Linear and material non-linear thermal analysis (steady and unsteady);
  - Linear and non linear computational fluid dynamics (CFD);
  - Coupled and uncoupled analyses;
  - Optimization.

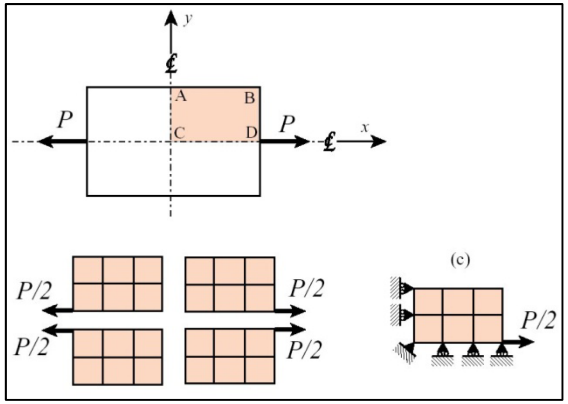









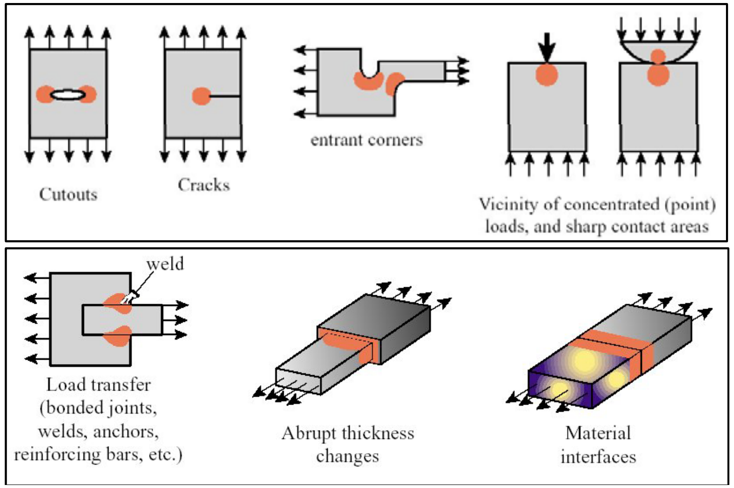
FINITE ELEMENT METHOD INTRODUCTION

- Finite Element approximation:
  - Dependent on domain analysis.
- finite element geometry:
  - Bad and acceptable.
- Simplify analysis, using symmetry conditions on:
  - Geometry ;
  - Loading conditions.
- Usual singularities:
  - Material behaviour: cracking and crushing;
  - Material interfaces: bonding conditions;
  - Geometry singularities.



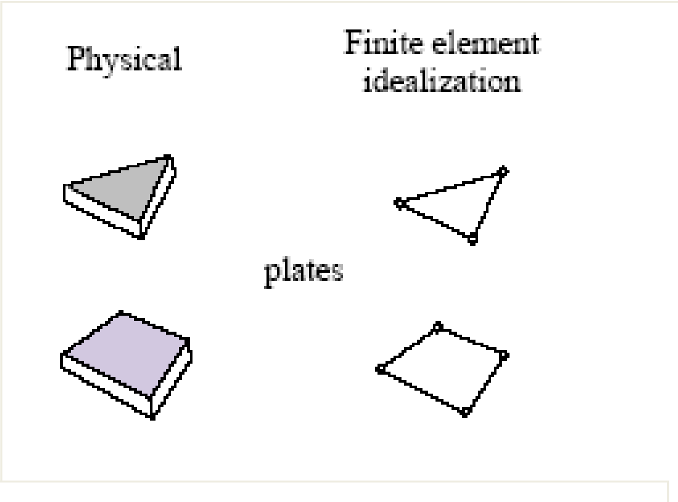


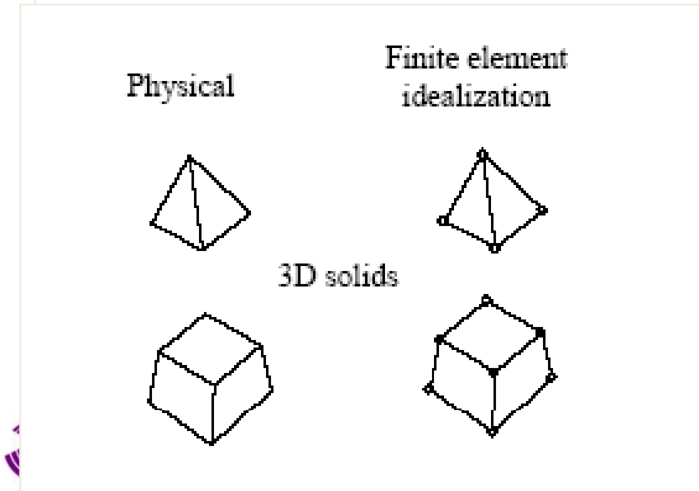




FINITE ELEMENT METHOD INTRODUCTION

- Level of approach:
  - Depends on the physical system.





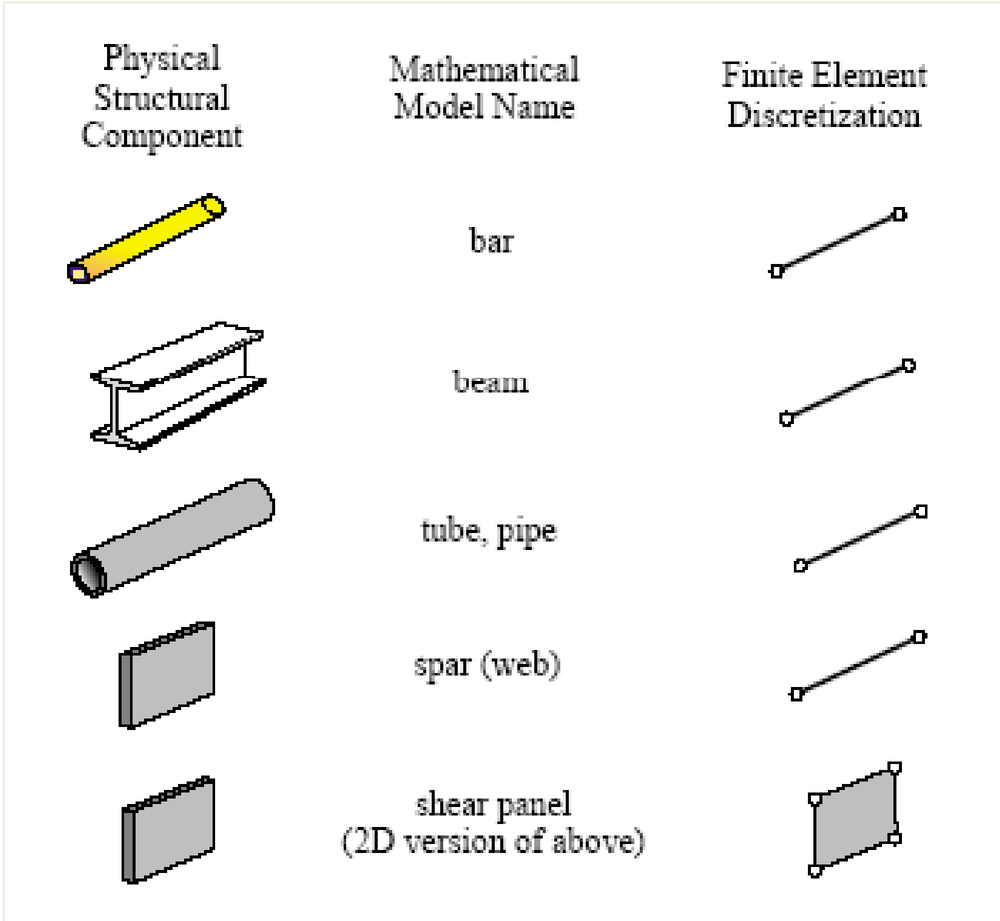
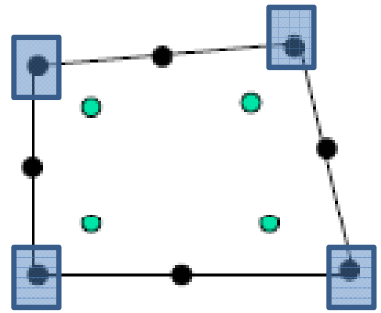
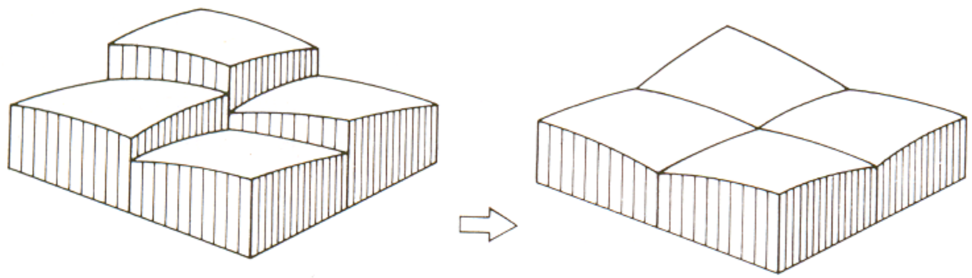


Figure 7.4. Examples of primitive structural elements.

FINITE ELEMENT METHOD INTRODUCTION



- A finite element is defined by:
  - His geometry;
  - The node coordinates;
  - The interpolation node coordinates;
  - The number of degrees of freedom;
  - The nodal variables definition (displacement, rotation , temperature, pressure);
  - The polynomial approach;
  - The type of continuity that nodal variables should satisfy in element and boundary: C0, C1, C2:
    - If the approaching variable and derivatives, are to be continuous over the entire element, the interpolation functions should be continuous up to the desired derivative order. (Important to calculate stress field, that depends on displacement derivatives).
    - If the approaching variable and derivatives, are to be continuous over the boundary , the Interpolation function and his derivatives up to the desired order, must depend on a unique way from the nodal variables.



FINITE ELEMENT METHOD INTRODUCTION

- Pre-requisites:
  - Apply the acquired knowledge and competences of:
    - differential and integral calculus;
    - numerical methods;
    - Programming;
    - mechanics of materials;
    - solid mechanics.
  - Understand oral and written English.
- Main stages in every Finite Element Analysis:
  - Pre-processor:
    - Geometric modelling (geometric primitives and other modelling tools), followed by mathematical modelling (mesh nodes and finite elements).
    - Mathematical modelling (mesh: nodes and finite elements).
    - Material behaviour, Boundary conditions.
  - Processor
    - Solver (direct, iterative, etc.)
  - Post-processor
    - Allows to watch the results.

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FINITE ELEMENT METHOD INTRODUCTION

- Solution convergence:
  - Numerical approximation to exact solutions (analytical).
  - To assert convergence is to claim the existence of a limit, which may be itself unknown. For any fixed standard of accuracy, you can always be sure to be within it, provided you have gone far enough
- Convergence happens when the discretization error becomes almost zero.

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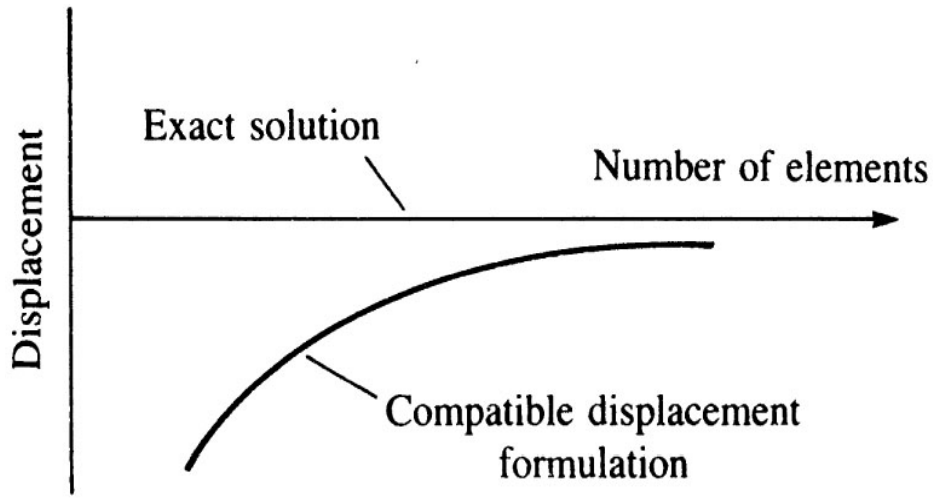


SOLID MECHANICS INTRODUCTION

- Constitutive Laws:
  - Mechanical (Generalized Hooke's Law):
    - In Stiffness Form.
    - In Compliance Form .
  - For isotropic materials (A);
  - For anisotropic materials (B);
  - For orthotropic materials (C).

A

$$\begin{bmatrix}\sigma_{xx}\\ \sigma_{yy}\\ \sigma_{zz}\\ \sigma_{yz}\\ \sigma_{zx}\\ \sigma_{xy}\end{bmatrix}=\frac{E}{(1+\nu)(1-2\nu)}\begin{bmatrix}1-\nu & \nu & \nu & 0 & 0 & 0\\ \nu & 1-\nu & \nu & 0 & 0 & 0\\ \nu & \nu & 1-\nu & 0 & 0 & 0\\ 0 & 0 & 0 & 1-2\nu & 0 & 0\\ 0 & 0 & 0 & 0 & 1-2\nu & 0\\ 0 & 0 & 0 & 0 & 0 & 1-2\nu\end{bmatrix}\begin{bmatrix}\varepsilon_{xx}\\ \varepsilon_{yy}\\ \varepsilon_{zz}\\ \varepsilon_{yz}\\ \varepsilon_{zx}\\ \varepsilon_{xy}\end{bmatrix}$$

A

$$\begin{bmatrix}\varepsilon_{xx}\\ \varepsilon_{yy}\\ \varepsilon_{zz}\\ \varepsilon_{yz}\\ \varepsilon_{zx}\\ \varepsilon_{xy}\end{bmatrix}=\frac{1}{E}\begin{bmatrix}1 & -\nu & -\nu & 0 & 0 & 0\\ -\nu & 1 & -\nu & 0 & 0 & 0\\ -\nu & -\nu & 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 1+\nu & 0 & 0\\ 0 & 0 & 0 & 0 & 1+\nu & 0\\ 0 & 0 & 0 & 0 & 0 & 1+\nu\end{bmatrix}\begin{bmatrix}\sigma_{xx}\\ \sigma_{yy}\\ \sigma_{zz}\\ \sigma_{yz}\\ \sigma_{zx}\\ \sigma_{xy}\end{bmatrix}$$

A

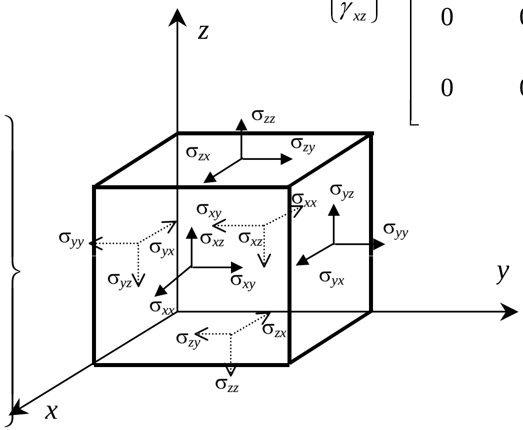
$$\begin{Bmatrix}\varepsilon_{xx}\\ \varepsilon_{yy}\\ \varepsilon_{zz}\\ \gamma_{xy}\\ \gamma_{yz}\\ \gamma_{xz}\end{Bmatrix}=\begin{bmatrix}\frac{1}{E} & -\frac{\nu}{E} & -\frac{\nu}{E} & 0 & 0 & 0\\ -\frac{\nu}{E} & \frac{1}{E} & -\frac{\nu}{E} & 0 & 0 & 0\\ -\frac{\nu}{E} & -\frac{\nu}{E} & \frac{1}{E} & 0 & 0 & 0\\ 0 & 0 & 0 & \frac{2(1+\nu)}{E} & 0 & 0\\ 0 & 0 & 0 & 0 & \frac{2(1+\nu)}{E} & 0\\ 0 & 0 & 0 & 0 & 0 & \frac{2(1+\nu)}{E}\end{bmatrix}\times\begin{Bmatrix}\sigma_{xx}\\ \sigma_{yy}\\ \sigma_{zz}\\ \tau_{xy}\\ \tau_{yz}\\ \tau_{xz}\end{Bmatrix}$$

C

$$\begin{Bmatrix}\varepsilon_{xx}\\ \varepsilon_{yy}\\ \varepsilon_{zz}\\ \gamma_{xy}\\ \gamma_{yz}\\ \gamma_{xz}\end{Bmatrix}=\begin{bmatrix}\frac{1}{E_x} & -\frac{\nu_{yx}}{E_y} & -\frac{\nu_{zx}}{E_z} & 0 & 0 & 0\\ -\frac{\nu_{xy}}{E_x} & \frac{1}{E_y} & -\frac{\nu_{zy}}{E_z} & 0 & 0 & 0\\ -\frac{\nu_{xz}}{E_x} & -\frac{\nu_{yz}}{E_y} & \frac{1}{E_z} & 0 & 0 & 0\\ 0 & 0 & 0 & \frac{1}{G_{xy}} & 0 & 0\\ 0 & 0 & 0 & 0 & \frac{1}{G_{yz}} & 0\\ 0 & 0 & 0 & 0 & 0 & \frac{1}{G_{xz}}\end{bmatrix}\times\begin{Bmatrix}\sigma_{xx}\\ \sigma_{yy}\\ \sigma_{zz}\\ \tau_{xy}\\ \tau_{yz}\\ \tau_{xz}\end{Bmatrix}$$

B

$$\begin{Bmatrix}\varepsilon_{xx}\\ \varepsilon_{yy}\\ \varepsilon_{zz}\\ \gamma_{xy}\\ \gamma_{yz}\\ \gamma_{xz}\end{Bmatrix}=\begin{bmatrix}C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16}\\ C_{21} & C_{22} & C_{23} & C_{24} & C_{25} & C_{26}\\ C_{31} & C_{32} & C_{33} & C_{34} & C_{35} & C_{36}\\ C_{41} & C_{42} & C_{43} & C_{44} & C_{45} & C_{46}\\ C_{51} & C_{52} & C_{53} & C_{54} & C_{55} & C_{56}\\ C_{61} & C_{62} & C_{63} & C_{64} & C_{65} & C_{66}\end{bmatrix}\times\begin{Bmatrix}\sigma_{xx}\\ \sigma_{yy}\\ \sigma_{zz}\\ \tau_{xy}\\ \tau_{yz}\\ \tau_{xz}\end{Bmatrix}$$



$$\{\sigma\}=[D]\{\varepsilon\}$$

SOLID MECHANICS INTRODUCTION

- Relation between strain and displacement (1st order, linear statics).
  - u,v,w (displacements).

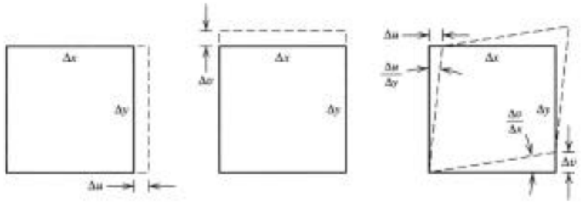
$$\{\varepsilon\}=\begin{Bmatrix}\varepsilon_{xx}\\ \varepsilon_{yy}\\ \varepsilon_{zz}\\ \gamma_{yz}\\ \gamma_{xz}\\ \gamma_{xy}\end{Bmatrix} \quad \{\varepsilon\}=[L]\{U\} \quad \{U\}=\begin{Bmatrix}u\\ v\\ w\end{Bmatrix}$$

$$\varepsilon_{xx}=\frac{\partial u}{\partial x} \qquad \gamma_{yz}=\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}$$

$$\varepsilon_{yy}=\frac{\partial v}{\partial y} \qquad \gamma_{xz}=\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}$$

$$\varepsilon_{zz}=\frac{\partial w}{\partial z} \qquad \gamma_{xy}=\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}$$

$$[L]=\begin{bmatrix}\frac{\partial}{\partial x} & 0 & 0\\ 0 & \frac{\partial}{\partial y} & 0\\ 0 & 0 & \frac{\partial}{\partial z}\\ 0 & \frac{\partial}{\partial z} & \frac{\partial}{\partial y}\\ \frac{\partial}{\partial z} & 0 & \frac{\partial}{\partial x}\\ \frac{\partial}{\partial z} & \frac{\partial}{\partial x} & 0\end{bmatrix}$$



- Relation between strain and displacement (2nd order, geometrically non-linear statics).

$$\{\varepsilon\}=\begin{Bmatrix}\varepsilon_{xx}\\ \varepsilon_{yy}\\ \varepsilon_{zz}\\ \gamma_{yz}\\ \gamma_{xz}\\ \gamma_{xy}\end{Bmatrix} \quad \{\varepsilon\}=[L]\{U\}+[NL]\{U\} \quad \{U\}=\begin{Bmatrix}u\\ v\\ w\end{Bmatrix}$$

$$[NL]=\begin{bmatrix}\frac{1}{2}\left(\frac{\partial}{\partial x}\right)^2 & \frac{1}{2}\left(\frac{\partial}{\partial x}\right)^2 & \frac{1}{2}\left(\frac{\partial}{\partial x}\right)^2\\ \frac{1}{2}\left(\frac{\partial}{\partial y}\right)^2 & \frac{1}{2}\left(\frac{\partial}{\partial y}\right)^2 & \frac{1}{2}\left(\frac{\partial}{\partial y}\right)^2\\ \frac{1}{2}\left(\frac{\partial}{\partial z}\right)^2 & \frac{1}{2}\left(\frac{\partial}{\partial z}\right)^2 & \frac{1}{2}\left(\frac{\partial}{\partial z}\right)^2\\ \frac{\partial}{\partial y}\frac{\partial}{\partial z} & \frac{\partial}{\partial y}\frac{\partial}{\partial z} & \frac{\partial}{\partial y}\frac{\partial}{\partial z}\\ \frac{\partial}{\partial x}\frac{\partial}{\partial z} & \frac{\partial}{\partial x}\frac{\partial}{\partial z} & \frac{\partial}{\partial x}\frac{\partial}{\partial z}\\ \frac{\partial}{\partial x}\frac{\partial}{\partial y} & \frac{\partial}{\partial x}\frac{\partial}{\partial y} & \frac{\partial}{\partial x}\frac{\partial}{\partial y}\end{bmatrix}$$

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SOLID MECHANICS INTRODUCTION

- Dynamic equilibrium:

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} + F_x = \rho \ddot{u}$$

$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} + F_y = \rho \ddot{v}$$

$$\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} + F_z = \rho \ddot{w}$$

$$[L]^T \{\sigma\} + \{F\} = [M]\{\ddot{U}\} \Rightarrow [L]^T [D]\{\varepsilon\} + \{F\} = [M]\{\ddot{U}\}$$

$$[L]^T [D][L]\{U\} + \{F\} = [M]\{\ddot{U}\}$$

- Static equilibrium:

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} + F_x = 0$$

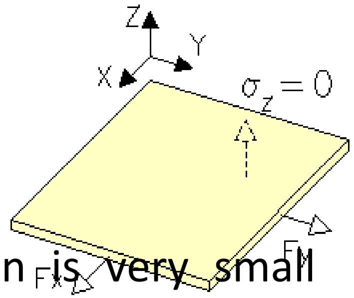
$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} + F_y = 0$$

$$\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} + F_z = 0$$

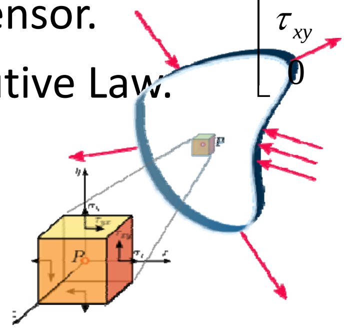
$$[L]^T \{\sigma\} + \{F\} = 0 \Rightarrow [L]^T [D]\{\varepsilon\} + \{F\} = 0$$

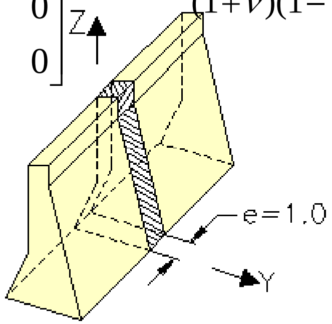
$$[L]^T [D][L]\{U\} + \{F\} = 0$$

SOLID MECHANICS INTRODUCTION



- Plane stress state: stress strain relation for isotropic materials:
  - This usually occurs in structural elements where one dimension is very small compared to the other two, i.e. the element is flat or thin.
  - The stresses are negligible with respect to the smaller dimension as they are not able to develop within the material and are small compared to the in-plane stresses.
  - Stress tensor. $\begin{bmatrix} \sigma_{xx} & \tau_{xy} & 0 \\ \tau_{xy} & \sigma_{yy} & 0 \\ 0 & 0 & 0 \end{bmatrix}$
  - Strain tensor. $\begin{bmatrix} \varepsilon_{xx} & \varepsilon_{xy} & 0 \\ \varepsilon_{xy} & \varepsilon_{yy} & 0 \\ 0 & 0 & \varepsilon_{zz} \end{bmatrix}$
  - Constitutive Law. $\mathbf{D} = \dfrac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1-\nu)/2 \end{bmatrix}$
- Plane strain state: stress strain relation for isotropic materials:
  - If one dimension is very large compared to the others, the principal strain in the direction of the longest dimension is constrained and can be assumed as zero, yielding a plane strain condition.
  - Stress tensor. $\begin{bmatrix} \sigma_{xx} & \tau_{xy} & 0 \\ \tau_{xy} & \sigma_{yy} & 0 \\ 0 & 0 & \sigma_{zz} \end{bmatrix}$
  - Strain tensor. $\begin{bmatrix} \varepsilon_{xx} & \varepsilon_{xy} & 0 \\ \varepsilon_{xy} & \varepsilon_{yy} & 0 \\ 0 & 0 & 0 \end{bmatrix}$
  - Constitutive Law. $\mathbf{D} = \dfrac{E(1-\nu)}{(1+\nu)(1-2\nu)}\begin{bmatrix} 1 & \dfrac{\nu}{1-\nu} & 0 \\ \dfrac{\nu}{1-\nu} & 1 & 0 \\ 0 & 0 & \dfrac{1-2\nu}{2(1-\nu)} \end{bmatrix}$





SOLID MECHANICS INTRODUCTION

- Thin plate theory (Kirchhoff):
  - The behavior of plates is similar to that of beams. They both carry transverse loads by bending action.
  - Strain displacement relation.

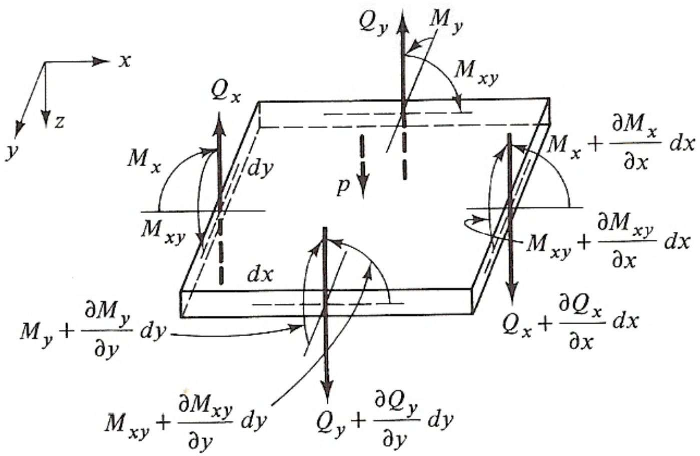
$$\{\varepsilon\}=\begin{Bmatrix}\varepsilon_{xx}\\ \varepsilon_{yy}\\ 2\varepsilon_{xy}\end{Bmatrix}=\begin{Bmatrix}\dfrac{\partial u}{\partial x}\\ \dfrac{\partial v}{\partial y}\\ \dfrac{\partial u}{\partial y}+\dfrac{\partial v}{\partial x}\end{Bmatrix}=-z\begin{Bmatrix}\dfrac{\partial^2 w}{\partial x^2}\\ \dfrac{\partial^2 w}{\partial y^2}\\ 2\dfrac{\partial^2 w}{\partial x\partial y}\end{Bmatrix}$$

  - Stress displacement relation.

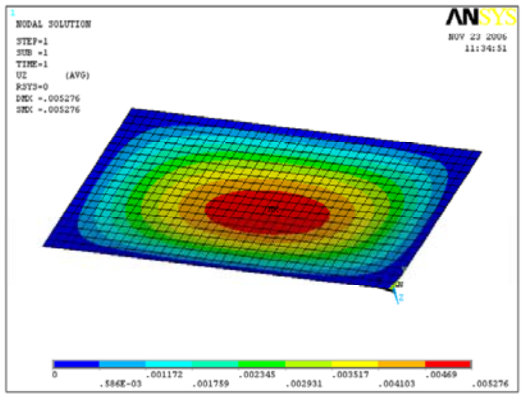
$$\{\sigma\}=\begin{Bmatrix}\sigma_{xx}\\ \sigma_{yy}\\ \sigma_{xy}\end{Bmatrix}=[D]\{\varepsilon\}=-\frac{Ez}{1-\nu^2}\begin{bmatrix}1 & \nu & 0\\ \nu & 1 & 0\\ 0 & 0 & \dfrac{1-\nu}{2}\end{bmatrix}\begin{Bmatrix}\dfrac{\partial^2}{\partial x^2}\\ \dfrac{\partial^2}{\partial y^2}\\ 2\dfrac{\partial^2}{\partial x\partial y}\end{Bmatrix}w$$

  - Matrix formulation

$$\{\sigma\}=[D]\{\varepsilon\}=-z[D]\{L\}w$$

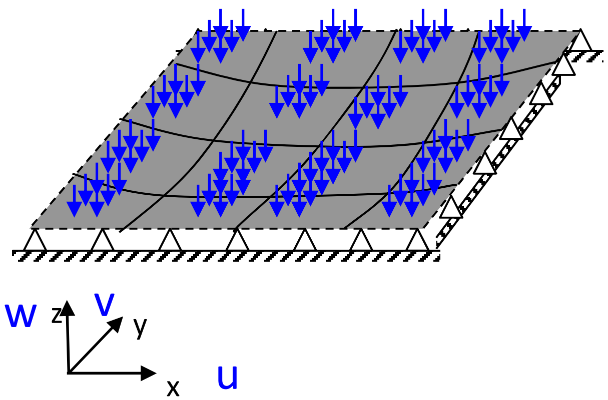


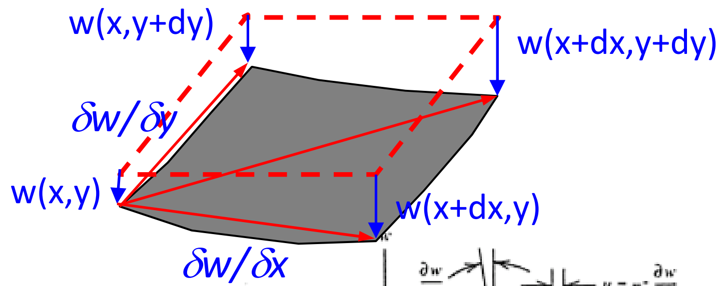
$$[\sigma]=\begin{bmatrix}\sigma_{xx} & \sigma_{xy} & 0\\ \sigma_{yx} & \sigma_{yy} & 0\\ 0 & 0 & 0\end{bmatrix}$$



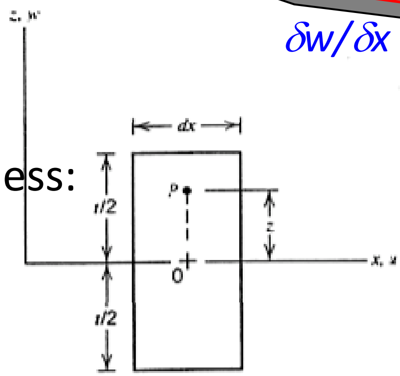
SOLID MECHANICS INTRODUCTION

- Thin plate theory (Kirchhoff):
  - Plates undergo bending which can be represented by the deflection (w) of the middle plane of the plate.
  - The middle plane of the plate undergoes deflections w(x,y). The top and bottom surfaces of the plate undergo deformations almost like a rigid body along with the middle surface. The normal stress in the direction of the plate thickness (z) is assumed to be negligible.
  - Thin plate theory - does not include transverse shear deformations.

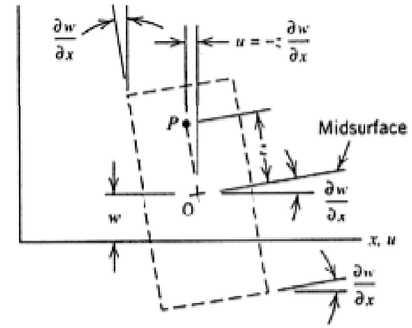




  - Infinitesimal slice of a thin plate thickness:
    - Rectangular angles are preserved



Before loading



After loading

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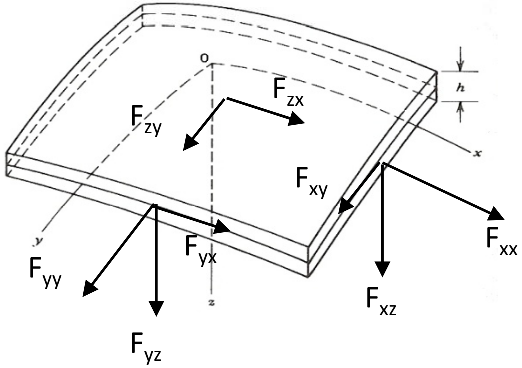
SOLID MECHANICS INTRODUCTION

- Thick plate theory (Mindlin):
  - Strain displacement relation.

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \\ \sigma_{yz} \\ \sigma_{zx} \end{Bmatrix} = \frac{E}{1+\nu} \begin{bmatrix} \frac{1}{1-\nu} & \frac{\nu}{1-\nu} & 0 & 0 & 0 \\ \frac{\nu}{1-\nu} & \frac{1}{1-\nu} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} \end{bmatrix} \times \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{xy} \\ \varepsilon_{yz} \\ \varepsilon_{zx} \end{Bmatrix}$$



$$[\sigma] = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & 0 \end{bmatrix}$$

  - Stress generalized displacement relation.

$$\sigma_{xx} = -\frac{E}{1-\nu^2} z \left(\frac{\partial \theta_x}{\partial x} + \nu \frac{\partial \theta_y}{\partial y} \right)$$

$$\sigma_{yy} = -\frac{E}{1-\nu^2} z \left(\frac{\partial \theta_y}{\partial y} + \nu \frac{\partial \theta_x}{\partial x} \right)$$

$$\sigma_{xy} = -\frac{E}{2(1+\nu)} z \left(\frac{\partial \theta_x}{\partial y} + \frac{\partial \theta_y}{\partial x} \right)$$

$$\sigma_{xz} = \frac{E}{2(1+\nu)} \left(-\theta_x + \frac{\partial w}{\partial x} \right)$$

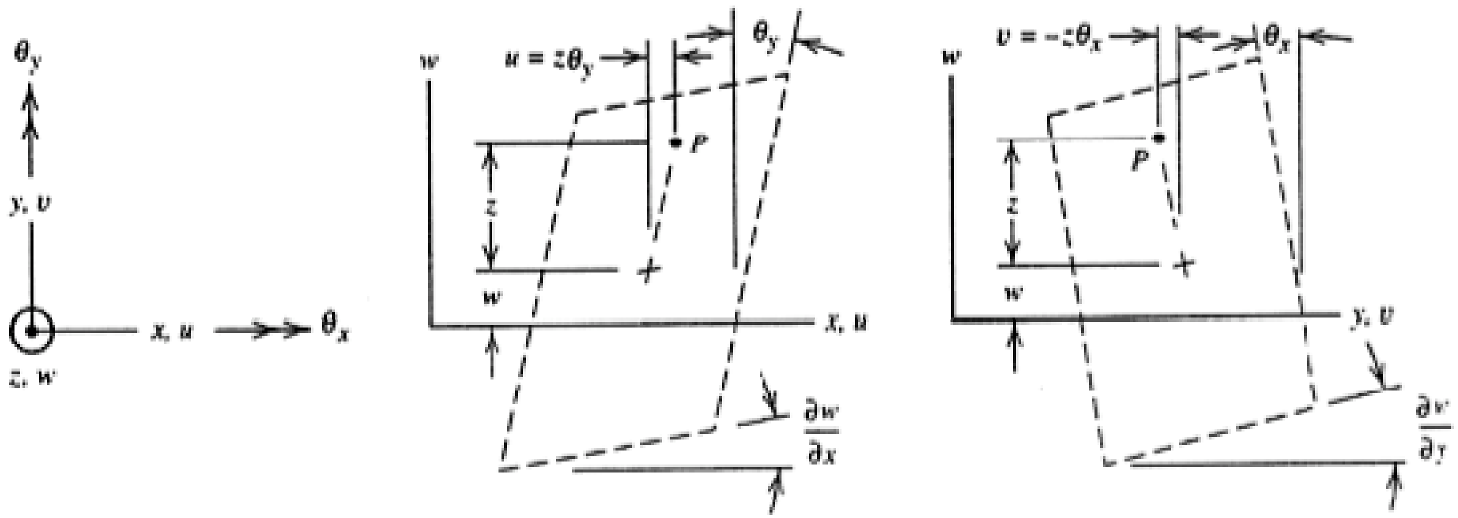
$$\sigma_{yz} = \frac{E}{2(1+\nu)} \left(-\theta_y + \frac{\partial w}{\partial y} \right)$$

SOLID MECHANICS INTRODUCTION

- Thick plate theory (Mindlin):
  - The transverse shear deformation effects are included by relaxing the assumption that plane sections remain perpendicular to middle surface, i.e., the right angles in the BPS element are no longer preserved.
  - Planes initially normal to the middle surface may experience different rotations than the middle surface itself
  - Analogy is the Timoshenko beam theory.

$$\begin{Bmatrix} w \\ \theta_x \\ \theta_y \end{Bmatrix} = \sum_{i=1}^{n} \begin{bmatrix} N_i & 0 & 0 \\ 0 & N_i & 0 \\ 0 & 0 & N_i \end{bmatrix} \begin{Bmatrix} w_i \\ \theta_{xi} \\ \theta_{yi} \end{Bmatrix};$$

Displacement field



SOLID MECHANICS INTRODUCTION

- Yield Criteria for Brittle Materials:
  - Maximum Principal Stress Failure Criteria: Fracture will occur when tensile stress is greater than ultimate tensile strength.

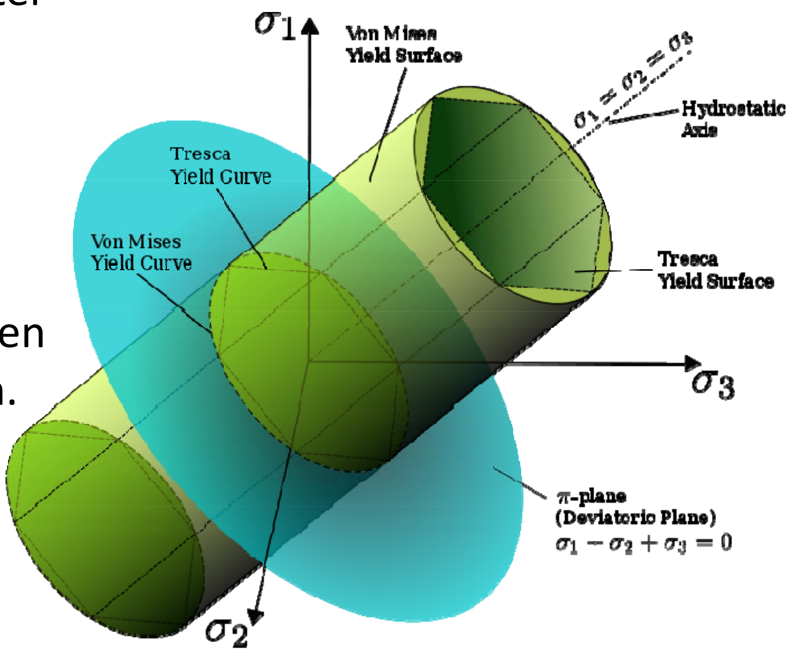
$$\sigma_1 > \sigma_u$$

- Yield Criteria for Ductile Materials:
  - Tresca Failure Criteria: Yielding will occur when shear stress is greater than shear yield strength.

$$\frac{\sigma_1 - \sigma_3}{2} > \frac{\sigma_y}{2} \quad \text{or} \quad \sigma_1 - \sigma_3 > \sigma_y$$

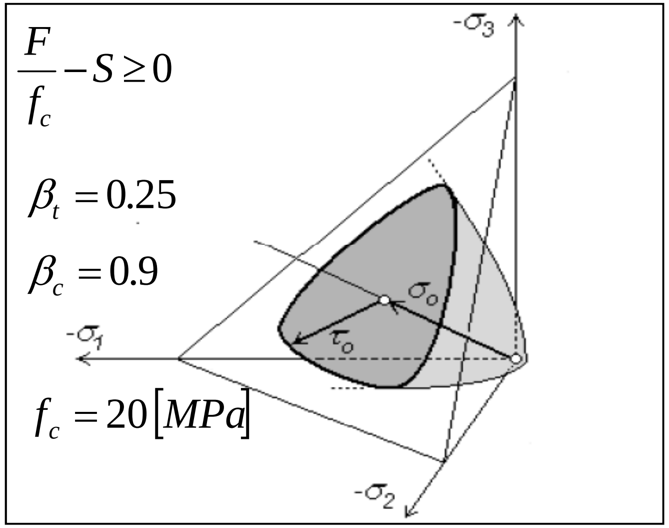
  - von Mises Failure Criteria: Yielding will occur when the von Mises stress is greater than yield strength.

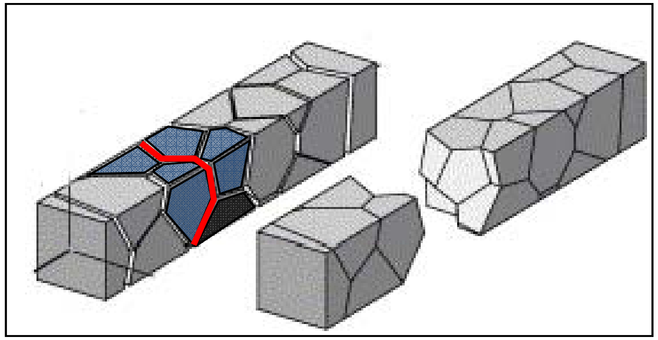
$$\sigma_e = \sqrt{\frac{1}{2}\left[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2\right]} > \sigma_y$$



FRACTURE MECHANICS INTRODUCTION - concrete

- Concrete material should present yield surface criteria:
  - Mohr Coulomb or Drucker Prager.
- It must present also flow rules that may be associated or not:
  - This flow rules defines the orientation of plastic strain. If this orientation is orthogonal to the yield surface than is consider an associative rule.
- Continuous damage theory:
  - Continuum mechanics provides a mean of modelling at the macroscopic level the material damage that occurs at the microscopic level.
  - Damage criterion for concrete (Willam and Warnkle, 1974). Cracks are treated as a "smeared band" of cracks, rather than discrete cracks. The presence of a crack at an integration point is represented through modification of the stress-strain relations, introducing a plane of weakness in a direction normal to the crack face.



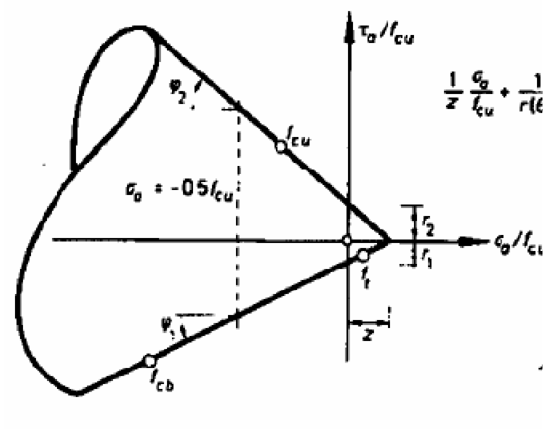


FRACTURE MECHANICS INTRODUCTION - concrete

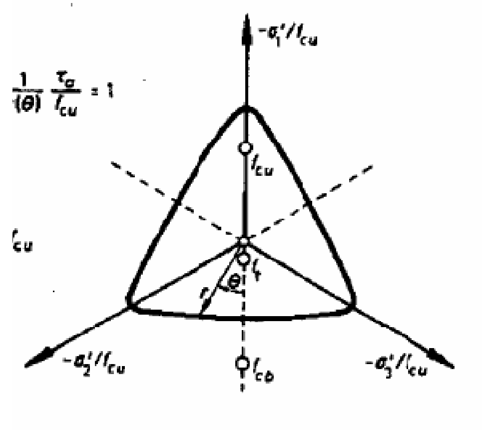
- The failure surface is defined as:

$$\frac{1}{z}\frac{\sigma_a}{f_{cu}} + \frac{1}{r(\theta)}\frac{\tau_a}{f_{cu}} = 1$$

  - σa and τa, are average stress components;
  - Z is the apex of the surface ;
  - *f cu* = uniaxial compressive strength.
  - The opening angles of the hydrostatic cone are defined by φ1 and φ2 . The free parameters of the failure surface *z* and *r* , are identified from the uniaxial compressive strength (*f cu*), biaxial compressive strength (*fcb*), and uniaxial tension strength (*f t*).
- The Willam and Warnke (1974) mathematical model of the failure surface for the concrete has the following advantages:
  - Close fit of experimental data in the operating range;
  - Simple identification of model parameters from standard test data;
  - Smoothness (e.g. continuous surface with continuously varying tangent planes);
  - Convexity (e.g. monotonically curved surface without inflection points).



Hydrostatic Section (θ = 0°)



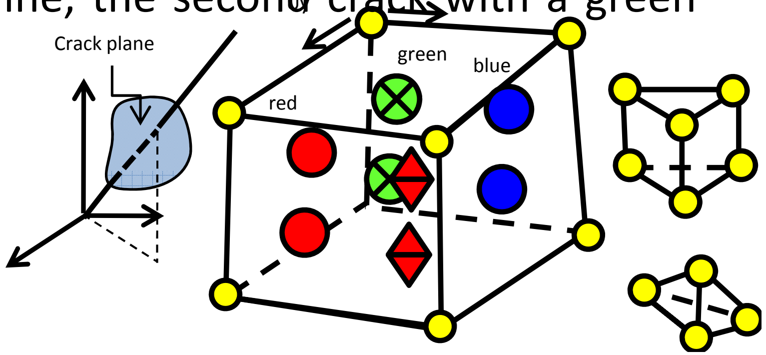
Deviatoric Section (σ_a = -0.5f_{cu})

FRACTURE MECHANICS INTRODUCTION – concrete in ANSYS

- The criterion for failure of concrete due to multiaxial stress state may be represented by this inequality:
  - F is a function of the principal stress state, S is the failure surface and fc is the uniaxial crushing strength.

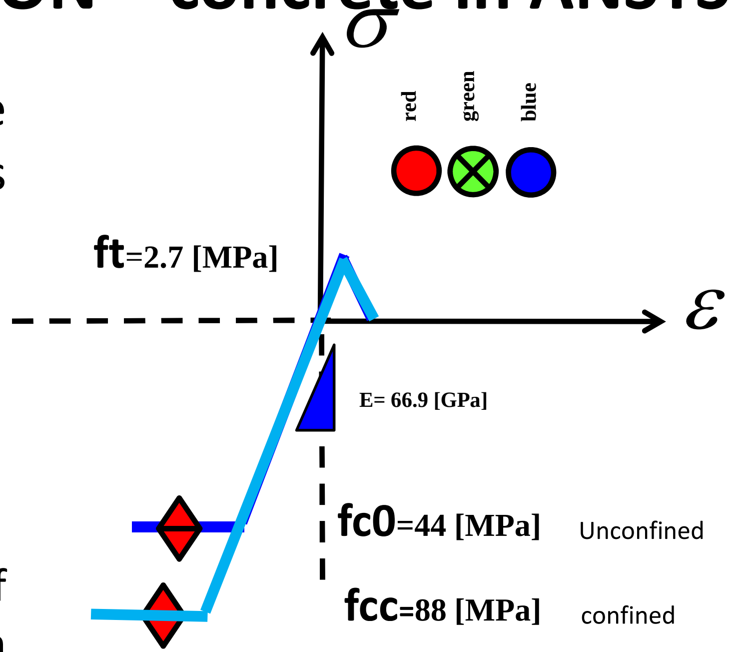
$$\frac{F}{f_c} - S \geq 0$$

- If the inequality is not verified:
  - There is no attendant cracking or crushing.
- Otherwise:
  - The material will crack if any principal stress is of tensile type, while crushing will occur if all principal stresses are compressive type.
  - Failure by crackings is represented with a circle outline in the plane of the crack.
  - Failure by crushing is represented by an octahedron outline.
  - If the crack has opened and then closed, the circle outline will have an X through it.
  - Each integration point can crack in up to three different planes. The first crack at an integration point is shown with a red circle outline, the second crack with a green outline, and the third crack with a blue outline.

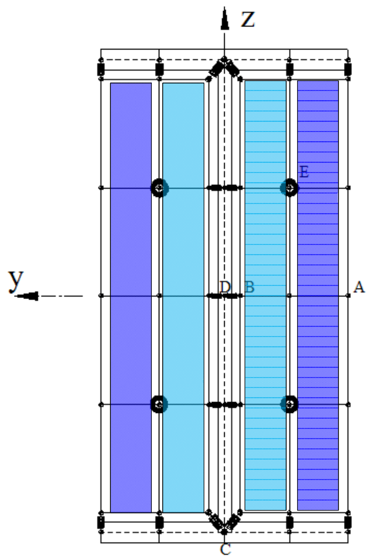


FRACTURE MECHANICS INTRODUCTION – concrete in ANSYS

- A total of five input strength parameters are needed to define the failure surface as well as an ambient hydrostatic stress state .
  - Ultimate uniaxial tensile strength (ft)
  - Ultimate uniaxial compressive strength (fc)
  - Ultimate biaxial compressive strength (fcb)
  - Ambient hydrostatic stress state (σah)
  - Ultimate compressive strength for a state of biaxial compression superimposed on hydrostatic stress state (f1)
  - Ultimate compressive strength for a state of uniaxial compression superimposed on hydrostatic stress state (f2)
- However the failure surface can be specified with a minimum of two constants:
  - f_t and f c.
  - The other parameters default: fcb=1.2 fc, f1=1.45fc, f2=1.725 fc.

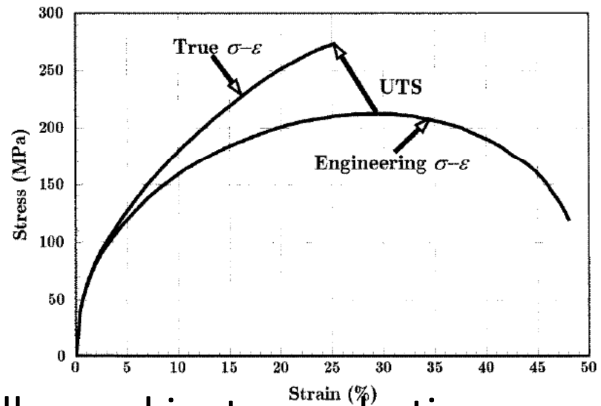


$$\frac{f_{cc}}{f_{co}} = -1.254 + 2.254\sqrt{1 + 7.94\alpha_c \frac{f_1}{f_{co}}} - 2\alpha_c \frac{f_1}{f_{co}}$$



PLASTICITY

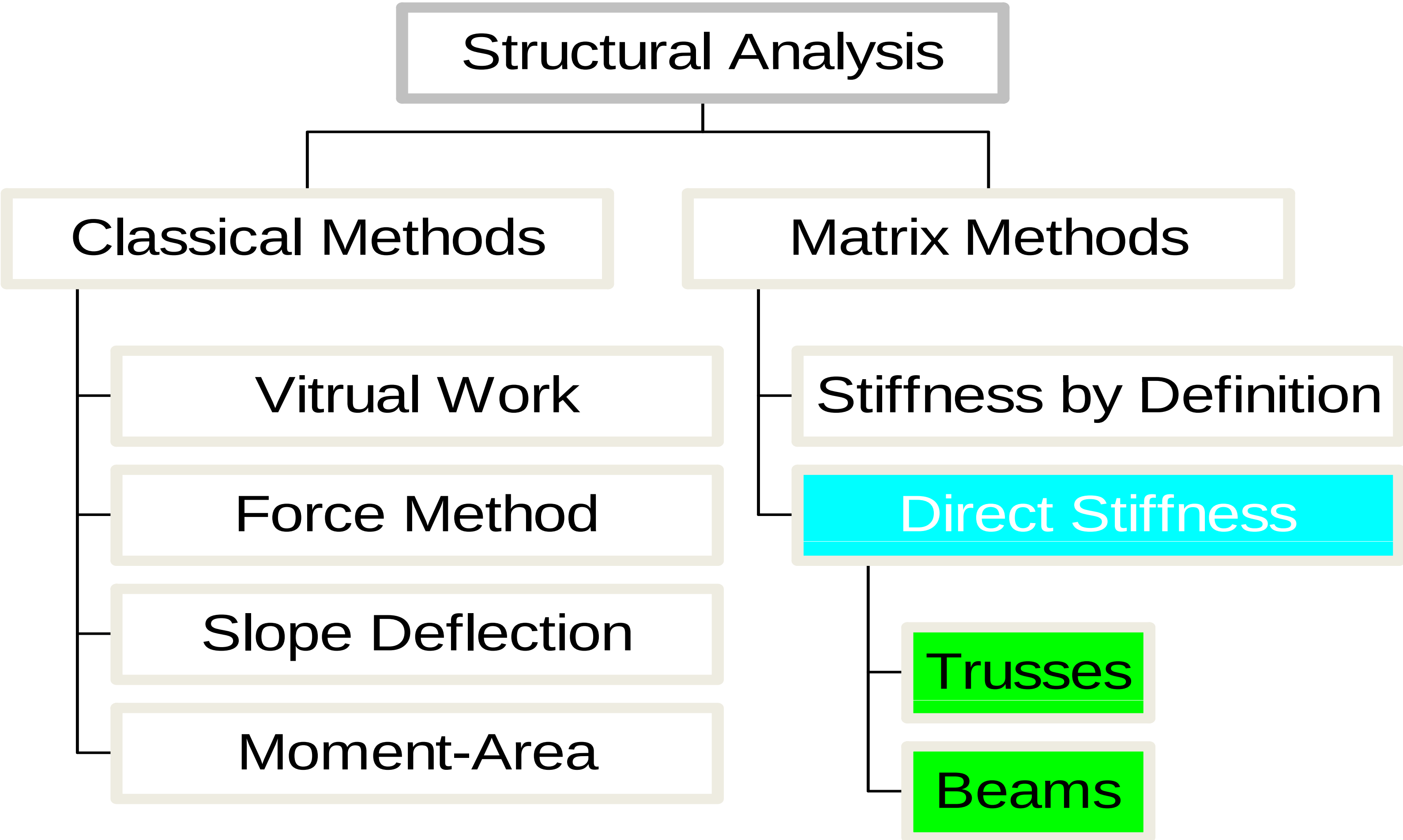
- Plasticity theory provides a mathematical relation that characterizes the elasto plastic behaviour of different materials.
- The main parameters during plastic analysis are:
  - Yield criterion (determines the stress level at which yielding is initiated. For complex stress tensors, this function depends on several stress components, which may be interpreted as an equivalent stress).
  - Flow rule (determines the direction of plastic straining).
  - Hardening rule (describes the changing of the yield surface with progressive yielding):
    - ISOTROPIC hardening.
    - KINEMATICS hardening.



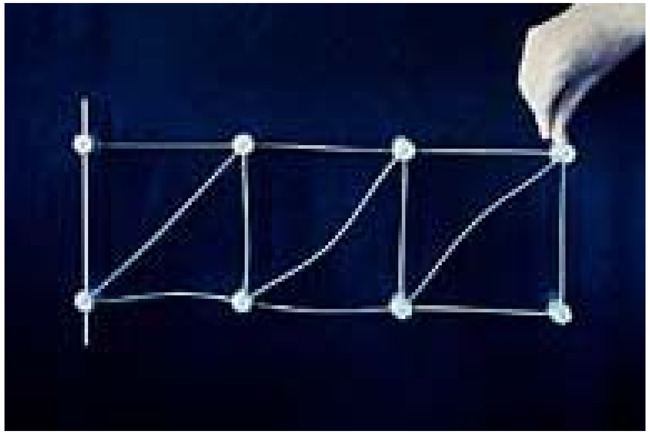
- Stress strain curves:
  - For finite element analysis with plasticity, what we really need is stress-plastic strain curve.
  - convert the Engineering stress strain to true stress strain, using equations.
    - True Strain e=ln(1+E) ; E = engineering strain
    - True stresss s=S(1+E) ;S= engineering stress

DIRECT STIFFNESS METHOD

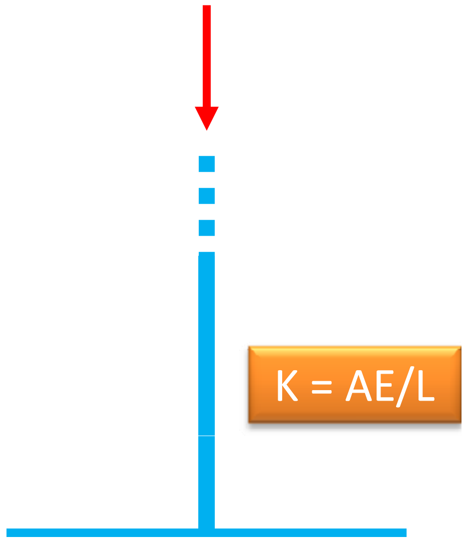
- Introduction to structural analysis:



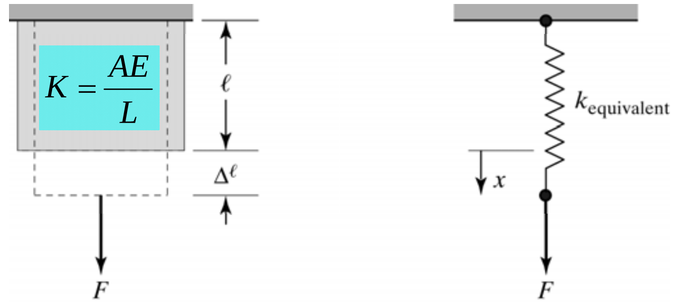
DIRECT STIFFNESS METHOD



- Truss analysis:
  - Finite elements to be used: Bars.
  - Composed of slender, lightweight members .
  - All loading occurs on joints.
  - No moments or rotations in the joints.
  - Axial Force Members (Tension (+) , Compression (-)).
- Stiffness:
  - Kij = the amount of force required at i to cause a unit displacement at j, with displacements at all other DOF = zero
  - A function of:
    - System geometry.
    - Material properties (E, A).
    - Boundary conditions (Pinned, Roller or Free for a truss).
    - NOT a function of external loads.



DIRECT STIFFNESS METHOD

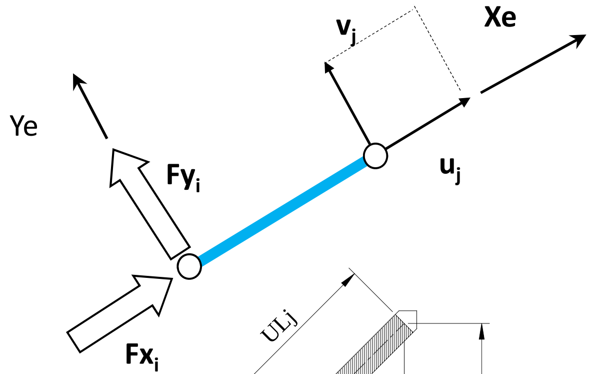


- From mechanics of materials (Strength of materials):
  - Element stiffness may be calculated according to:
    - Spring behaviour:

$$F = K \times \delta \quad \Leftrightarrow \quad K = \frac{f}{\delta}$$

    - Axial deformation of a structural element (stress and deformation definitions):

$$\delta = \frac{FL}{AE} \quad \Leftrightarrow \delta\left(\frac{AE}{L}\right) = F$$



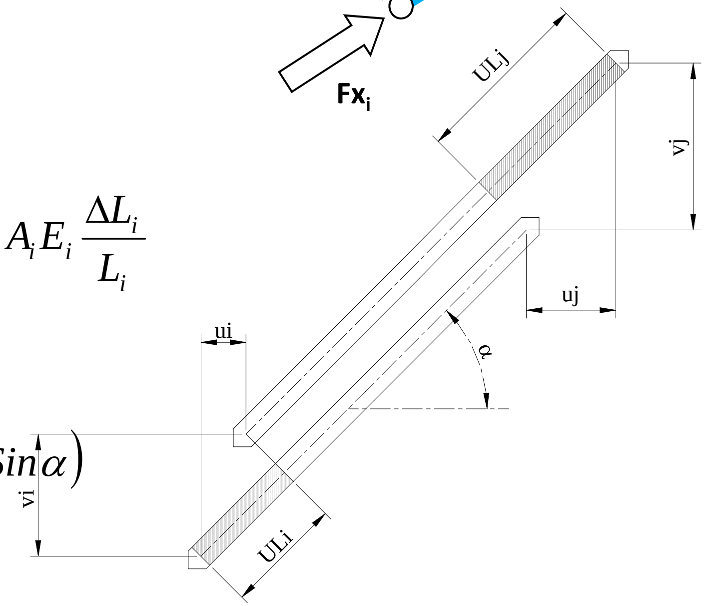
    - Internal effort (normal)
      - Local coordinates

$$N = \frac{AE}{L} \times \delta$$

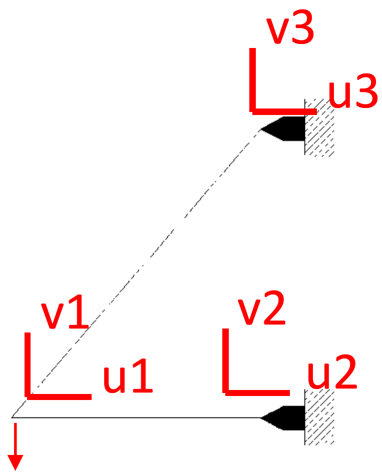
$$N_i = A_i E_i \frac{\Delta L_i}{L_i}$$

      - Global coordinates

$$N_k = \frac{A_k E_k}{L_k}\left(u_j Cos\alpha + v_j Sin\alpha - u_i Cos\alpha - v_i Sin\alpha\right)$$



DIRECT STIFFNESS METHOD



- We can create a stiffness matrix that accounts for the material and geometric properties of the structure
- A square, symmetric matrix Kij = Kji
- Diagonal terms always positive
- The stiffness matrix is independent of the loads acting on the structure.
- Many loading cases can be tested without recalculating the stiffness matrix.

K_{11}	K_{12}	K_{13}	K_{14}	K_{15}	K_{16}	u_1		R_1
K_{21}	K_{22}	K_{23}	K_{24}	K_{25}	K_{26}	v_1		R_2
K_{31}	K_{32}	K_{33}	K_{34}	K_{35}	K_{36}	u_2	=	R_3
K_{41}	K_{42}	K_{43}	K_{44}	K_{45}	K_{46}	v_2		R_4
K_{51}	K_{52}	K_{53}	K_{54}	K_{55}	K_{56}	u_3		R_5
K_{61}	K_{62}	K_{63}	K_{64}	K_{65}	K_{66}	v_3		R_6

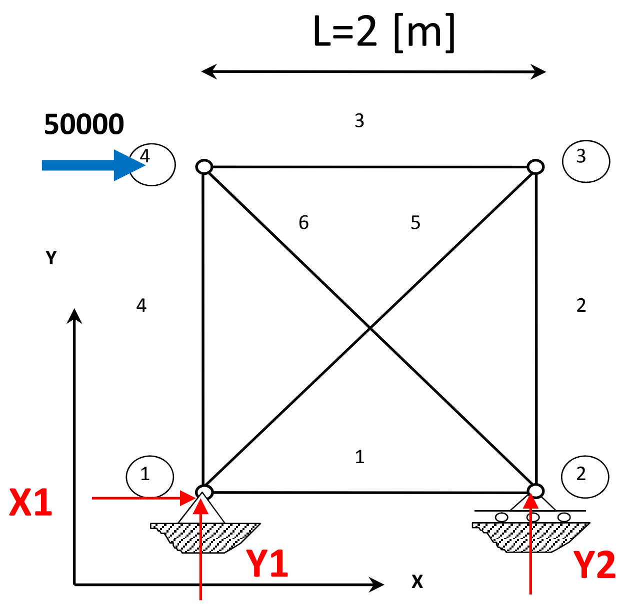
DIRECT STIFFNESS METHOD

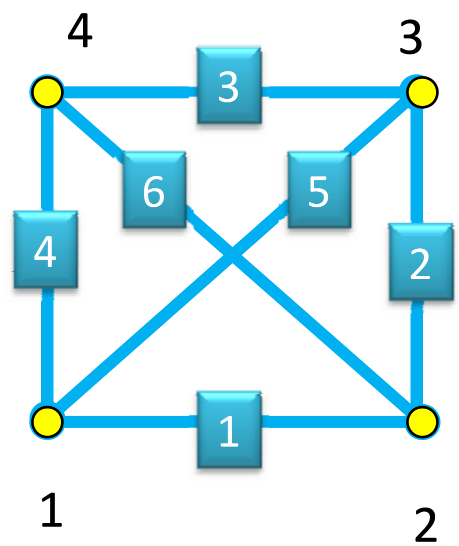
- Stiffness by Definition
  - 2 Degrees of Freedom
- Direct Stiffness
  - 6 Degrees of Freedom
  - DOFs 3,4,5,6 = 0
  - Unknown Reactions (to be solved) included in Loading Matrix
- Stiffness by Definition Solution in RED
- Direct Stiffness Solution in RED/YELLOW
- The fundamental Procedure:
  - Calculate the Stiffness Matrix.
  - Determine Local Stiffness Matrix, Ke.
  - Transform it into Global Coordinates, KG.
  - Assemble all matrices.
  - Solve for the Unknown Displacements .
  - Use unknown displacements to solve for the Unknown Reactions.
  - Calculate the Internal Forces.

$$\begin{bmatrix} K_{11} & K_{12} & K_{13} & K_{14} & K_{15} & K_{16} \\ K_{21} & K_{22} & K_{23} & K_{24} & K_{25} & K_{26} \\ K_{31} & K_{32} & K_{33} & K_{34} & K_{35} & K_{36} \\ K_{41} & K_{42} & K_{43} & K_{44} & K_{45} & K_{46} \\ K_{51} & K_{52} & K_{53} & K_{54} & K_{55} & K_{56} \\ K_{61} & K_{62} & K_{63} & K_{64} & K_{65} & K_{66} \end{bmatrix} \begin{bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{bmatrix} = \begin{bmatrix} R_1 \\ R_2 \\ R_3 \\ R_4 \\ R_5 \\ R_6 \end{bmatrix}$$

DIRECT STIFFNESS METHOD – case 1

- Thematic exercise:
  - First, decompose the entire structure into a set of finite elements.
  - Build a stiffness matrix for each element (6 Here).
  - Later, transform all of the local stiffness element matrices into global stiffness element matrix.
  - Assembly all the element global stiffness matrix.
  - Solve problem:
    - E=2.1E11 [N/m2]
    - A=0.001 [m2]

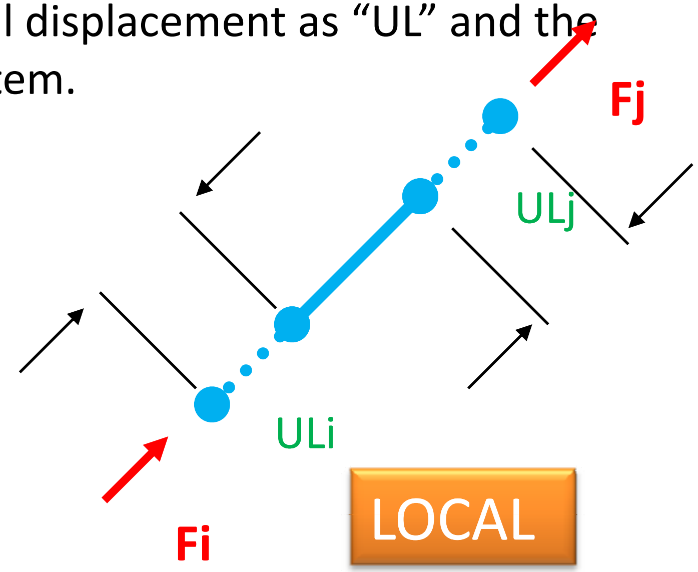




Element	Node i	Node j	Angle
1	1	2	0
2	2	3	90
3	3	4	180
4	1	4	90
5	1	3	45
6	2	4	135

DIRECT STIFFNESS METHOD – case 1

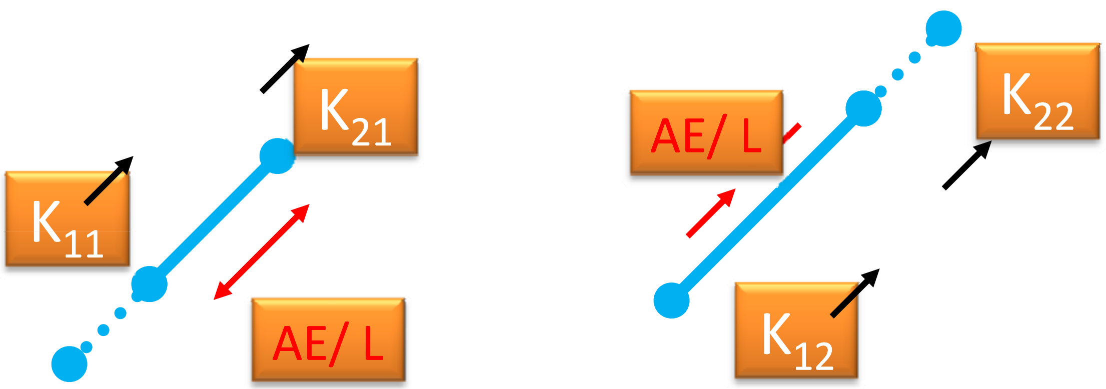
- Element Stiffness Matrix in Local Coordinates:
  - Remember Kij= the amount of force required at i to cause a unit displacement at j, with displacements at all other DOF = zero.
  - For a truss element (which has 2 DOF), the axial displacement as "UL" and the internal force as "F" in the local coordinate system.



$$K_{11} \times UL_i + K_{12} \times UL_j = F_i$$

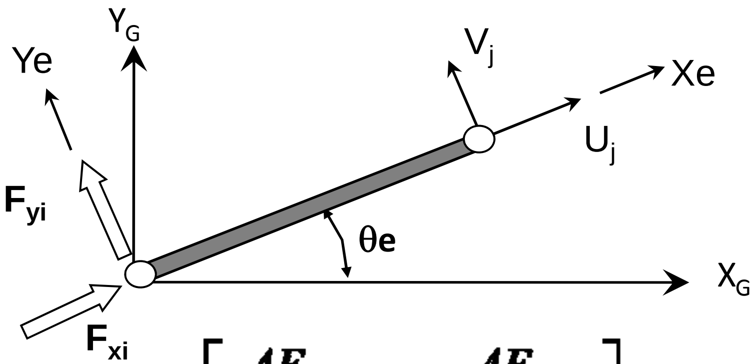
$$K_{21} \times UL_i + K_{22} \times UL_j = F_j$$

$$\begin{bmatrix} \frac{EA}{L} & -\frac{EA}{L} \\ -\frac{EA}{L} & \frac{EA}{L} \end{bmatrix} \begin{Bmatrix} UL_i \\ UL_j \end{Bmatrix} = \begin{Bmatrix} F_i \\ F_j \end{Bmatrix}$$

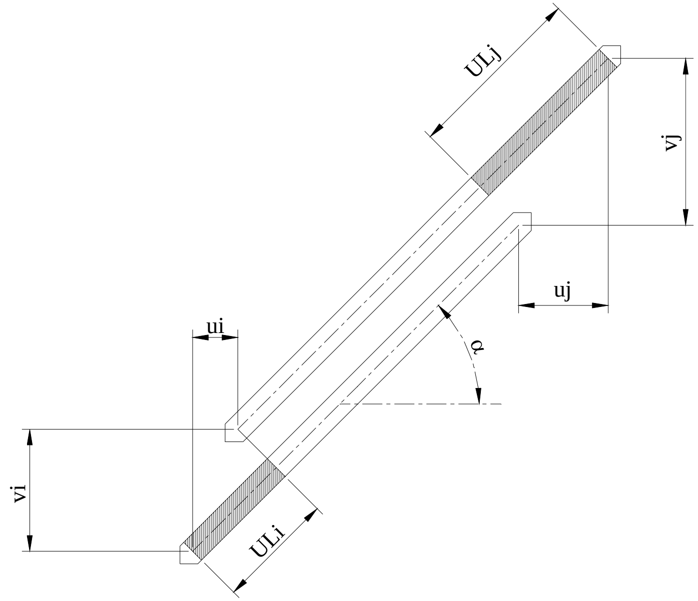


DIRECT STIFFNESS METHOD – case 1

- Element Stiffness Matrix in Local Coordinates (expanded to matrix dimension)



$$[K_e]_{local} = \begin{bmatrix} \frac{AE}{Le} & 0 & \frac{-AE}{Le} & 0 \\ 0 & 0 & 0 & 0 \\ \frac{-AE}{Le} & 0 & \frac{AE}{Le} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

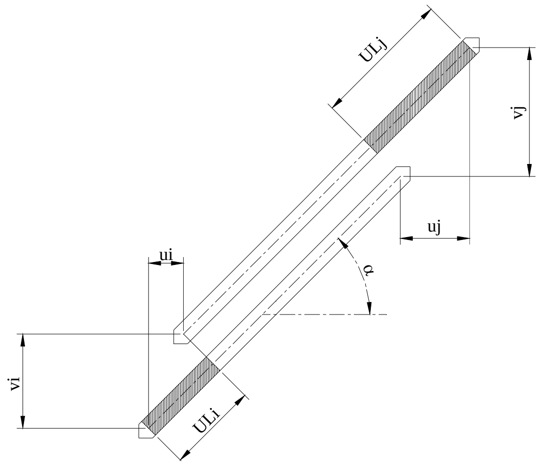


- Element Stiffness Matrix in Global Coordinates

$$[K_e]_{global} = \begin{bmatrix} \frac{AE}{L}\cos^2(\theta) & \frac{AE}{L}\cos(\theta)\sin(\theta) & -\frac{AE}{L}\cos^2(\theta) & -\frac{AE}{L}\cos(\theta)\sin(\theta) \\ \frac{AE}{L}\cos(\theta)\sin(\theta) & \frac{AE}{L}\sin^2(\theta) & -\frac{AE}{L}\cos(\theta)\sin(\theta) & -\frac{AE}{L}\sin^2(\theta) \\ -\frac{AE}{L}\cos^2(\theta) & -\frac{AE}{L}\cos(\theta)\sin(\theta) & \frac{AE}{L}\cos^2(\theta) & \frac{AE}{L}\cos(\theta)\sin(\theta) \\ -\frac{AE}{L}\cos(\theta)\sin(\theta) & -\frac{AE}{L}\sin^2(\theta) & \frac{AE}{L}\cos(\theta)\sin(\theta) & \frac{AE}{L}\sin^2(\theta) \end{bmatrix}$$

DIRECT STIFFNESS METHOD – case 1

- Relation between local and global displacement values, using angle of inclination:



$$u_j = U_j \cos(\theta) - V_j \sin(\theta)$$
$$v_j = U_j \sin(\theta) + V_j \cos(\theta)$$

$$u_i = U_i \cos(\theta) - V_i \sin(\theta)$$
$$v_i = U_i \sin(\theta) + V_i \cos(\theta)$$

$$\begin{Bmatrix} u_i \\ v_i \\ u_j \\ v_j \end{Bmatrix} = \begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 & 0 \\ \sin(\theta) & \cos(\theta) & 0 & 0 \\ 0 & 0 & \cos(\theta) & -\sin(\theta) \\ 0 & 0 & \sin(\theta) & \cos(\theta) \end{bmatrix} \begin{Bmatrix} U_i \\ V_i \\ U_j \\ V_j \end{Bmatrix}$$

- Relation between local and global displacement values, using nodal coordinates:

$$\begin{Bmatrix} u_i \\ v_i \\ u_j \\ v_j \end{Bmatrix} = \begin{bmatrix} l & -m & 0 & 0 \\ m & l & 0 & 0 \\ 0 & 0 & l & -m \\ 0 & 0 & m & l \end{bmatrix} \begin{Bmatrix} U_i \\ V_i \\ U_j \\ V_j \end{Bmatrix}$$

$$l = \frac{x_j - x_i}{L^{(e)}} \qquad L^{(e)} = \sqrt{(y_j - y_i)^2 + (x_j - x_i)^2}$$

$$l = \frac{y_j - y_i}{L^{(e)}}$$

DIRECT STIFFNESS METHOD – case 1

- Element stiffness matrix in global coordinates:

$$\left[K^e_{global}\right]=\left[T_{local->global}\right]\left[K^e_{local}\right]\left[T_{local->global}\right]^t$$

$$\left[K^e_{global}\right]=\frac{EA}{L^{(e)}}\begin{bmatrix} l^2 & lm & -l^2 & -lm \\ lm & m^2 & -lm & -m^2 \\ -l^2 & -lm & l^2 & lm \\ -lm & -m^2 & lm & m^2 \end{bmatrix}$$

- Element stress in local coordinates:

$$\sigma = E\frac{U_j - U_i}{L^{(e)}} = \frac{E}{L^{(e)}}\langle -1 \quad 1\rangle \begin{Bmatrix} U_i \\ U_j \end{Bmatrix}$$

- Element stress in global coordinates:

$$\sigma = \frac{E}{L^{(e)}}\langle -1 \quad 1\rangle \begin{bmatrix} l & m & 0 & 0 \\ 0 & 0 & l & m \end{bmatrix} \begin{Bmatrix} u_i \\ v_i \\ u_j \\ v_j \end{Bmatrix} = \frac{E}{L^{(e)}}\langle -l \quad -m \quad l \quad m\rangle \begin{Bmatrix} u_i \\ v_i \\ u_j \\ v_j \end{Bmatrix}$$

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DIRECT STIFFNESS METHOD – case 1

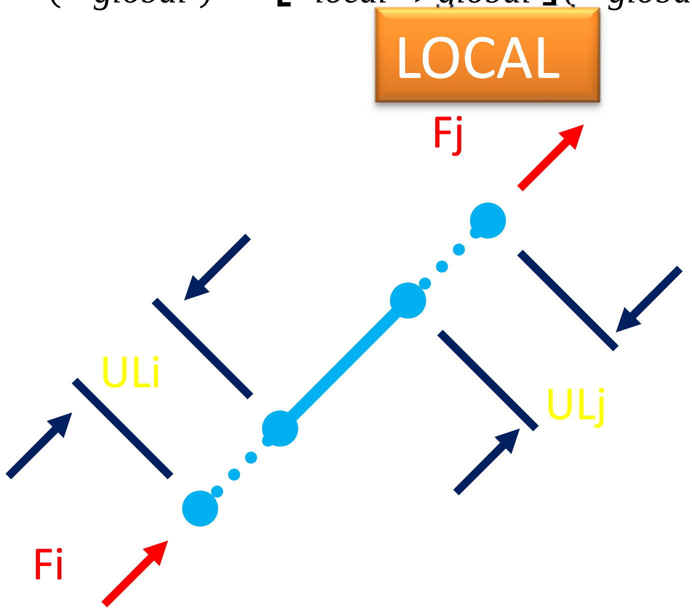
- Structures are composed of many members with many orientations
- The Element stiffness matrix must be transformed from a local to a global coordinate system.

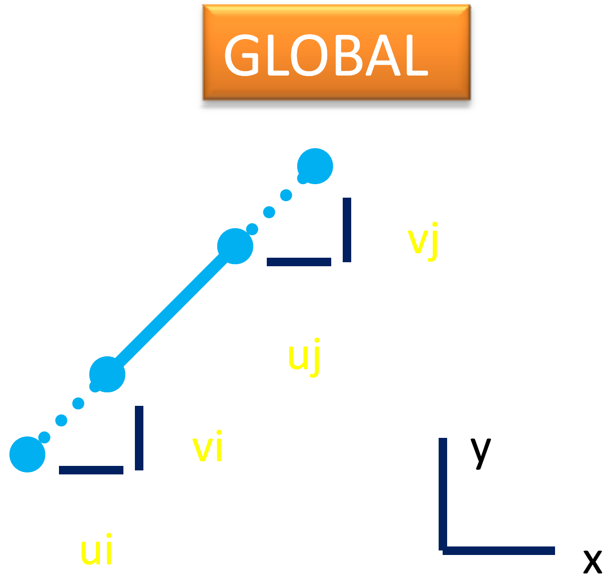
$$\left[K^{e}_{global}\right]=\left[T_{local->global}\right]\left[K^{e}_{local}\right]\left[T_{local->global}\right]^{t}$$

$$\left[T_{local->global}\right]=\begin{bmatrix}\cos(\theta_e) & -\sin(\theta_e) & 0 & 0\\ \sin(\theta_e) & \cos(\theta_e) & 0 & 0\\ 0 & 0 & \cos(\theta_e) & -\sin(\theta_e)\\ 0 & 0 & \sin(\theta_e) & \cos(\theta_e)\end{bmatrix}$$

- The load vector must be transformed from a local to a global coordinate system.

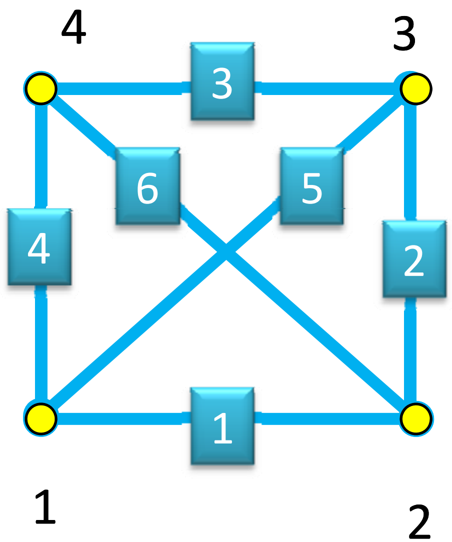
$$\left\{F^{e}_{global}\right\}=\left[T_{local->global}\right]\left\{F^{e}_{global}\right\}$$





DIRECT STIFFNESS METHOD – case 1

- Element stiffness matrix in global coordinates.



Element 1

$$\frac{AE}{L_1}\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Element 2

$$\frac{AE}{L_2}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}$$

Element 3

$$\frac{AE}{L_3}\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Element 4

$$\frac{AE}{L_4}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}$$

Element 5

$$\frac{AE}{L_5}\begin{bmatrix} 1/2 & 1/2 & -1/2 & -1/2 \\ 1/2 & 1/2 & -1/2 & -1/2 \\ -1/2 & -1/2 & 1/2 & 1/2 \\ -1/2 & -1/2 & 1/2 & 1/2 \end{bmatrix}$$

Element 6

$$\frac{AE}{L_6}\begin{bmatrix} 1/2 & -1/2 & -1/2 & 1/2 \\ -1/2 & 1/2 & 1/2 & -1/2 \\ -1/2 & 1/2 & 1/2 & -1/2 \\ 1/2 & -1/2 & -1/2 & 1/2 \end{bmatrix}$$

DIRECT STIFFNESS METHOD – case 1

- Assembling element matrices:
  - Each element matrix must be positioned in respect to the global degree of freedom that is related to.

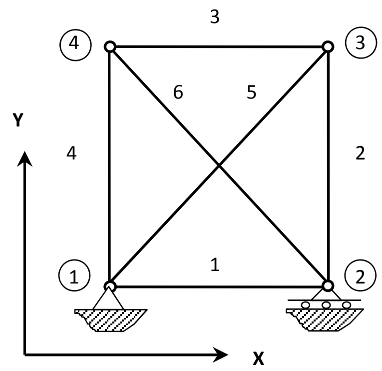
	u1	v1	u2	v2	u3	v3	u4	v4
u1	$K^e_1+K^e_5+K^e_4$	$K^e_1+K^e_5+K^e_4$	K^e_1	K^e_1	K^e_5	K^e_5	K^e_4	K^e_4
v1	$K^e_1+K^e_5+K^e_4$	$K^e_1+K^e_5+K^e_4$	K^e_1	K^e_1	K^e_5	K^e_5	K^e_4	K^e_4
u2	K^e_1	K^e_1	$K^e_1+K^e_2+K^e_6$	$K^e_1+K^e_2+K^e_6$	K^e_2	K^e_2	K^e_6	K^e_6
v2	K^e_1	K^e_1	$K^e_1+K^e_2+K^e_6$	$K^e_1+K^e_2+K^e_6$	K^e_2	K^e_2	K^e_6	K^e_6
u3	K^e_5	K^e_5	K^e_2	K^e_2	$K^e_2+K^e_3+K^e_5$	$K^e_2+K^e_3+K^e_5$	K^e_3	K^e_3
v3	K^e_5	K^e_5	K^e_2	K^e_2	$K^e_2+K^e_3+K^e_5$	$K^e_2+K^e_3+K^e_5$	K^e_3	K^e_3
u4	K^e_4	K^e_4	K^e_6	K^e_6	K^e_3	K^e_3	$K^e_3+K^e_6+K^e_4$	$K^e_3+K^e_6+K^e_4$
v4	K^e_4	K^e_4	K^e_6	K^e_6	K^e_3	K^e_3	$K^e_3+K^e_6+K^e_4$	$K^e_3+K^e_6+K^e_4$

	u1	v1	u2	v2	u3	v3	u4	v4
u1	$1/L_1+1/(2L_5)+0$	$0+1/(2L_5)+0$	$-1/L_1$	0	$-1/(2L_5)$	$-1/(2L_5)$	0	0
v1	$0+1/(2L_5)+0$	$0+1/(2L_5)+1/L_4$	0	0	$-1/(2L_5)$	$-1/(2L_5)$	0	$-1/L_4$
u2	$-1/L_1$	0	$1/L_1+0+1/(2L_6)$	$0+0-1/(2L_6)$	0	0	$-1/(2L_6)$	$-1/(2L_6)$
v2	0	0	$0+0-1/(2L_6)$	$0+1/L_2+1/(2L_6)$	K^e_2	K^e_2	$1/(2L_6)$	$-1/(2L_6)$
u3	$-1/(2L_5)$	$-1/(2L_5)$	0	0	$0+1/L_3+1/(2L_5)$	$0+0+1/(2L_5)$	$-1/L_3$	0
v3	$-1/(2L_5)$	$-1/(2L_5)$	0	$-1/L_2$	$0+0+1/(2L_5)$	$1/L_2+0+1/(2L_5)$	0	0
u4	0	0	$-1/(2L_6)$	$1/(2L_6)$	$-1/L_3$	0	$1/L_3+1/(2L_6)+0$	$0-1/(2L_6)+0$
v4	0	$-1/L_4$	$1/(2L_6)$	$-1/(2L_6)$	0	0	$0-1/(2L_6)+0$	$0+1/(2L_6)+1/L_4$

DIRECT STIFFNESS METHOD – case 1

- Introducing boundary conditions:
  - Eliminate lines and columns where displacements are known (method 1)
  - Solve the remaining system of algebraic equations.

0.67677	0.17677	-0.5	0	-0.17677	-0.17677	0	0	0
0.17677	0.67677	0	0	-0.17677	-0.17677	0	-0.5	0
-0.5	0	0.67677	-0.17677	0	0	-0.17677	0.17677	0
0	0	-0.17677	0.67677	0	-0.5	0.17677	-0.17677	0
-0.17677	-0.17677	0	0	0.67677	0.17677	-0.5	0	0
-0.17677	-0.17677	0	-0.5	0.17677	0.67677	0	0	0
0	0	-0.17677	0.17677	-0.5	0	0.67677	-0.17677	50000
0	-0.5	0.17677	-0.17677	0	0	-0.17677	0.67677	0



0.67677	0	0	-0.17677	0.17677	0
0	0.67677	0.17677	-0.5	0	0
0	0.17677	0.67677	0	0	0
-0.17677	-0.5	0	0.67677	-0.17677	50000
0.17677	0	0	-0.17677	0.67677	0

$$u2 = 0.000238\,(m)$$
$$u3 = 0.000911\,(m)$$
$$v3 = -0.000238\,(m)$$
$$u4 = 0.0011496\,(m)$$
$$v4 = 0.000238\,(m)$$

METHODS FOR INTRODUCING BOUNDARY CONDITIONS

- Method 1:
  - Assuming the following system of equations
    - fi represent external forces or reactions at prescribed displacements.
  - Assuming any nodal known displacement, for example u2=ů2.

$$\begin{cases} k_{11}u_1 & + & k_{12}u_2 & + & k_{13}u_3 & + & \ldots & + & k_{1n}u_n & = & f_1 \\ k_{21}u_1 & + & k_{22}u_2 & + & k_{23}u_3 & + & \ldots & + & k_{2n}u_n & = & f_2 \\ k_{31}u_1 & + & k_{32}u_2 & + & k_{33}u_3 & + & \ldots & + & k_{3n}u_n & = & f_3 \\ \vdots & & \vdots & & \vdots & & & & \vdots & & \vdots \\ k_{n1}u_1 & + & k_{n2}u_2 & + & k_{n3}u_3 & + & \ldots & + & k_{nn}u_n & = & f_n \end{cases}$$

  - Eliminate the line and column number associated with prescribed degree of freedom, and modify second member of each equation according to:

$$\begin{cases} k_{11}u_1 & + & k_{13}u_3 & + & \ldots & + & k_{1n}u_n & = & f_1 - k_{12}\dot{u}_2 \\ k_{31}u_1 & + & k_{33}u_3 & + & \ldots & + & k_{2n}u_n & = & f_2 - k_{32}\dot{u}_2 \\ \vdots & & \vdots & & & & \vdots & & \vdots \\ k_{n1}u_1 & + & k_{n3}u_3 & + & & + & k_{3n}u_n & = & f_n - k_{n2}\dot{u}_2 \end{cases}$$

  - Solve the equation number that corresponds to each prescribed value:

$$f_2 = k_{21}u_1 + k_{22}\dot{u}_2 + k_{23}u_3 + \ldots + k_{2n}u_n$$

METHODS FOR INTRODUCING BOUNDARY CONDITIONS

- Method 2 (Penalty):
  - Assuming the following system of equations
    - fi represent external forces or reactions at prescribed displacements.
  - Assuming any nodal known displacement, for example u2=ů2.

$$\begin{cases} k_{11}u_1 & + & k_{12}u_2 & + & k_{13}u_3 & + & \dots & + & k_{1n}u_n & = & f_1 \\ k_{21}u_1 & + & k_{22}u_2 & + & k_{23}u_3 & + & \dots & + & k_{2n}u_n & = & f_2 \\ k_{31}u_1 & + & k_{32}u_2 & + & k_{33}u_3 & + & \dots & + & k_{3n}u_n & = & f_3 \\ \vdots & & \vdots & & \vdots & & & & \vdots & & \vdots \\ k_{n1}u_1 & + & k_{n2}u_2 & + & k_{n3}u_3 & + & \dots & + & k_{nn}u_n & = & f_n \end{cases}$$

  - Multiply the element from diagonal corresponding to the known variable and also the independent member, as shown.

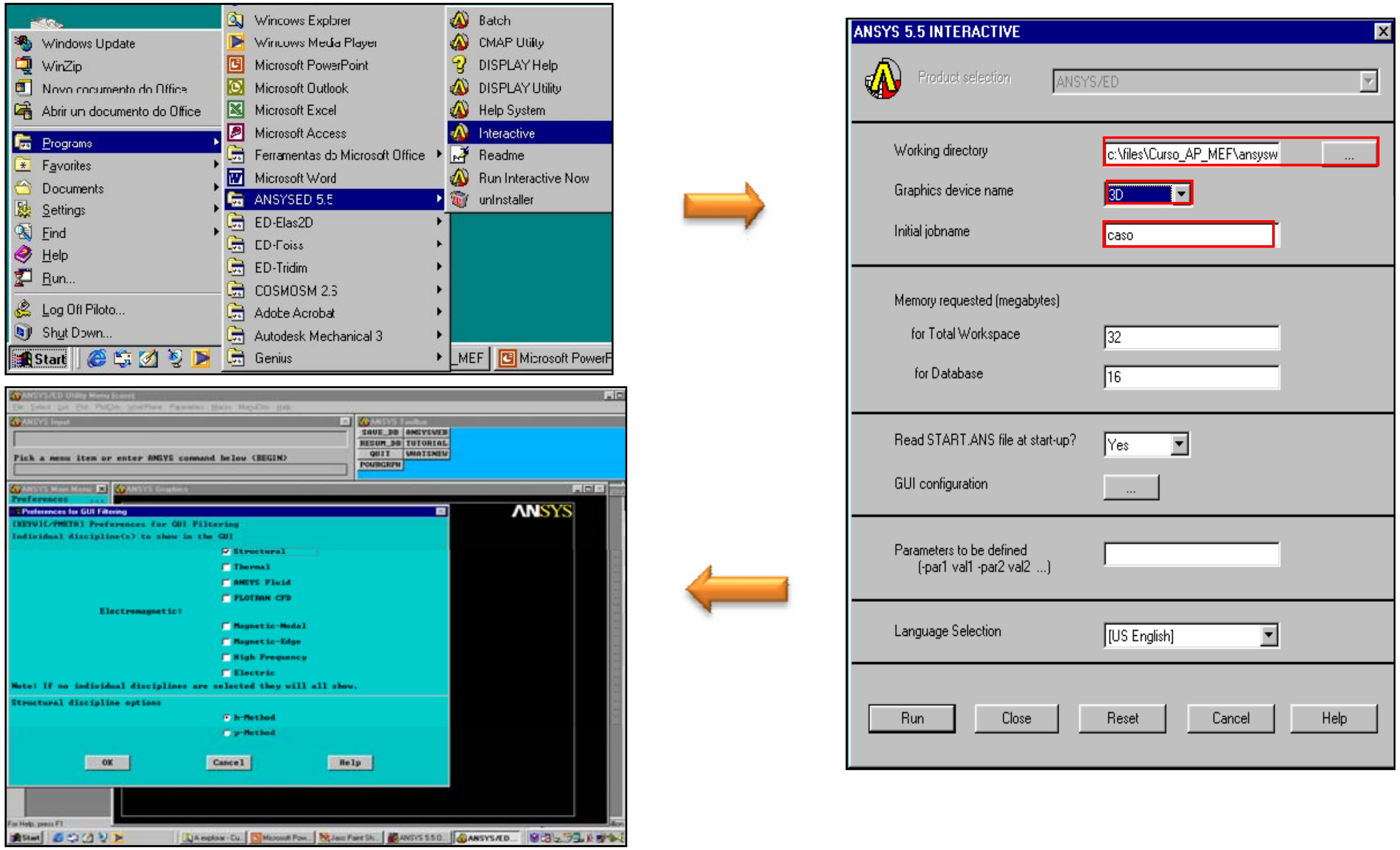
$$\begin{cases} k_{11}u_1 & + & k_{12}u_2 & + & k_{13}u_3 & + & \dots & + & k_{1n}u_n & = & f_1 \\ k_{21}u_1 & + & (10^{20})k_{22}u_2 & + & k_{23}u_3 & + & \dots & + & k_{2n}u_n & = & (10^{20})k_{22}\dot{u}_2 \\ k_{31}u_1 & + & k_{32}u_2 & + & k_{33}u_3 & + & \dots & + & k_{3n}u_n & = & f_3 \\ \vdots & & \vdots & & \vdots & & & & \vdots & & \vdots \\ k_{n1}u_1 & + & k_{n2}u_2 & + & k_{n3}u_3 & + & \dots & + & k_{nn}u_n & = & f_n \end{cases}$$

  - The equation that corresponds to specified DOF, allow to conclude:

$$(10^{20})k_{22}u_2 = (10^{20})k_{22}\dot{u}_2 \quad \Leftrightarrow \quad u_2 = \dot{u}_2$$

PRACTICE – CASE 1

- Practice with ANSYS.
  - Specify file management;
  - Filter the analysis (structural).
  - Specify method of approximation:
    - P=increase precision of polynomials, increasing degree of interpolation functions.
    - H= increasing precision with mesh refinement, holding degree of polynomial.

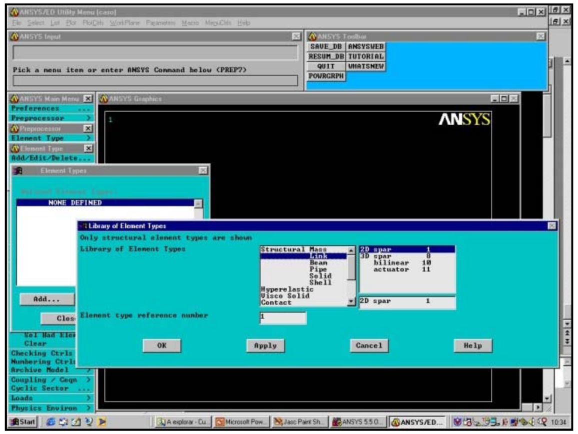


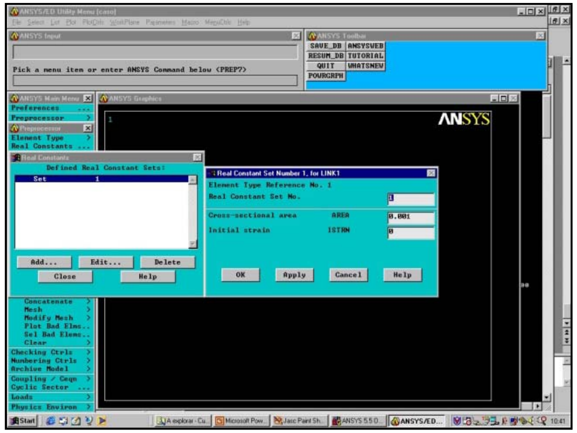
PRACTICE – CASE 1

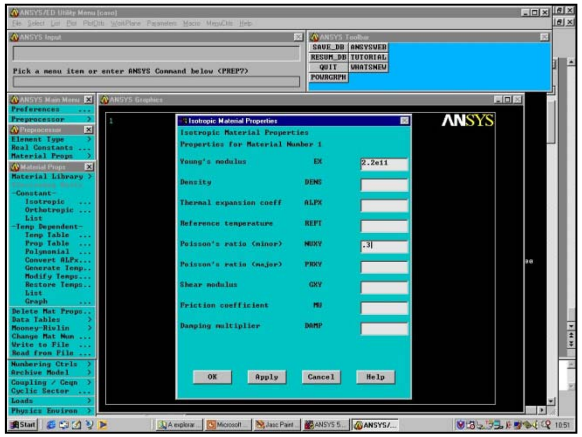
- Specify finite element:
  - LINK 2D Spar.
- Real constants:
  - Every thing that is constant during simulation.
    - ISTRN (Initial strain).
    - AREA (cross section area).

$$ISTRN = \frac{\delta}{L}$$

- Materials:
  - Steel: Homogeneous, isotropic.

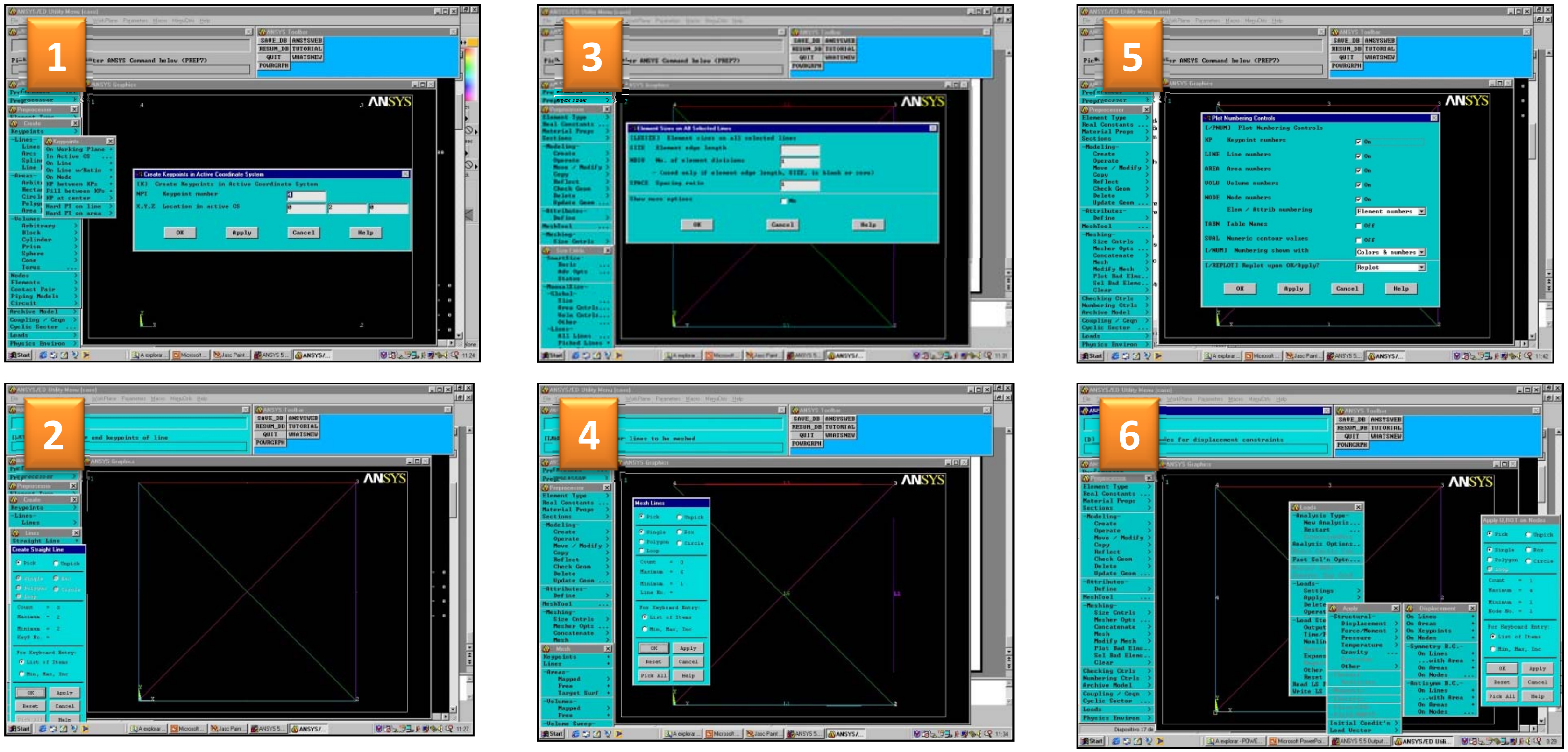






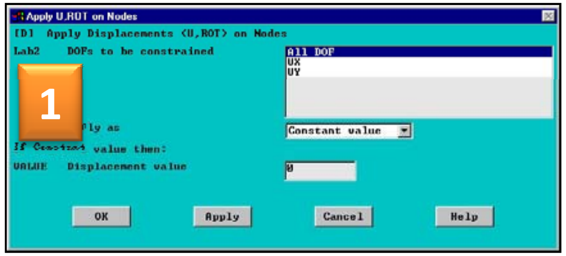
PRACTICE – CASE 1

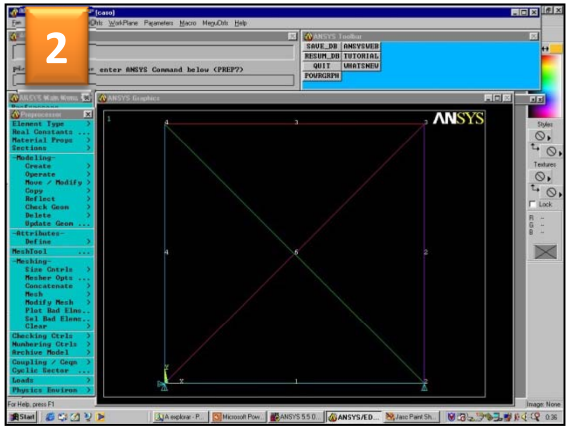
- Modelling: Geometric modelling using primitives (keypoints and lines):
  - Keypoints by coordinates in active coordinate system;
  - Lines may be created using end keypoints;
  - Mesh size selected, using line divisions;
  - Use appropriate command to mesh the geometry.
  - Impose boundary conditions.

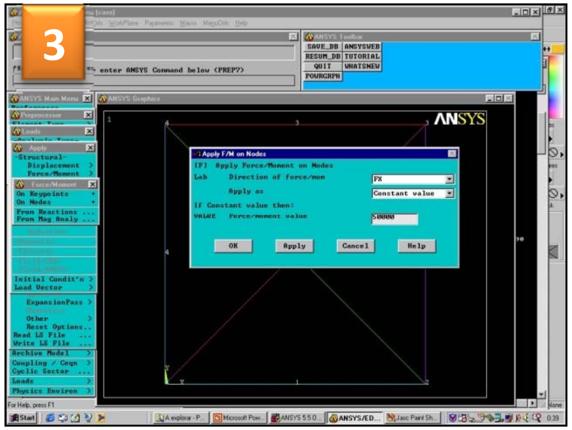


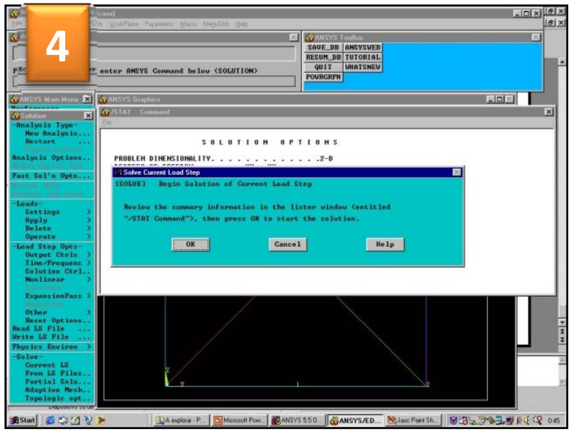
PRACTICE – CASE 1

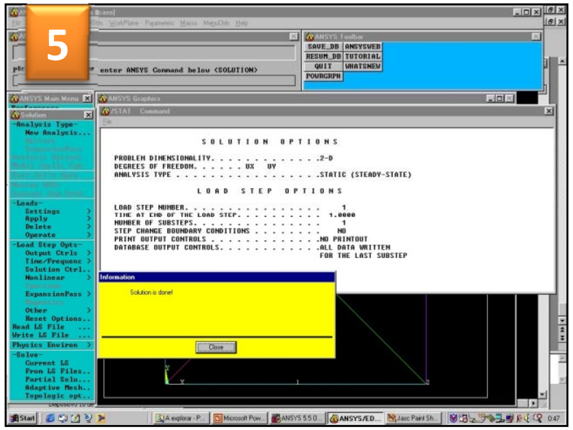
- Preparing the solution:
  - Window for boundary conditions inputs (displacement).
  - Plot boundary conditions (Plot CTRL).
  - Window for boundary conditions inputs (loads).
  - Command to start solution.
  - End of solution phase.





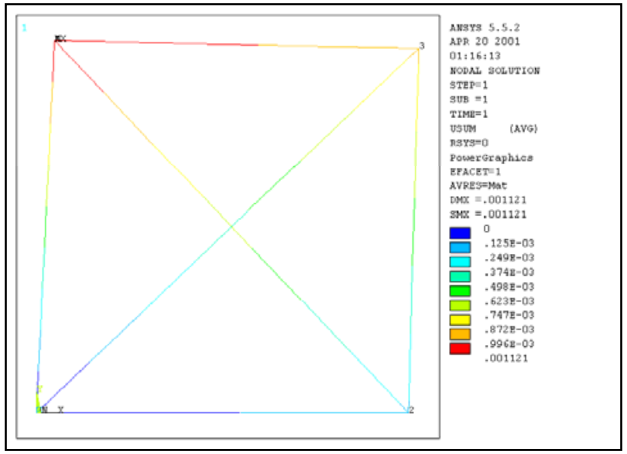


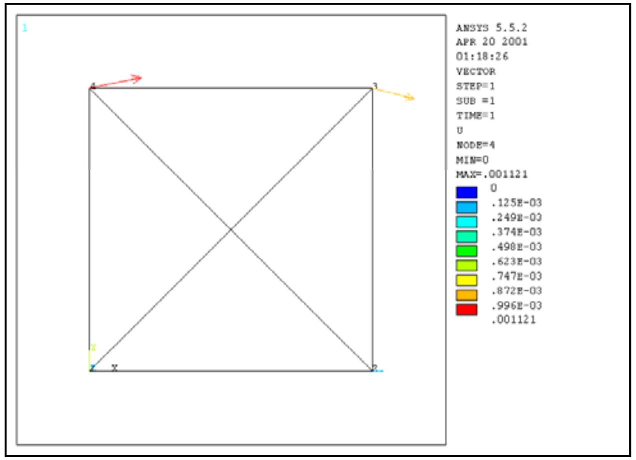




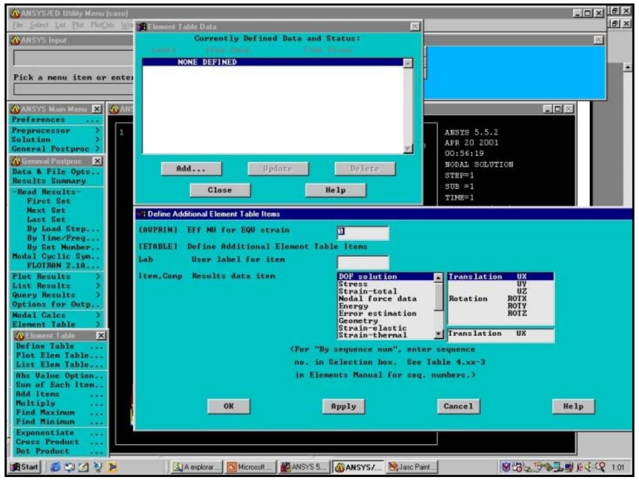
PRACTICE – CASE 1

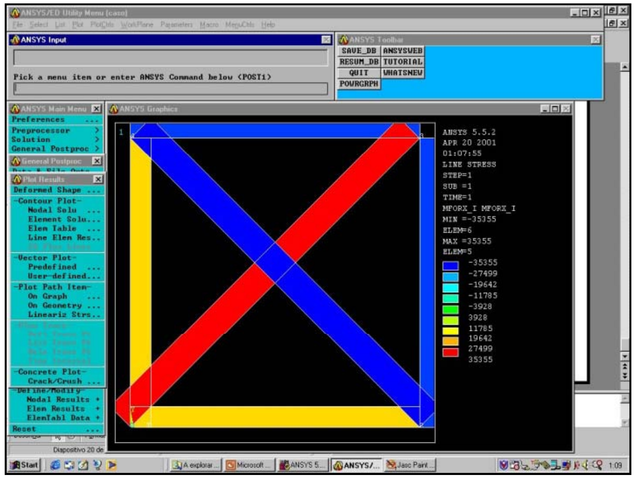
- Post-processor:
  - Displacement contour;
  - Vector displacement results (nodal);





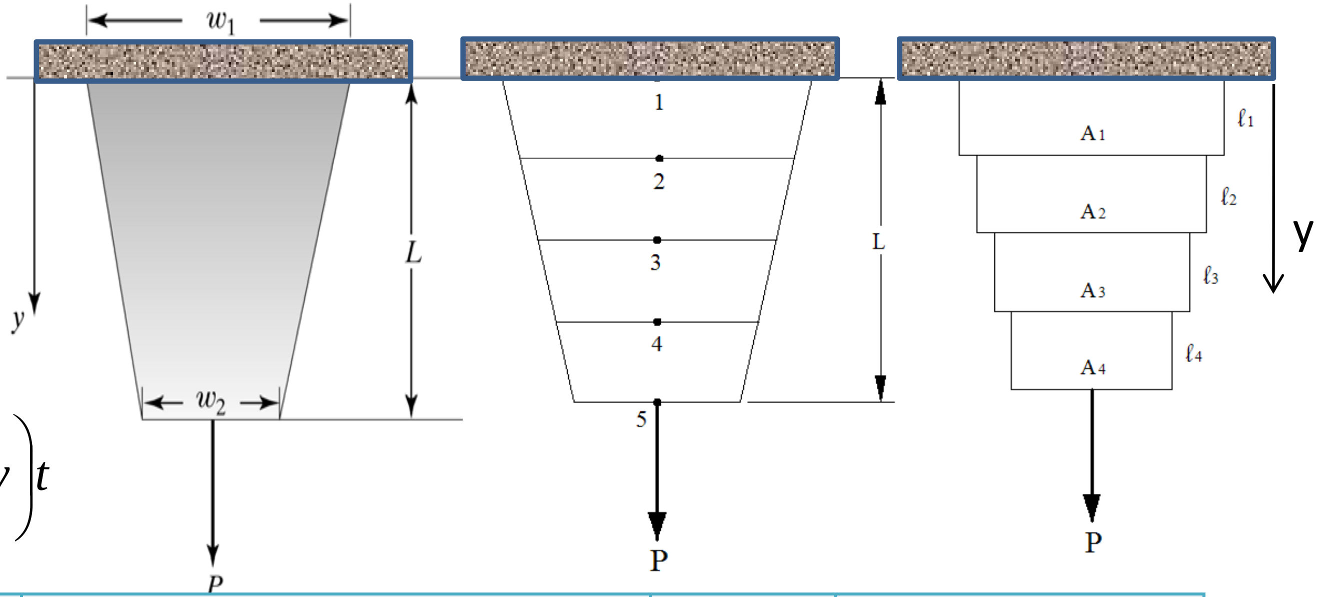
  - Axial load in members, using Element Table.





DIRECT STIFFNESS METHOD – case 2

- For the Element submitted to axial load, knowing that w1=2 [m], w2=1 [m], thickness t=0.125 [m], Length L =10 [m] and Load P=1000 [N].
- Determine the vertical displacement at nodes 1,2,3,4,5.

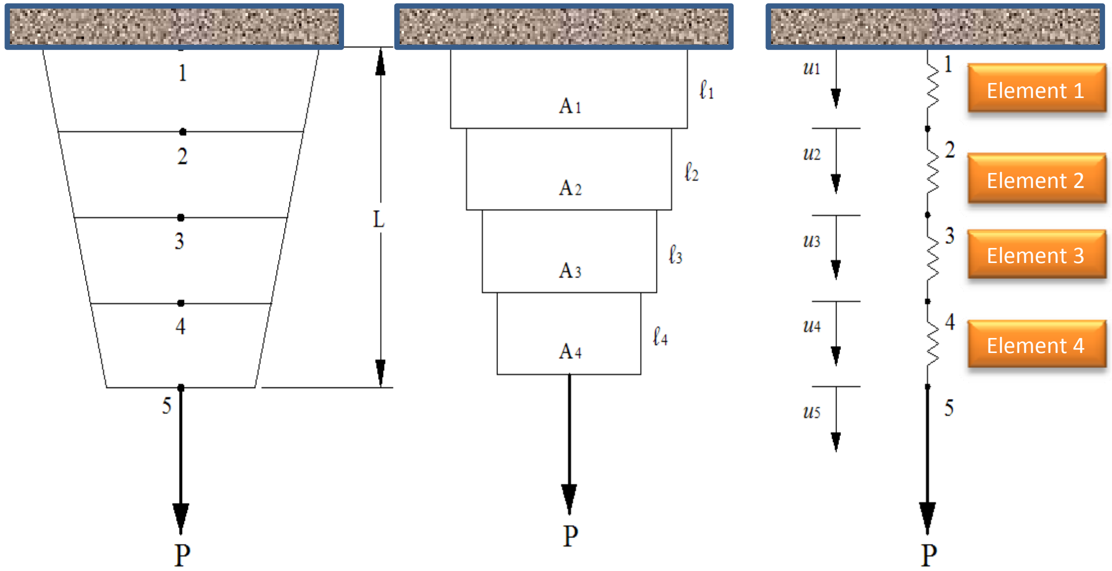


$$A(y) = \left(w_1 + \left(\frac{w_2 - w_1}{L} \right) y \right) t$$

Element (e_i)	Nodes	Cross section area (A) [m²]	Length (l_i) [m]	Elastic Modulus(E_i) [N/m²]
1	1 2	[A(y=0)+A(y=2.5)]/2=A(el1)=0.234375	2.5	$10.4x10^6$
2	2 3	[A(y=2.5)+A(y=5)]/2=A(el2)=0.203125	2.5	$10.4x10^6$
3	3 4	[A(y=5)+A(y=7.5)]/2=A(el3)=0.171875	2.5	$10.4x10^6$
4	4 5	[A(y=7.5)+A(y=10)]/2=A(el4)=0.140625	2.5	$10.4x10^6$

DIRECT STIFFNESS METHOD – case 2

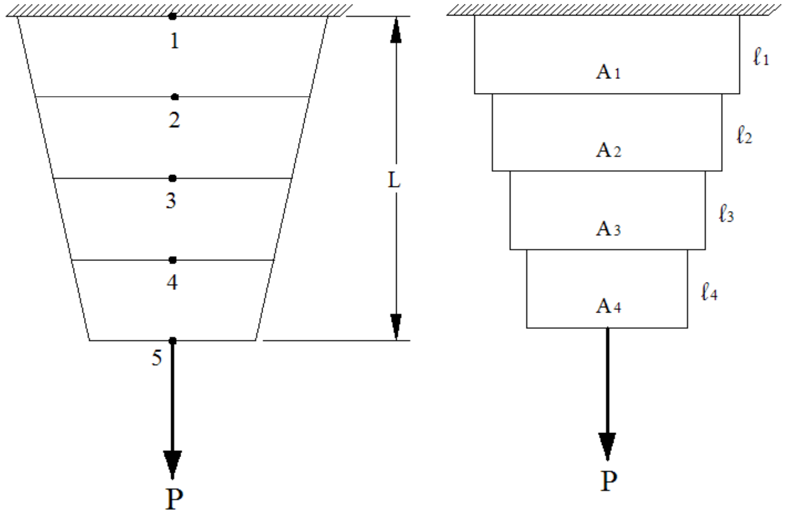
- Solution method:
  - First, decompose the entire structure into a set of finite elements.
  - Build a stiffness matrix for each element (4 Here).
  - Assembly all the element global stiffness matrix.



DIRECT STIFFNESS METHOD – case 2

- Stiffness matrix for each element

$$k_{eq} = \frac{\frac{(A_{i+1} + A_i)}{2} E}{L}$$



$$[K]^{(1)} = \begin{bmatrix} k_1 & -k_1 \\ -k_1 & k_1 \end{bmatrix} \quad [K]^{(2)} = \begin{bmatrix} k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} \quad [K]^{(3)} = \begin{bmatrix} k_3 & -k_3 \\ -k_3 & k_3 \end{bmatrix} \quad [K]^{(4)} = \begin{bmatrix} k_3 & -k_3 \\ -k_3 & k_3 \end{bmatrix}$$

$$[K]^{(1G)} = \begin{bmatrix} k_1 & -k_1 & 0 & 0 & 0 \\ -k_1 & k_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{matrix} \quad [K]^{(2G)} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & k_2 & -k_2 & 0 & 0 \\ 0 & -k_2 & k_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{matrix}$$

$$[K]^{(3G)} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & k_3 & -k_3 & 0 \\ 0 & 0 & -k_3 & k_3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{matrix} \quad [K]^{(4G)} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & k_4 & -k_4 \\ 0 & 0 & 0 & -k_4 & k_4 \end{bmatrix} \begin{matrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{matrix}$$

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DIRECT STIFFNESS METHOD – case 2

- Assembly all the element global stiffness matrix.

$$[K]^{(G)} = [K]^{(1G)} + [K]^{(2G)} + [K]^{(3G)} + [K]^{(4G)}$$

$$[K]^{(G)} = \begin{bmatrix} k_1 & -k_1 & 0 & 0 & 0 \\ -k_1 & k_1 + k_2 & -k_2 & 0 & 0 \\ 0 & -k_2 & k_2 + k_3 & -k_3 & 0 \\ 0 & 0 & -k_3 & k_3 + k_4 & -k_4 \\ 0 & 0 & 0 & -k_4 & k_4 \end{bmatrix}$$

- Apply boundary conditions.
- Solve the algebraic system of equations for u2,u3,u4 and u5.
- Solve the first equation to calculate reaction forces.

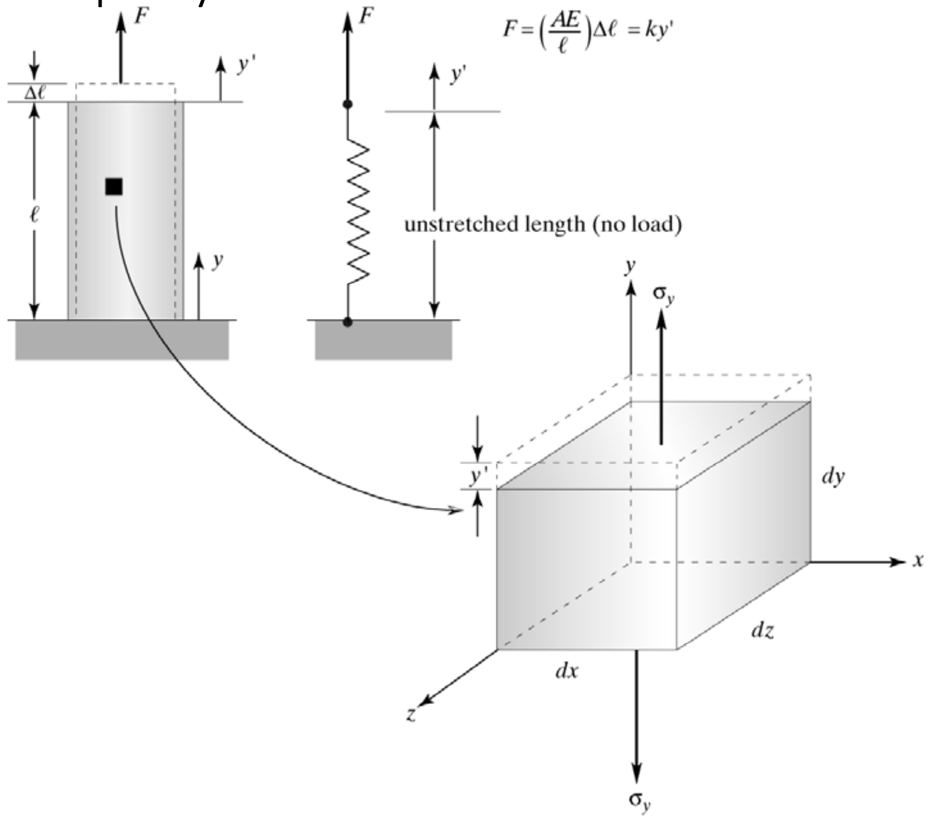
$$\begin{bmatrix} +k_1 & -k_1 & 0 & 0 & 0 \\ -k_1 & +k_1 + k_2 & -k_2 & 0 & 0 \\ 0 & -k_2 & +k_2 + k_3 & -k_3 & 0 \\ 0 & 0 & -k_3 & +k_3 + k_4 & -k_4 \\ 0 & 0 & 0 & -k_4 & +k_4 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{Bmatrix} = \begin{Bmatrix} -R_1 \\ 0 \\ 0 \\ 0 \\ P \end{Bmatrix}$$

$$\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0.001026 \\ 0.002210 \\ 0.003608 \\ 0.005317 \end{Bmatrix} m$$

VARIATIONAL METHODS

- Energy methods:
  - Common approximation in solid mechanics to be use in finite element analysis;
  - External forces cause deformation into the element. During deformation, the work developed by external forces is sustained by elastic energy.
  - Total potential energy (Π) is equal to two parts:
    - Deformation energy;
    - The energy corresponding to the work developed by external forces.

$$\Pi_p = U + W$$



VARIATIONAL METHODS

- Minimum potential energy theorem:
  - Assuming any virtual displacement (u) to the deformed configuration, satisfying the cinematic conditions , the energy variation with respect to this displacement is equal to zero (Minimum energy).

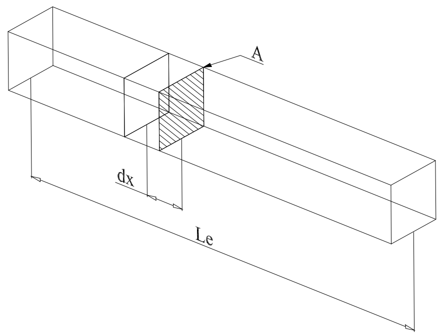
$$\frac{\partial \Pi_p}{\partial u} = \frac{\partial U}{\partial u} + \frac{\partial W}{\partial u} = 0 \qquad \Leftrightarrow \qquad \delta U = \delta W$$

  - Deformation energy, depends on stress stain field.

$$\delta U = \int_{volume} \langle \sigma \rangle \{\delta \varepsilon\} . dv \qquad \delta U = \int_{volume} [E] \langle \varepsilon \rangle \{\delta \varepsilon\} . dv$$

  - Work developed by external forces, depends on the displacement field.

$$\delta W = \langle \delta u \rangle \{F\}$$



VARIATIONAL METHODS (BAR ELEMENT)

- To determine the equilibrium configuration of a uniform bar submitted to external loading:
  - A displacement field is required to satisfy the minimum potential energy.
- The polynomial approximation may be the simplest solution:

$$u(x) = \sum_{i=1}^{n} a_i x^i = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n$$

  - For example assume 1st order approximation:

$$u(x) = a_0 + a_1 x$$

  - In matrix formulation:

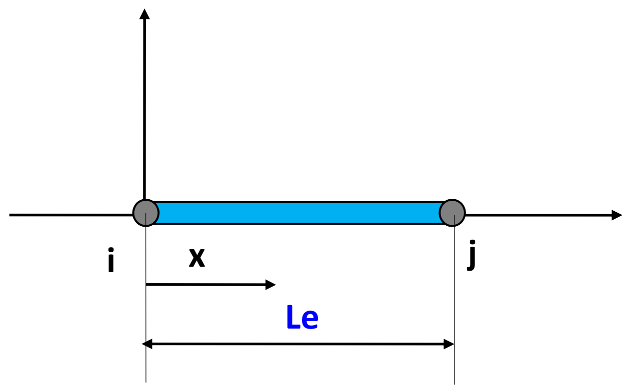
$$u(x) = \langle 1 \quad x \rangle \begin{Bmatrix} a_0 \\ a_1 \end{Bmatrix}$$

VARIATIONAL METHODS (BAR ELEMENT)

- The polynomial approximation may be the simplest solution (cont.)
  - Displacement field is valid from node i to node j.
  - Lets apply this solution to node i (x=0) and to node j (x=Le).

$$u_i = \langle 1 \quad 0 \rangle \begin{Bmatrix} a_0 \\ a_1 \end{Bmatrix}$$

$$u_j = \langle 1 \quad L_e \rangle \begin{Bmatrix} a_0 \\ a_1 \end{Bmatrix}$$



  - Group both expression in to a matrix formulation:

$$\begin{Bmatrix} u_i \\ u_j \end{Bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & Le \end{bmatrix} \begin{Bmatrix} a_0 \\ a_1 \end{Bmatrix}$$

  - Lets assume that node displacement are well known and try to find nodal parameters (a0 and a1), inverting the solution:

$$\left[\begin{array}{cc|cc} 1 & 0 & 1 & 0 \\ 1 & Le & 0 & 1 \end{array}\right] \Leftrightarrow \left[\begin{array}{cc|cc} 1 & 0 & 1 & 0 \\ 0 & Le & -1 & 1 \end{array}\right] \Leftrightarrow \left[\begin{array}{cc|cc} 1 & 0 & 1 & 0 \\ 0 & 1 & -1/Le & 1/Le \end{array}\right]$$

$$\begin{Bmatrix} a_0 \\ a_1 \end{Bmatrix} = \begin{bmatrix} 1 & 0 \\ -1/Le & 1/Le \end{bmatrix} \begin{Bmatrix} u_i \\ u_j \end{Bmatrix}$$

VARIATIONAL METHODS (BAR ELEMENT)

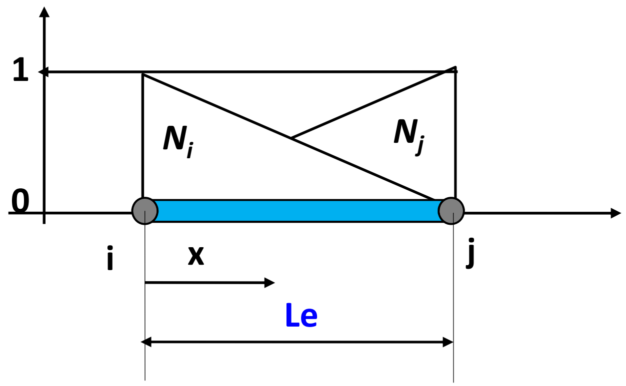
- The polynomial approximation may be the simplest solution (cont.)
  - Substitute the nodal parameters in to the assumed 1st order displacement field.
  - The interpolation functions (Ni, Nj) will appear.
  - Those functions are also called shape functions, because they may be used for interpolate the geometry coordinates.

$$u(x) = \langle 1 \quad x \rangle \begin{Bmatrix} a_0 \\ a_1 \end{Bmatrix}$$

$$= \langle 1 \quad x \rangle \begin{bmatrix} 1 & 0 \\ -1/Le & 1/Le \end{bmatrix} \begin{Bmatrix} u_i \\ u_j \end{Bmatrix}$$

$$= \left\langle \left(1 - \frac{x}{Le}\right) \quad \left(\frac{x}{Le}\right) \right\rangle \begin{Bmatrix} u_i \\ u_j \end{Bmatrix}$$

$$= \langle Ni \quad Nj \rangle \begin{Bmatrix} u_i \\ u_j \end{Bmatrix}$$



MINIMUM POTENTIAL ENERGY METHOD

- Apply the minimum potential energy method to a bar finite element, assuming:
  - Displacement field:

$$u(x) = \langle Ni \quad Nj \rangle \begin{Bmatrix} u_i \\ u_j \end{Bmatrix}$$

  - Strain field (constant values are expected over the entire element, because the derivatives of those interpolating functions are constant values):

$$\{\varepsilon(x)\} = \frac{d}{dx} u(x) = \left\langle \left(\frac{dN_i}{dx} \right) \quad \frac{dN_j}{dx} \right\rangle \begin{Bmatrix} u_i \\ u_j \end{Bmatrix}$$

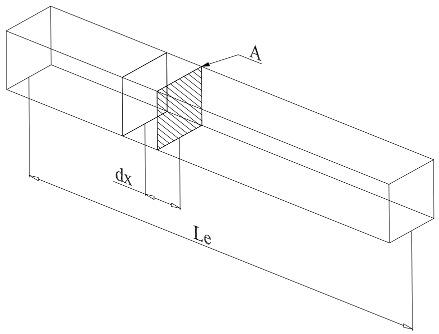
  - Virtual strain field:

$$\langle \delta\varepsilon(x) \rangle = \frac{d}{dx} \delta u(x) = \langle \delta u_i \quad \delta u_j \rangle \begin{Bmatrix} \left(\frac{dN_i}{dx} \right) \\ \left(\frac{dN_j}{dx} \right) \end{Bmatrix}$$

MINIMUM POTENTIAL ENERGY METHOD

- Apply the minimum potential energy method to a bar finite element, assuming:

$$\delta U = \delta W$$



$$\int_{volume} [E] \langle \varepsilon \rangle \{\delta \varepsilon\} . dv = \langle \delta u \rangle \{F\}$$

$$\int_{volume} [E] \{\delta \varepsilon\} \langle \varepsilon \rangle . dv = \langle \delta u \rangle \{F\}$$

  – The existence of only one finite element (total volume equal to one finite element volume):

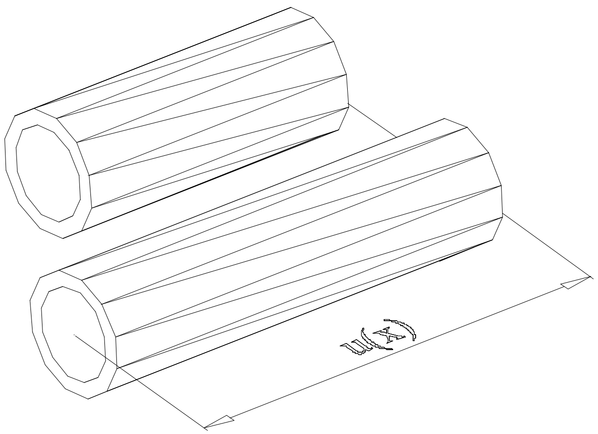
$$A \int_0^{Le} [E] \left(\langle \delta u_i \quad \delta u_j \rangle \begin{Bmatrix} \frac{\partial N_i}{\partial x} \\ \frac{\partial N_j}{\partial x} \end{Bmatrix} \right) \times \left(\left\langle \frac{\partial N_i}{\partial x} \quad \frac{\partial N_j}{\partial x} \right\rangle \begin{Bmatrix} u_i \\ u_j \end{Bmatrix} \right) . dx = \langle \delta u \rangle \{F\}$$

$$\langle \delta u_i \quad \delta u_j \rangle EA \begin{bmatrix} \int_0^{Le} \left(\frac{\partial N_i}{\partial x} \frac{\partial N_i}{\partial x} \right) . dx & \int_0^{Le} \left(\frac{\partial N_i}{\partial x} \frac{\partial N_j}{\partial x} \right) . dx \\ \int_0^{Le} \left(\frac{\partial N_j}{\partial x} \frac{\partial N_i}{\partial x} \right) . dx & \int_0^{Le} \left(\frac{\partial N_j}{\partial x} \frac{\partial N_j}{\partial x} \right) . dx \end{bmatrix} \begin{Bmatrix} u_i \\ u_j \end{Bmatrix} = \langle \delta u \rangle \{F\}$$

MINIMUM POTENTIAL ENERGY METHOD

- Apply the minimum potential energy method to a bar finite element (cont.)
  - The stiffness matrix for bar element will appear.

$$EA\begin{bmatrix} \int_0^{Le}\left(-\frac{1}{Le}\left(-\frac{1}{Le}\right)\right).dx & \int_0^{Le}\left(-\frac{1}{Le}\left(\frac{1}{Le}\right)\right).dx \\ \int_0^{Le}\left(\frac{1}{Le}\left(-\frac{1}{Le}\right)\right).dx & \int_0^{Le}\left(\frac{1}{Le}\left(\frac{1}{Le}\right)\right).dx \end{bmatrix}\begin{Bmatrix} u_i \\ u_j \end{Bmatrix} = \{F\}$$



$$EA\begin{bmatrix} \frac{1}{Le} & -\frac{1}{Le} \\ -\frac{1}{Le} & \frac{1}{Le} \end{bmatrix}\begin{Bmatrix} u_i \\ u_j \end{Bmatrix} = \begin{Bmatrix} F_i \\ F_j \end{Bmatrix}$$

MINIMUM POTENTIAL ENERGY METHOD

- Apply the minimum potential energy method to a bar finite element (cont.)
  - Assuming more elements over the entire volume, the assembling is necessary.

$$\int_{\substack{Total\\Volume}} dV = \sum_{Elements} \left[\int_{\substack{Element\\Volume}} \ldots dVe \right]$$

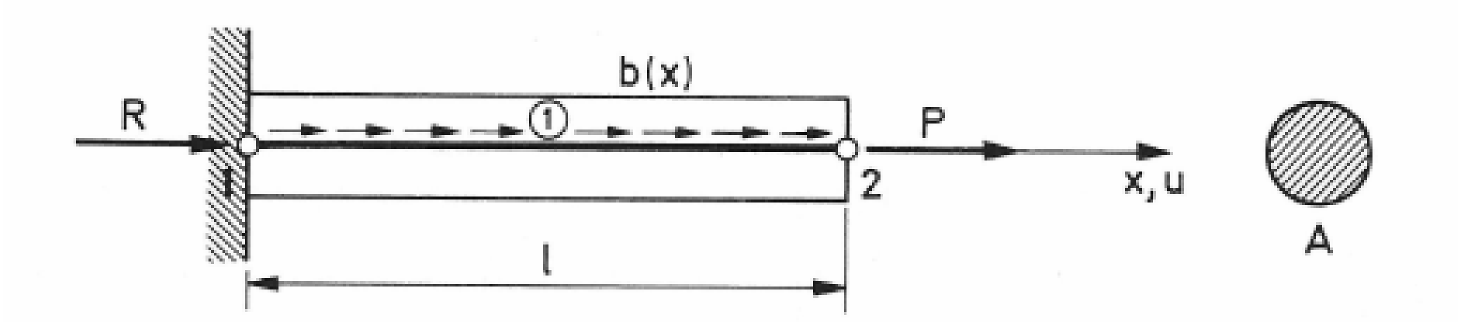
  - Assume different external load, such as, distributed load. In this case, the virtual work due to distributed load, may be calculated according:

$$\delta W = \int_{Le} b(x).\langle \delta u \rangle dve$$

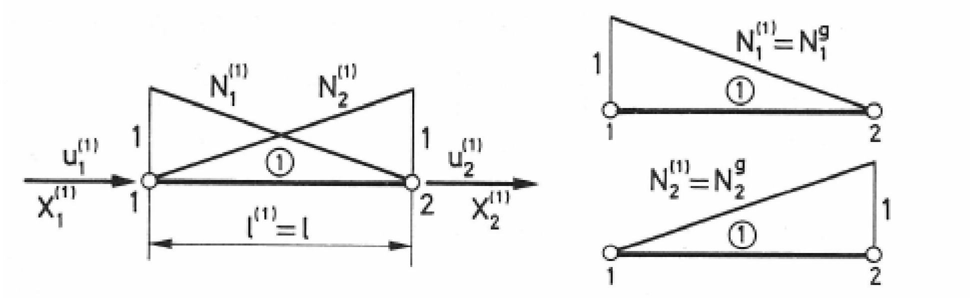
$$= \int_{0}^{Le} b \begin{Bmatrix} N_i \\ N_j \end{Bmatrix} dx.\langle \delta u \rangle$$

MINIMUM POTENTIAL ENERGY METHOD– case 1

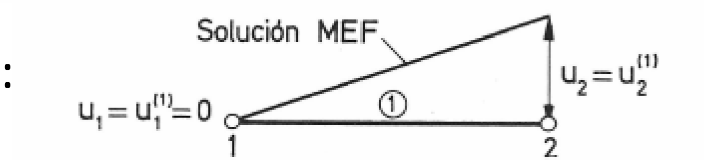
- Assume ONE finite element to mesh this structural element, of uniform cross section area A, constant elastic modulus (E), submitted to the loading conditions.



- Assume the following interpolating functions.

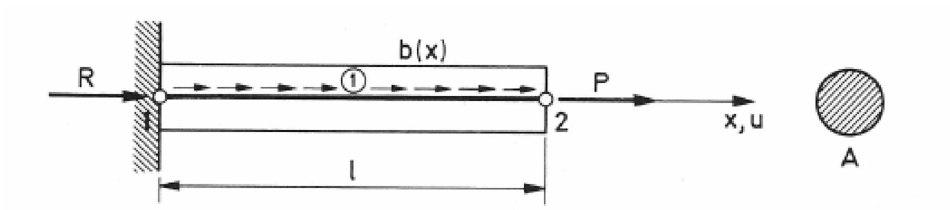


- The expected FE solution is:



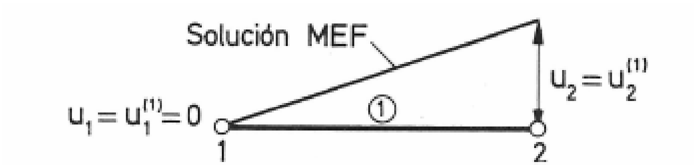
MINIMUM POTENTIAL ENERGY METHOD– case 1

- Applying the minimum potential energy:
  - The algebraic system is:



$$\begin{bmatrix} \frac{AE}{L} & \frac{AE}{L} \\ \frac{AE}{L} & \frac{AE}{L} \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} R + b(\frac{L}{2}) \\ P + b(\frac{L}{2}) \end{Bmatrix}$$

  - Imposing boundary conditions:
    - U1=0
  - Solving system of ONE equation:
    - $$u_2 = \frac{\left(P + \frac{bL}{2}\right)L}{AE}$$

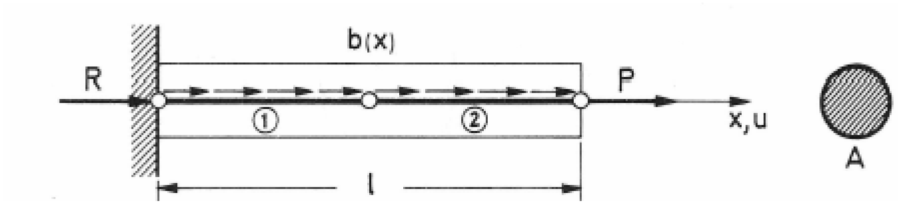


  - Solution inside finite element:

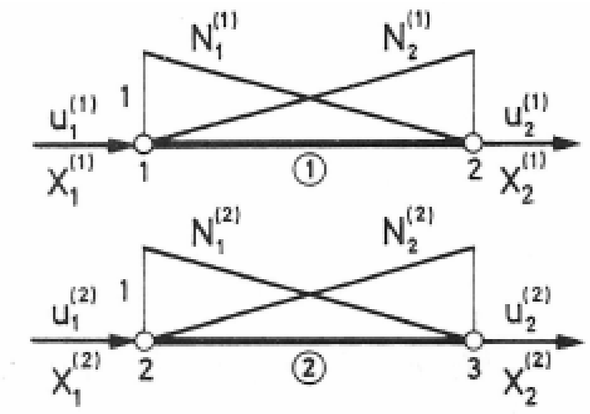
$$u(x) = \left\langle \left(1 - \frac{x}{L}\right) \quad \left(\frac{x}{L}\right) \right\rangle \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \left\langle \left(1 - \frac{x}{L}\right) \quad \left(\frac{x}{L}\right) \right\rangle \begin{Bmatrix} 0 \\ u_2 \end{Bmatrix} = \left(\frac{x}{L}\right) \left[\frac{\left(P + \frac{bL}{2}\right)L}{AE} \right]$$

MINIMUM POTENTIAL ENERGY METHOD– case 2

- Assume TWO finite element to mesh this structural element, of uniform cross section area A, constant elastic modulus (E), submitted to the loading conditions.

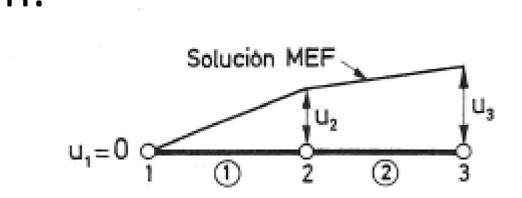


- Shape functions for each element:



$$u = \sum_{i=1}^{2} N_i^{(e)} u_i^{(e)}$$

- The expected solution:



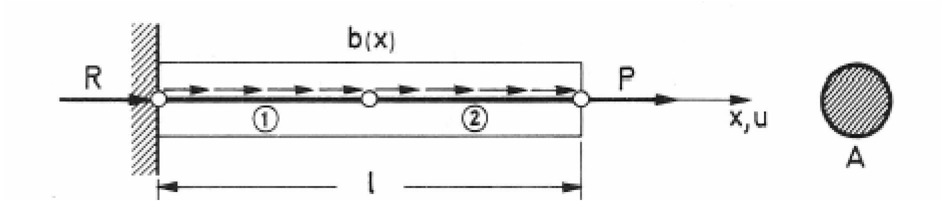
MINIMUM POTENTIAL ENERGY METHOD– case 2

- Assembling stiffness matrices and load vector:

$$[K]=\begin{bmatrix} K_{11}^1 & K_{12}^1 & 0 \\ K_{21}^1 & K_{22}^1+K_{11}^2 & K_{12}^2 \\ 0 & K_{12}^2 & K_{22}^2 \end{bmatrix}=\begin{bmatrix} \frac{AE}{(L/2)} & -\frac{AE}{(L/2)} & 0 \\ -\frac{AE}{(L/2)} & \frac{AE}{(L/2)}+\frac{AE}{(L/2)} & -\frac{AE}{(L/2)} \\ 0 & -\frac{AE}{(L/2)} & \frac{AE}{(L/2)} \end{bmatrix}$$

- Recall virtual work due to distributed load:

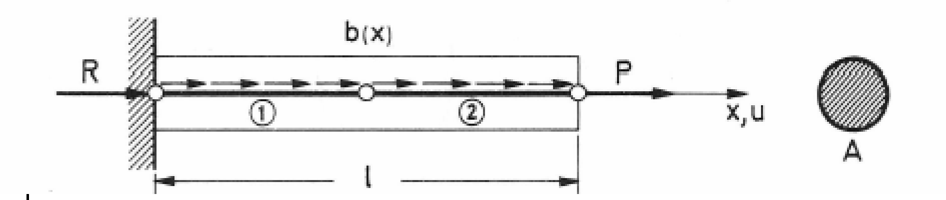
$$\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}=\begin{Bmatrix} R+bL/4 \\ bL/4+bL/4 \\ P+bL/4 \end{Bmatrix}$$



- Solve Algebraic system of equations, using MAPLE 13 symbolic Manipulator:
  - > restart; with(linalg);
  - > with(LinearAlgebra);
  - > K := Matrix(2, 2, [[AE/((1/2)*L)+AE/((1/2)*L), -AE/((1/2)*L)], [-EA/((1/2)*L), EA/((1/2)*L)]]);
  - > F := Matrix(2, 1, [[(1/4)*bL+(1/4)*bL], [P+(1/4)*bL]]);
  - > B := MatrixInverse[GF27](K);
  - > U := B.F;

MINIMUM POTENTIAL ENERGY METHOD– case 2

- Solution for displacement:



$$\{F\} = \begin{Bmatrix} 0 \\ \dfrac{L}{2EA}\left[P + \dfrac{3}{4}bL\right] \\ \dfrac{L}{2EA}\left[2P + bL\right] \end{Bmatrix}$$

- Reaction Solution:

$$R = (-P + bL)$$

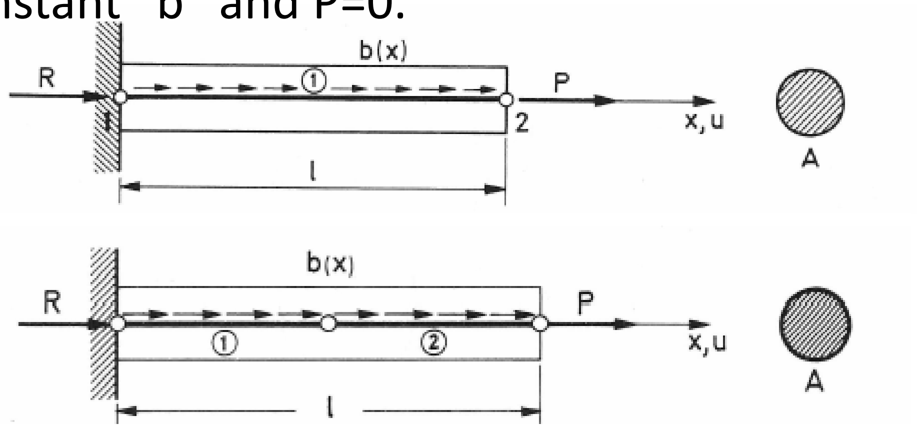
- Analytical solution:

$$u(x) = \frac{1}{EA}\left[-\frac{bx^2}{2} + (P + bL)x\right]$$

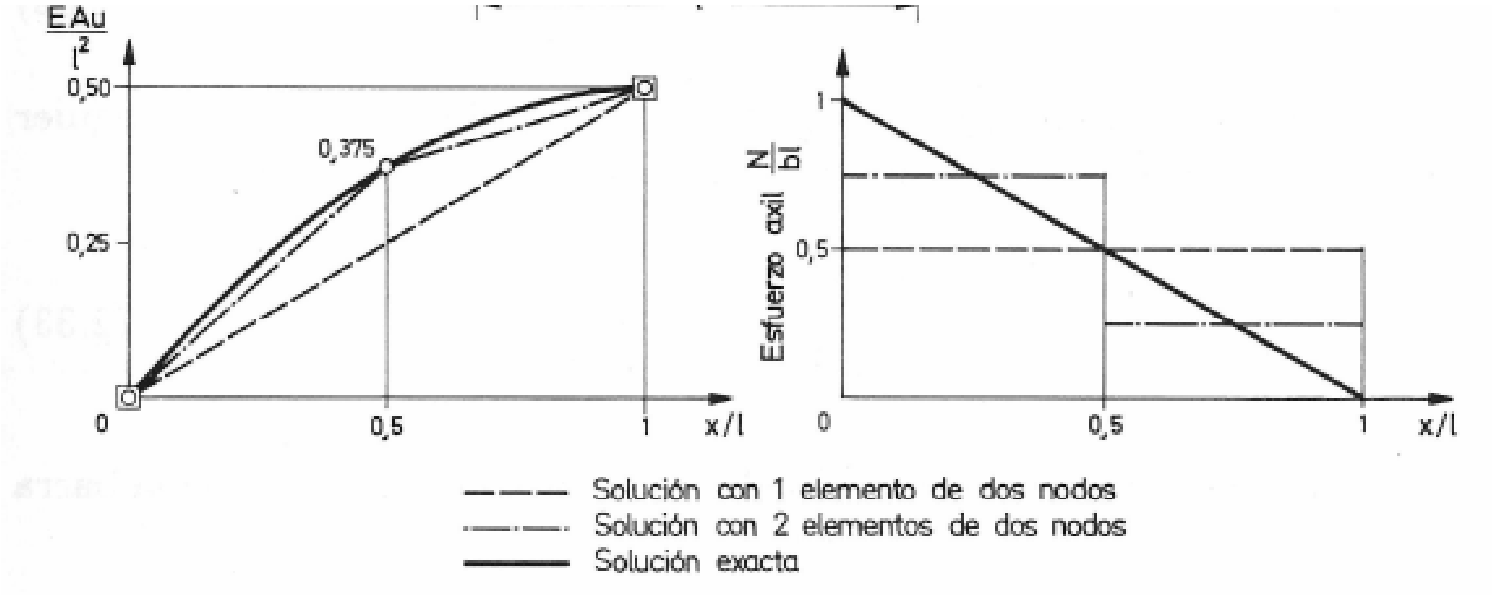
MINIMUM POTENTIAL ENERGY METHOD– case 3

- Convergence analysis for the case of constant "b" and P=0.
  - Analytical solution:

$$u(x) = \frac{1}{EA}\left[-\frac{bx^2}{2} + (P + bL)x\right]$$

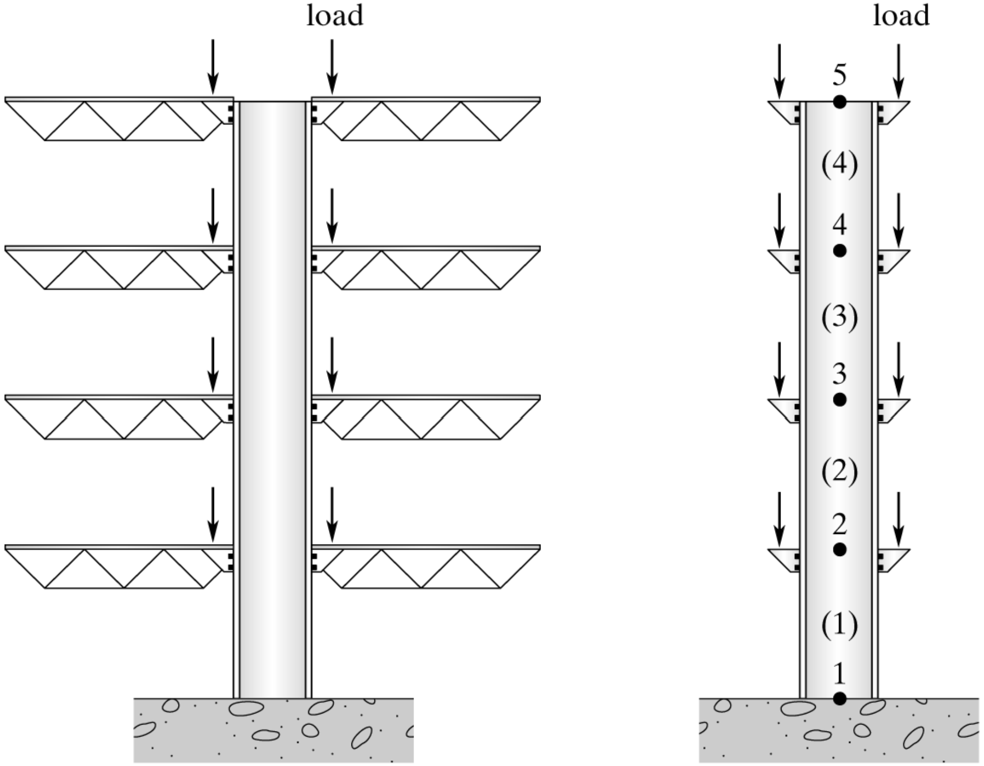


  - Comparison of displacement solution for FEA with 1 and 2 FE.



VARIATIONAL METHODS (BAR ELEMENT)

- Continuous members:
  – Linear interpolating functions to approximate solution for a column in a four-story buiding.



VARIATIONAL METHODS (BAR ELEMENT)

- Approximate element solution, linearly.

$$u^e = c_1 + c_2 Y$$

- Displacement field is valid for node i to node j.

$$u = u_i \quad in \quad Y = Y_i \qquad u_i = c_1 + c_2 Y_i$$

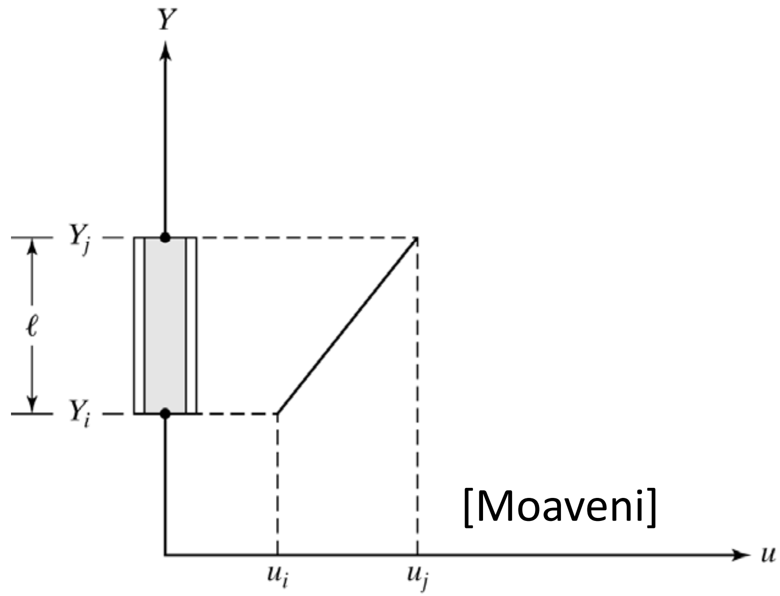
$$u = u_j \quad in \quad Y = Y_j \qquad u_j = c_1 + c_2 Y_j$$

- Lets assume that node displacement (ui, uj) are well known and try to find nodal parameters (c1 and c2), inverting the solution:

$$c_1 = \frac{u_i Y_j - u_j Y_i}{Y_j - Y_i} \qquad c_2 = \frac{u_j - u_i}{Y_j - Y_i}$$

- Solution for element displacement (1st order):

$$u^e = \frac{u_i Y_j - u_j Y_i}{Y_j - Y_i} + \frac{u_j - u_i}{Y_j - Y_i} Y$$

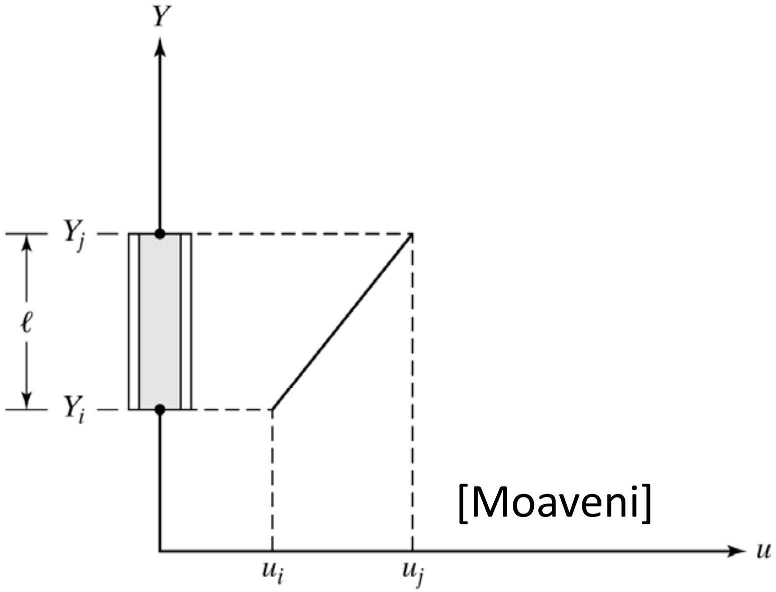


[Moaveni]

VARIATIONAL METHODS (BAR ELEMENT)

- The polynomial approximation may be the simplest solution (cont.)
  - Substitute the nodal parameters in to the assumed 1st order displacement field.
  - The interpolation functions (Ni, Nj) will appear (in global coordinates).
  - Those functions are also called shape functions, because they may be used for interpolate the geometry coordinates.

$$u^e = \left(\frac{Y_j - Y}{Y_j - Y_i}\right)u_i + \left(\frac{Y - Y_i}{Y_j - Y_i}\right)u_j$$



[Moaveni]

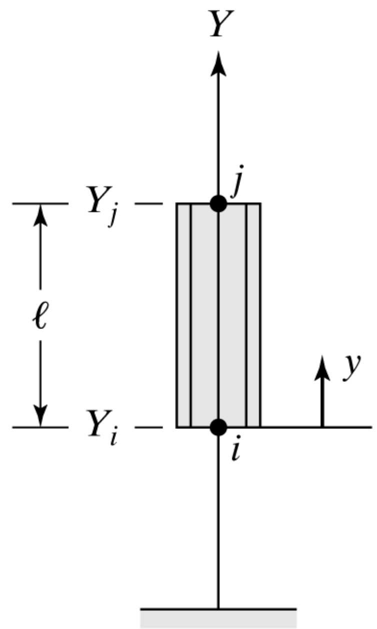
$$N_i = \frac{Y_j - Y}{Y_j - Y_i} = \frac{Y_j - Y}{l}$$

$$N_j = \frac{Y - Y_i}{Y_j - Y_i} = \frac{Y - Y_i}{l}$$

$$u^e = N_i u_i + N_j u_j \qquad u^e = \begin{bmatrix} N_i & N_j \end{bmatrix} \begin{Bmatrix} u_i \\ u_j \end{Bmatrix} \qquad \{u^e\} = [N]\{u\}$$

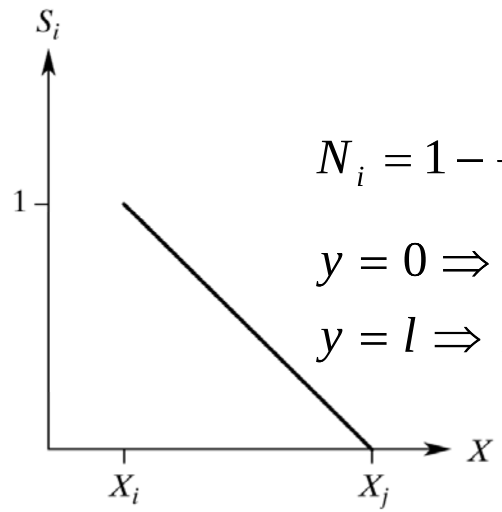
VARIATIONAL METHODS (BAR ELEMENT)

- The interpolation functions may be determined in local coordinates:
  - Substitution of variables: $Y = Y_i + y \quad (0 \le y \le l)$



$$N_i = \frac{Y_j - Y}{l} = \frac{Y_j - (Y_i + y)}{l} = 1 - \frac{y}{l}$$

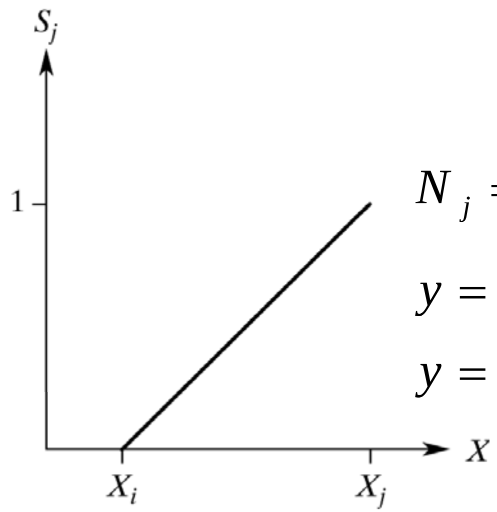
$$N_j = \frac{Y - Y_i}{l} = \frac{(Y_i + y) - Y_i}{l} = \frac{y}{l}$$



$$N_i = 1 - \frac{y}{l}$$

$$y = 0 \Rightarrow N_i = 1$$

$$y = l \Rightarrow N_i = 0$$



$$N_j = \frac{y}{l}$$

$$y = 0 \Rightarrow N_j = 0$$

$$y = l \Rightarrow N_j = 1$$

$N_1 = 1$	$N_2 = 0$	in node 1
$N_1 = 0$	$N_2 = 1$	in node 2

$N_1 + N_2 = 1$

VARIATIONAL METHODS (BAR ELEMENT)

- Apply the minimum potential energy method to a bar finite element:

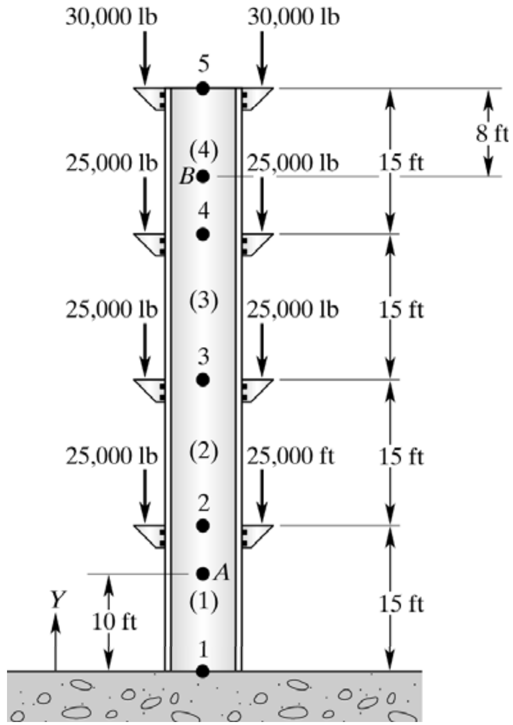
$$EA\begin{bmatrix} \frac{1}{Le} & -\frac{1}{Le} \\ -\frac{1}{Le} & \frac{1}{Le} \end{bmatrix}\begin{Bmatrix} u_i \\ u_j \end{Bmatrix} = \begin{Bmatrix} F_i \\ F_j \end{Bmatrix}$$

- Assemble the solution.

$$[K]^G\{u\} = \{F\}$$

VARIATIONAL METHODS (BAR ELEMENT) - case 4

- Determine vertical displacements in each story of a four story building.
- Determine stress at elements, knowing that E=29E6lb/in^2 e A=39.7in^2



$$[K]^e = [K]^1 = [K]^2 = [K]^3 = [K]^4 = \frac{AE}{l}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \frac{39.7 \times 20E6}{1.5 \times 12}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$[K]^G = 6.396E6 \begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ -1 & 1+1 & -1 & 0 & 0 \\ 0 & -1 & 1+1 & -1 & 0 \\ 0 & 0 & -1 & 1+1 & -1 \\ 0 & 0 & 0 & -1 & 1 \end{bmatrix} \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \\ F_5 \end{Bmatrix} = \begin{Bmatrix} R \\ 50000 \\ 50000 \\ 50000 \\ 60000 \end{Bmatrix}$$

– Displacements:

$$\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0.03283 \\ 0.05784 \\ 0.07504 \\ 0.08442 \end{Bmatrix}$$

– Axial stresses determined from:

$$\sigma_e = \frac{E(u_j - u_i)}{l}$$

$$\sigma_1 = -5289 \quad \sigma_2 = -4029 \quad \sigma_3 = -2771 \quad \sigma_4 = -1511 \quad [lb/in^2]$$

VARIATIONAL METHODS (GENERAL ELEMENT)

- Minimum potential energy theorem:

$$\frac{\partial \Pi_p}{\partial u} = \frac{\partial U}{\partial u} + \frac{\partial W}{\partial u} = 0 \qquad \Leftrightarrow \qquad \delta U = \delta W$$

  - Assuming the interpolating functions Ni, Nj

$$u^e = N_i u_i + N_j u_j$$

  - Strain field:

$$\varepsilon = \frac{\partial u}{dy} = \frac{\partial}{dy}\left[N_i u_i + N_j u_j\right] = \frac{\partial}{dy}\left[N_i + N_j\right]\begin{Bmatrix} u_i \\ u_j \end{Bmatrix} = \frac{\partial}{dy}[N]\{u\} = [B]\{u\}$$

  - Internal energy EQUALS external work:

$$\delta u = \delta w \Leftrightarrow \int_{volume} [E]\{\delta\varepsilon\}\langle\varepsilon\rangle.dv = \langle\delta u\rangle\{F\} \qquad [K]\{u\} = \{f\}$$

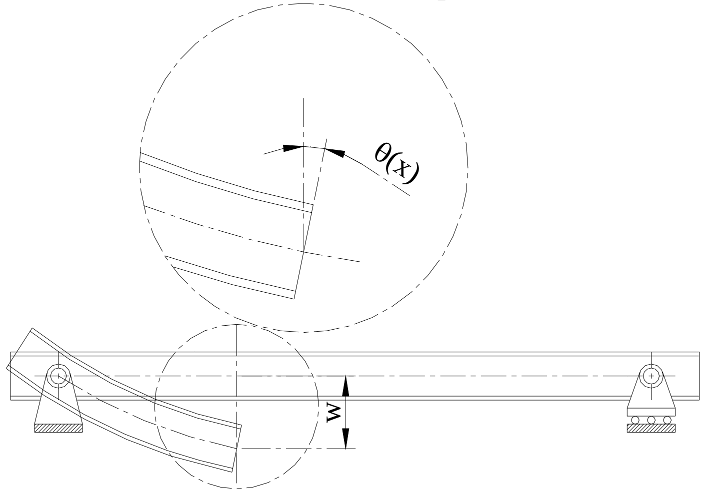
$$[K] = \int_{\Omega} [B]^T [D][B] d\Omega \qquad \{f\} = \int_{\Omega} [N]^T \{b\} d\Omega$$

  - Solution:

$$\{u\} = [K]^{-1}\{f\} \qquad \{\sigma\} = [D]\{\varepsilon\} = [D][B]\{u\}$$

VARIATIONAL METHODS (BEAM ELEMENT)

- Henky Mindlin Theory:
  - Longitudinal displacement: u
  - Vertical displacement : W (1st main variable)
  - Transversal coordinate: y
  - Section rotation : θ ((2nd main variable).



- Hypotheses:
  - Rigidy body in transversal direction;
  - Cross section remains plane and not warped;
  - Before deformation, cross section normal is orthogonal;

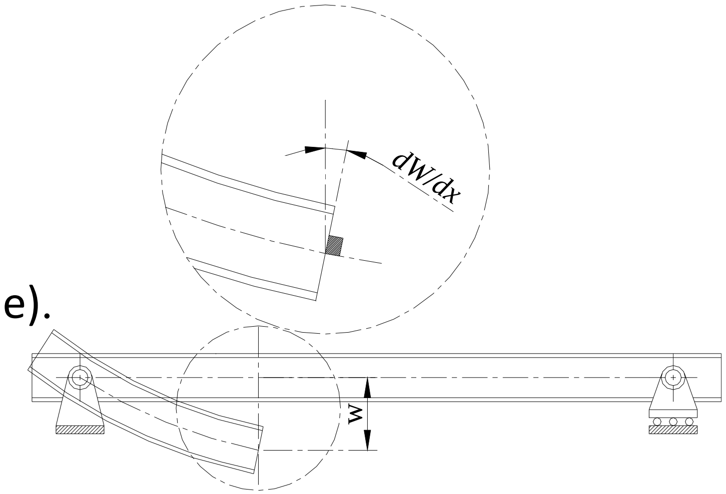
$$u(x,y) = u(x) - y Sin\theta(x)$$
$$W(x,y) = W(x) - y[1 - Cos\theta(x)]$$

- Additional hypothesis to small displacements (Linear theory)
  - Due to small rotations, sen(θ)->(θ), while cos(θ)->1
  - Axial displacement depend on cross section rotation.

$$u(x,y) = u(x) - y\theta(x)$$
$$W(x,y) = W(x)$$

VARIATIONAL METHODS (BEAM ELEMENT)

- Euler Bernoulli theory:
  - Longitudinal displacement: u
  - Vertical displacement : W (main variable)
  - Transversal coordinate: y
  - Section rotation : W' (derivative of main variable).



- Hypotheses:
  - Rigidy body in transversal direction;
  - Cross section remains plane and not warped;
  - After and Before deformation, cross section normal is orthogonal;

- Additional hypothesis to small displacements (Linear theory)
  - Due to small rotations, sen(θ)->(θ), while cos(θ)->1 .
  - Axial displacement depend on transversal displacement.

$$u(x,y) = u(x) - yW'(x)$$

$$W(x,y) = W(x)$$

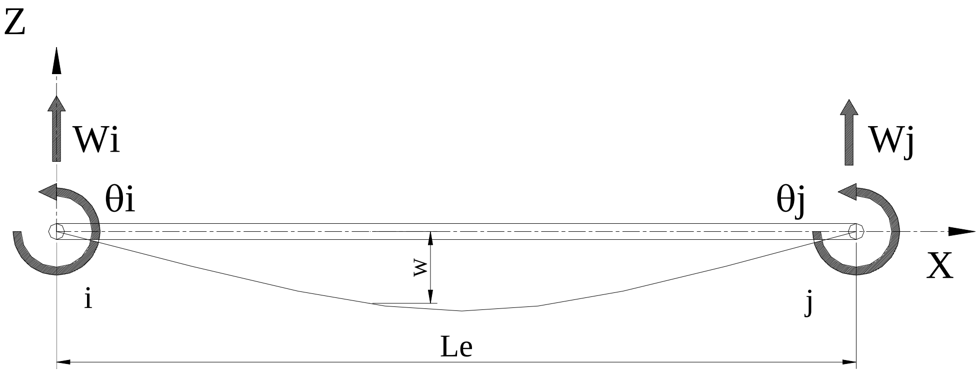
VARIATIONAL METHODS (EULER B. BEAM ELEMENT)

- To determine the equilibrium configuration of a uniform bar submitted to external loading:
  - A displacement field is required to satisfy the minimum potential energy.
- The polynomial approximation may be the simplest solution:
  - For example assume 3rd order approximation (Hermite polynomial, C1 type):

$$W(x) = b_1 + x b_2 + x^2 b_3 + x^3 b_4$$

$$\theta(x) = \frac{dW(x)}{dx} = b_2 + 2x b_3 + 3x^2 b_4$$

  - In matrix formulation.



$$\begin{Bmatrix} W \\ \theta \end{Bmatrix} = \begin{bmatrix} 1 & x & x^2 & x^3 \\ 0 & 1 & 2x & 3x^2 \end{bmatrix} \begin{Bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{Bmatrix}$$

  - Expand to nodal degrees of freedom

$$\begin{Bmatrix} W_i \\ \theta_i \\ W_j \\ \theta_j \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & L_e & L_e^2 & L_e^3 \\ 0 & 1 & 2L_e & 3L_e^3 \end{bmatrix} \begin{Bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{Bmatrix}$$

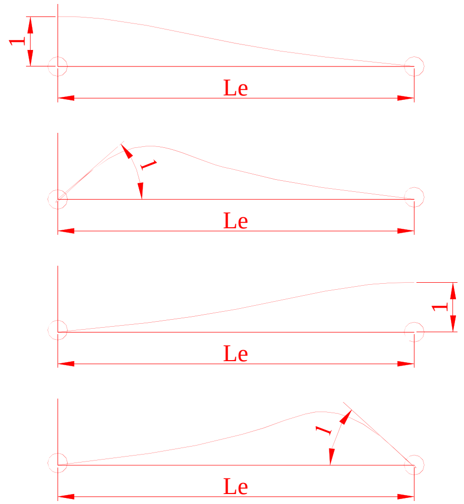
$$\begin{Bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \frac{-3}{L_e^2} & \frac{-2}{L_e} & \frac{-3}{L_e^2} & \frac{-1}{L_e} \\ \frac{2}{L_e^3} & \frac{1}{L_e^2} & \frac{-2}{L_e^3} & \frac{1}{L_e^2} \end{bmatrix} \begin{Bmatrix} W_i \\ \theta_i \\ W_j \\ \theta_j \end{Bmatrix}$$

VARIATIONAL METHODS (EULER B. BEAM ELEMENT)

- The polynomial approximation may be the simplest solution (cont.)
  - Substitute the nodal parameters in to the assumed 1st order displacement field.
  - The interpolation functions (Ni, Nj) will appear.
  - Those functions are also called shape functions, because they may be used for interpolate the geometry coordinates.

$$\begin{Bmatrix} W \\ \theta \end{Bmatrix} = \begin{bmatrix} N_1 & N_2 & N_3 & N_4 \\ \frac{\partial N_1}{\partial x} & \frac{\partial N_2}{\partial x} & \frac{\partial N_3}{\partial x} & \frac{\partial N_4}{\partial x} \end{bmatrix} \begin{Bmatrix} W_i \\ \theta_i \\ W_j \\ \theta_j \end{Bmatrix} = \begin{bmatrix} \left(1 - \frac{3x^2}{L_e^2} + \frac{2x^3}{L_e^3}\right) & \left(x - \frac{2x^2}{L_e} + \frac{x^3}{L_e^2}\right) & \left(\frac{3x^2}{L_e^2} - \frac{2x^3}{L_e^3}\right) & \left(\frac{-x^2}{L_e} + \frac{x^3}{L_e^2}\right) \\ \left(\frac{-6x}{L_e^2} + \frac{6x^2}{L_e^3}\right) & \left(1 - \frac{4x}{L_e} + \frac{3x^2}{L_e^2}\right) & \left(\frac{6x}{L_e^2} - \frac{6x^2}{L_e^3}\right) & \left(\frac{-2x}{L_e} + \frac{3x^2}{L_e^2}\right) \end{bmatrix} \begin{Bmatrix} W_i \\ \theta_i \\ W_j \\ \theta_j \end{Bmatrix}$$

$$\xi = \frac{2x}{\ell} - 1$$

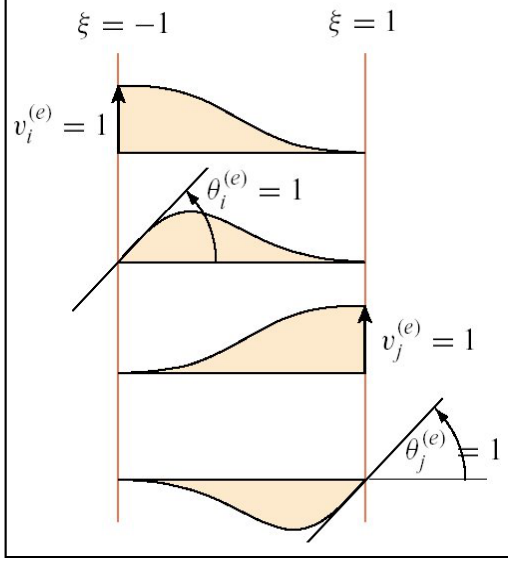


$$N_1 = 1 - \frac{3x^2}{L_e^2} + \frac{2x^3}{L_e^3}$$

$$N_2 = x - \frac{2x^2}{L_e} + \frac{x^3}{L_e^2}$$

$$N_3 = \frac{3x^2}{L_e^2} - \frac{2x^3}{L_e^3}$$

$$N_4 = \frac{-x^2}{L_e} + \frac{x^3}{L_e^2}$$



$$N_{vi}^{(e)}(\xi) = \tfrac{1}{4}(1-\xi)^2(2+\xi)$$

$$N_{\theta i}^{(e)}(\xi) = \tfrac{1}{8}\ell(1-\xi)^2(1+\xi)$$

$$N_{vj}^{(e)}(\xi) = \tfrac{1}{4}(1+\xi)^2(2-\xi)$$

$$N_{\theta j}^{(e)}(\xi) = -\tfrac{1}{8}\ell(1+\xi)^2(1-\xi)$$

VARIATIONAL METHODS (EULER B. BEAM ELEMENT)

- Apply the minimum potential energy method to a beam finite element, assuming:
  - U and V may be represented by null displacement variables;

$$\delta U = \delta W$$

$$\delta U = \int_{Vol} \langle \sigma \rangle \{\delta\varepsilon\} = \int_{Vol} \langle \varepsilon \rangle [D]^t \{\delta\varepsilon\}$$

$$\begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ 2\varepsilon_{xy} \\ 2\varepsilon_{yz} \\ 2\varepsilon_{zx} \end{Bmatrix} = \begin{bmatrix} \frac{\partial}{\partial x} & 0 & 0 \\ 0 & \frac{\partial}{\partial y} & 0 \\ 0 & 0 & \frac{\partial}{\partial z} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial z} & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} & 0 & \frac{\partial}{\partial x} \end{bmatrix} \begin{Bmatrix} U \\ V \\ W \end{Bmatrix} = \begin{bmatrix} \frac{\partial}{\partial x} & 0 & 0 \\ 0 & \frac{\partial}{\partial y} & 0 \\ 0 & 0 & \frac{\partial}{\partial z} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial z} & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} & 0 & \frac{\partial}{\partial x} \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ W \end{Bmatrix}$$

$$\{\delta\varepsilon\} = \left\langle \frac{\partial N_1}{\partial x} \quad \frac{\partial N_2}{\partial x} \quad \frac{\partial N_3}{\partial x} \quad \frac{\partial N_4}{\partial x} \right\rangle \begin{Bmatrix} \delta W_i \\ \delta\theta_i \\ \delta W_j \\ \delta\theta_j \end{Bmatrix}$$

$$2\varepsilon_{zx} = \frac{\partial W}{\partial x} = \theta(x) = \left\langle \left(\frac{-6x}{L_e^2} + \frac{6x^2}{L_e^3} \right) \quad \left(1 - \frac{4x}{L_e} + \frac{3x^2}{L_e^2} \right) \quad \left(\frac{6x}{L_e^2} - \frac{6x^2}{L_e^3} \right) \quad \left(\frac{-2x}{L_e} + \frac{3x^2}{L_e^2} \right) \right\rangle \begin{Bmatrix} W_i \\ \theta_i \\ W_j \\ \theta_j \end{Bmatrix}$$

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VARIATIONAL METHODS (EULER B. BEAM ELEMENT)

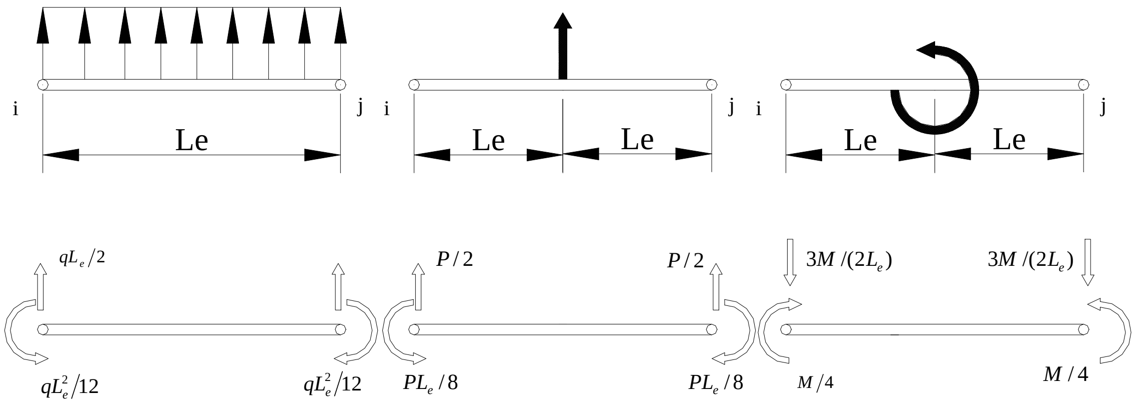
- Apply the minimum potential energy method to a beam finite element (cont.):
  - The work of external forces (F – concentrated forces, q - distributed forces).

$$\delta W = \int_{Sup} \langle F_k \rangle \{\delta_k\} + \int_{Sup} \langle q \rangle \{\delta_k\}$$

$$\delta U = \delta W$$

$$\int_{Sup} \langle F_k \rangle \{\delta_k\} = \sum_{k=1}^{n} \langle N \rangle_k \{F_k\}$$

$$\int_{Sup} \langle q \rangle \{\delta_k\} = \int_0^{L_e} \langle N_1 \quad N_2 \quad N_3 \quad N_4 \rangle q \begin{Bmatrix} \delta W_i \\ \delta\theta_i \\ \delta W_j \\ \delta\theta_j \end{Bmatrix} dx = \langle \delta W_i \quad \delta\theta_i \quad \delta W_j \quad \delta\theta_j \rangle \times q \times \begin{Bmatrix} \int_0^{L_e} \left(1 - \frac{3x^2}{L_e^2} + \frac{2x^3}{L_e^3}\right) dx \\ \int_0^{L_e} \left(x - \frac{2x^2}{L_e} + \frac{x^3}{L_e^2}\right) dx \\ \int_0^{L_e} \left(\frac{3x^2}{L_e^2} - \frac{2x^3}{L_e^3}\right) dx \\ \int_0^{L_e} \left(\frac{-x^2}{L_e} + \frac{x^3}{L_e^2}\right) dx \end{Bmatrix}$$



VARIATIONAL METHODS (EULER B. BEAM ELEMENT)

- Apply the minimum potential energy method to a beam finite element (cont.):

$$\delta U = \delta W$$

$$\delta U = \delta W \Leftrightarrow \int_{Vol} \langle \varepsilon \rangle [D]^t \{\delta\varepsilon\} = \int_{Sup} \langle F_k \rangle \{\delta_k\} + \int_{Sup} \langle q \rangle \{\delta_k\}$$

$$\int_{Vol} \left\langle \frac{\partial N_1}{\partial x} \quad \frac{\partial N_2}{\partial x} \quad \frac{\partial N_3}{\partial x} \quad \frac{\partial N_4}{\partial x} \right\rangle \begin{Bmatrix} W_i \\ \theta_i \\ W_j \\ \theta_j \end{Bmatrix} [D]^t \left\langle \frac{\partial N_1}{\partial x} \quad \frac{\partial N_2}{\partial x} \quad \frac{\partial N_3}{\partial x} \quad \frac{\partial N_4}{\partial x} \right\rangle \begin{Bmatrix} \delta W_i \\ \delta\theta_i \\ \delta W_j \\ \delta\theta_j \end{Bmatrix} dVol =$$

$$= \langle \delta W_i \quad \delta\theta_i \quad \delta W_j \quad \delta\theta_j \rangle \times q \times \begin{Bmatrix} \int_0^{L_e} \left(1 - \frac{3x^2}{L_e^2} + \frac{2x^3}{L_e^3}\right) dx \\ \int_0^{L_e} \left(x - \frac{2x^2}{L_e} + \frac{x^3}{L_e^2}\right) dx \\ \int_0^{L_e} \left(\frac{3x^2}{L_e^2} - \frac{2x^3}{L_e^3}\right) dx \\ \int_0^{L_e} \left(\frac{-x^2}{L_e} + \frac{x^3}{L_e^2}\right) dx \end{Bmatrix} + \langle \delta W_i \quad \delta\theta_i \quad \delta W_j \quad \delta\theta_j \rangle \sum_{k=1}^{n} \langle N \rangle_k \{F_k\}$$

VARIATIONAL METHODS (EULER B. BEAM ELEMENT)

- Apply the minimum potential energy method to a beam finite element (cont.):

$$\int_{Vol} \begin{Bmatrix} \frac{\partial N_1}{\partial x} \\ \frac{\partial N_2}{\partial x} \\ \frac{\partial N_3}{\partial x} \\ \frac{\partial N_4}{\partial x} \end{Bmatrix} [D] \left\langle \frac{\partial N_1}{\partial x} \quad \frac{\partial N_2}{\partial x} \quad \frac{\partial N_3}{\partial x} \quad \frac{\partial N_4}{\partial x} \right\rangle \begin{Bmatrix} W_i \\ \theta_i \\ W_j \\ \theta_j \end{Bmatrix} dVol = q \times \begin{Bmatrix} \int_0^{L_e} \left(1 - \frac{3x^2}{L_e^2} + \frac{2x^3}{L_e^3}\right) dx \\ \int_0^{L_e} \left(x - \frac{2x^2}{L_e} + \frac{x^3}{L_e^2}\right) dx \\ \int_0^{L_e} \left(\frac{3x^2}{L_e^2} - \frac{2x^3}{L_e^3}\right) dx \\ \int_0^{L_e} \left(\frac{-x^2}{L_e} + \frac{x^3}{L_e^2}\right) dx \end{Bmatrix} + \sum_{k=1}^{n} \langle N \rangle_k \{F_k\}$$

$$[K_e] \begin{Bmatrix} W_i \\ \theta_i \\ W_j \\ \theta_{ji} \end{Bmatrix} = \{F_e\}$$

$$[K_e] = \frac{EI}{L_e^3} \begin{bmatrix} 12 & 6L_e & -12 & 6L_e \\ 6L_e & 4L_e^2 & -6L_e & 2L_e^2 \\ -12 & -6L_e & 12 & -6L_e \\ 6L_e & 2L_e^2 & -6L_e & 4L_e^2 \end{bmatrix}$$

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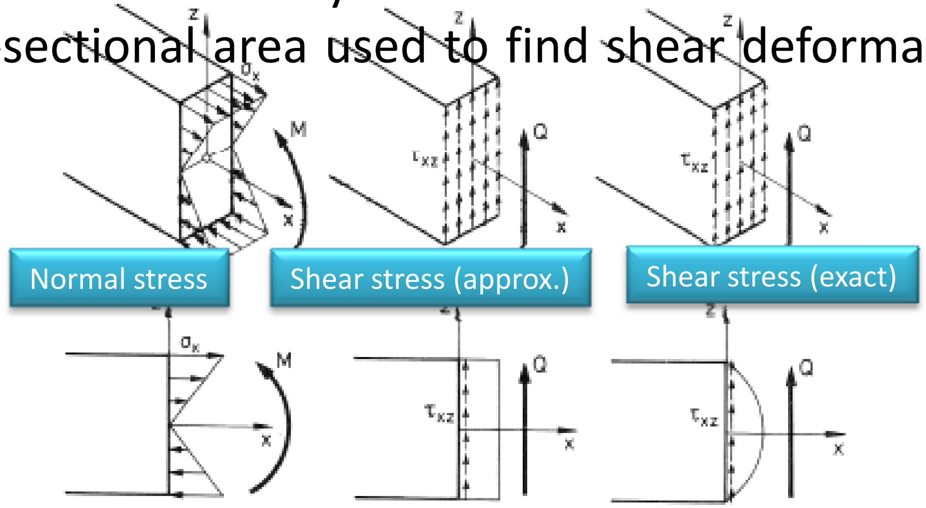
VARIATIONAL METHODS (TIMOSHENKO BEAM ELEMENT)

- Bending moment and shear are defined according to figure.
  - Axial stress variation in “y” is linear (exact solution).
  - Shear stress will be constant in “y” (approximate solution).
- The polynomial approximation may be linear, because the main variables are independent.
- Most FE software use beam elements from Timoshenko beam theory, which includes deflection due to shear strain, as well as that due to bending strain and rigid body motion.
- If the length of a beam is short (similar to its depth) then this assumption leads to better results.
- The Timoshenko finite element may revert to a Euler-Bernoulli finite element if the “shear area” (cross-sectional area used to find shear deformation) is not specified by the user.



VARIATIONAL METHODS (TIMOSHENKO BEAM ELEMENT

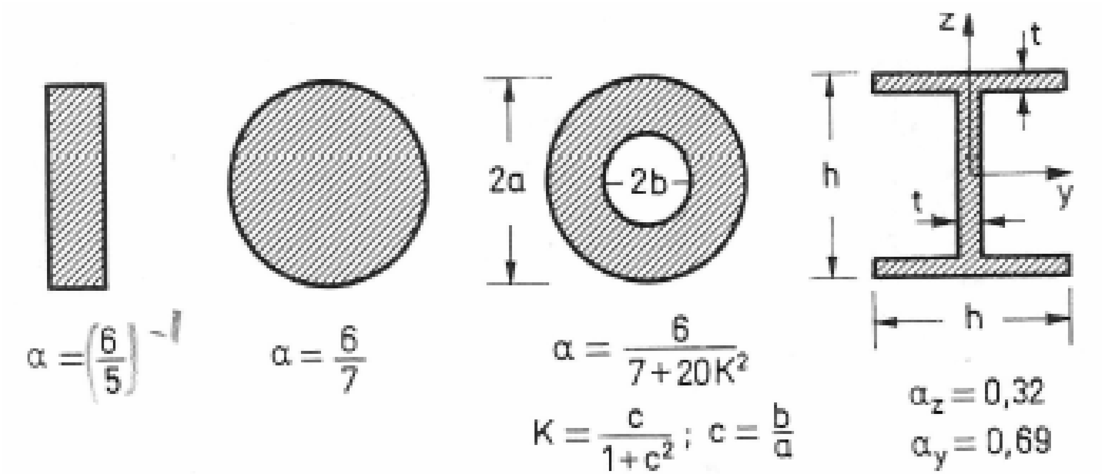
- To solve the problem of shear stress distribution:
  - The deformation energy should be correct in accordance to the theory, considering the shear stress equal to:

$$\tau_{xy} = \alpha \, G \, \gamma_{xy}$$

  - Shear should be determined in accordance:

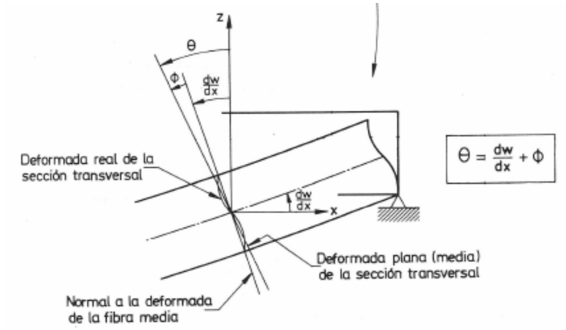
$$V = A \, \alpha \, G \, \gamma_{xy} = A^{*} G \, \gamma_{xy}$$

    - Where α is the shape factor and A* is the reduced area.
    - α depends on the cross section area.



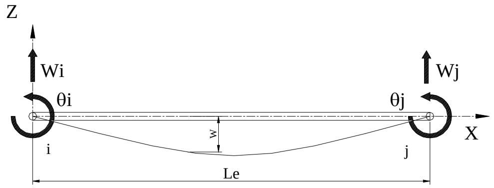
VARIATIONAL METHODS (TIMOSHENKO BEAM ELEMENT)

- Interpolating functions:
  - Assume linear function for both main variables.
- Displacement field:



$$w(x) = b_1 + b_2 x = \langle 1 \quad x \rangle \begin{Bmatrix} b_1 \\ b_2 \end{Bmatrix}$$

$$\theta(x) = b_3 + b_4 x = \langle 1 \quad x \rangle \begin{Bmatrix} b_3 \\ b_4 \end{Bmatrix}$$



- Assume nodal displacement variables:

$$\begin{Bmatrix} w_i \\ w_j \end{Bmatrix} \quad \begin{Bmatrix} \theta_i \\ \theta_j \end{Bmatrix}$$

$$w(\xi) = N_1(\xi)\, w_1 + N_2(\xi)\, w_2$$

$$\Theta(\xi) = N_1(\xi)\, \Theta_1 + N_2(\xi)\, \Theta_2$$

- Interpolating functions in local coordinates or in natural coordinates:

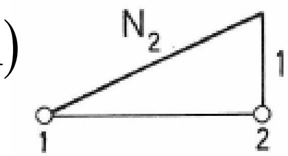
$$w(x) = \left\langle 1 \quad \frac{x}{L_e} \right\rangle \begin{Bmatrix} w_i \\ w_j \end{Bmatrix} = \langle N_i \quad N_j \rangle \begin{Bmatrix} w_i \\ w_j \end{Bmatrix}$$

$$\theta(x) = \left\langle 1 \quad \frac{x}{L_e} \right\rangle \begin{Bmatrix} \theta_i \\ \theta_j \end{Bmatrix} = \langle N_i \quad N_j \rangle \begin{Bmatrix} \theta_i \\ \theta_j \end{Bmatrix}$$

$$\xi = \frac{2x}{L_e} - 1$$

$$N_1 = \frac{1}{2}(1 - \xi)$$

$$N_2 = \frac{1}{2}(\xi + 1)$$



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VARIATIONAL METHODS (TIMOSHENKO BEAM ELEMENT)

- Displacement field:
  - Axial displacement due to bending: $u(x) = -y \times \theta(x)$
  - Transversal displacement: $w(x) = w(x)$
  - Rotation: $\theta(x) = \theta(x)$
- Strain field which produces strain energy:
  - Normal strain: $$\varepsilon_{xx} = \frac{\partial u}{\partial x} = -y \times \frac{\partial \theta(x)}{\partial x}$$
  - Shear strain: $$\gamma_{xy} = \frac{dw}{dx} + \frac{du}{dy} = \frac{dw}{dx} + \frac{\partial u}{\partial y} = \frac{dw}{dx} - \theta(x)$$
- Stress field:

$$\sigma_{xx} = E\,\varepsilon_{xx} = E\left(-y \times \frac{\partial \theta(x)}{\partial x}\right)$$

$$\tau_{xy} = G\,\gamma_{xy} = G\left[\frac{dw(x)}{dx} - \theta(x)\right]$$

- Apply the minimum potential energy method to a beam finite element:

$$\delta U = \delta W$$

VARIATIONAL METHODS (TIMOSHENKO BEAM ELEMENT)

- The deformation energy should consider :
  - Normal strain in "x" direction;
  - Shear strain in "y" direction.

$$\delta U = \delta W$$

$$\delta U = \int_{Vol} \langle \sigma \rangle \{\delta \varepsilon\}\, dv = \int_{Vol} \sigma_{xx} \times \delta\varepsilon_{xx} + \tau_{xy} \times \delta\gamma_{xy}\ dv = \int_{Vol} \delta\varepsilon_{xx} \times \sigma_{xx} + \delta\gamma_{xy} \times \tau_{xy}\ dv$$

- Approaching virtual strain field and also the real stress field in discrete format:

$$\delta\varepsilon_{xx} = -y \times \frac{d}{dx}\delta\theta(x) = -y \times \left\langle \frac{dN_i}{dx} \quad \frac{dN_j}{dx} \right\rangle \begin{Bmatrix} \delta\theta_i \\ \delta\theta_j \end{Bmatrix}$$

$$\delta\gamma_{xy} = \frac{d}{dx}\delta w - \delta\theta(x) = \left\langle \frac{dN_i}{dx} \quad \frac{dN_j}{dx} \right\rangle \begin{Bmatrix} \delta w_i \\ \delta w_j \end{Bmatrix} - \langle N_i \quad N_j \rangle \begin{Bmatrix} \delta\theta_i \\ \delta\theta_j \end{Bmatrix}$$

$$\sigma_{xx} = E\left(-y \times \frac{\partial\theta(x)}{\partial x}\right) = E\,(-y) \times \left\langle \frac{dN_i}{dx} \quad \frac{dN_j}{dx} \right\rangle \begin{Bmatrix} \theta_i \\ \theta_j \end{Bmatrix}$$

$$\tau_{xy} = G\left[\frac{dw(x)}{dx} - \theta(x)\right] = G\left[\left\langle \frac{dN_i}{dx} \quad \frac{dN_j}{dx} \right\rangle \begin{Bmatrix} w_i \\ w_j \end{Bmatrix} - \langle N_i \quad N_j \rangle \begin{Bmatrix} \theta_i \\ \theta_j \end{Bmatrix}\right]$$

VARIATIONAL METHODS (TIMOSHENKO BEAM ELEMENT)

- Approaching virtual strain field and also the real stress field in discrete format, using :
  - Assuming nodal variables: $\langle w_i \quad \theta_i \quad w_j \quad \theta_j \rangle^t$
  - An expansion for vector dimension should be applied.

$$\delta\varepsilon_{xx} = -y \times \left\langle 0 \quad \frac{dN_i}{dx} \quad 0 \quad \frac{dN_j}{dx} \right\rangle \bullet \begin{Bmatrix} \delta w_i \\ \delta\theta_i \\ \delta w_j \\ \delta\theta_j \end{Bmatrix} \qquad \sigma_{xx} = E\,(-y) \times \left\langle 0 \quad \frac{dN_i}{dx} \quad 0 \quad \frac{dN_j}{dx} \right\rangle \bullet \begin{Bmatrix} w_i \\ \theta_i \\ w_j \\ \theta_j \end{Bmatrix}$$

$$\delta\gamma_{xy} = \left\langle \frac{dN_i}{dx} \quad -N_i \quad \frac{dN_j}{dx} \quad -N_j \right\rangle \bullet \begin{Bmatrix} \delta w_i \\ \delta\theta_i \\ \delta w_j \\ \delta\theta_j \end{Bmatrix} \qquad \tau_{xy} = G \left[\left\langle \frac{dN_i}{dx} \quad N_i \quad \frac{dN_j}{dx} \quad N_j \right\rangle \bullet \begin{Bmatrix} w_i \\ \theta_i \\ w_j \\ \theta_j \end{Bmatrix} \right]$$

- The deformation energy may be calculated in discrete format :

$$\boxed{\delta U = \int_{Vol} \delta\varepsilon_{xx} \times \sigma_{xx} + \delta\gamma_{xy} \times \tau_{xy} \; dv}$$

$$\delta U = \delta W$$

VARIATIONAL METHODS (TIMOSHENKO BEAM ELEMENT)

- The deformation energy may be calculated in discrete format : $\delta U = \delta W$

$$\delta U = \int_{Vol} -y \begin{Bmatrix} 0 \\ \frac{dN_i}{dx} \\ 0 \\ \frac{dN_j}{dx} \end{Bmatrix} \bullet \langle \delta w_i \quad \delta\theta_i \quad \delta w_j \quad \delta\theta_j \rangle \times E(-y) \left\langle 0 \quad \frac{dN_i}{dx} \quad 0 \quad \frac{dN_j}{dx} \right\rangle \bullet \begin{Bmatrix} w_i \\ \theta_i \\ w_j \\ \theta_j \end{Bmatrix} dv +$$

$$+ \int_{Vol} \begin{Bmatrix} \frac{dN_i}{dx} \\ -N_i \\ \frac{dN_j}{dx} \\ -N_j \end{Bmatrix} \bullet \langle \delta w_i \quad \delta\theta_i \quad \delta w_j \quad \delta\theta_j \rangle \times G \left\langle \frac{dN_i}{dx} \quad -N_i \quad \frac{dN_j}{dx} \quad -N_j \right\rangle \bullet \begin{Bmatrix} w_i \\ \theta_i \\ w_j \\ \theta_j \end{Bmatrix} dv$$

- Assuming any virtual nodal variable, $\langle \delta w_i \quad \delta\theta_i \quad \delta w_j \quad \delta\theta_j \rangle$ consistent with boundary conditions.

$$\delta U = \int_0^{Le} E \int_A y^2 dA \begin{Bmatrix} 0 \\ \frac{dN_i}{dx} \\ 0 \\ \frac{dN_j}{dx} \end{Bmatrix} \left\langle 0 \quad \frac{dN_i}{dx} \quad 0 \quad \frac{dN_j}{dx} \right\rangle \begin{Bmatrix} w_i \\ \theta_i \\ w_j \\ \theta_j \end{Bmatrix} dx + \int_0^{Le} G \int_{A^*} dA \begin{Bmatrix} \frac{dN_i}{dx} \\ -N_i \\ \frac{dN_j}{dx} \\ -N_j \end{Bmatrix} \left\langle \frac{dN_i}{dx} \quad -N_i \quad \frac{dN_j}{dx} \quad -N_j \right\rangle \begin{Bmatrix} w_i \\ \theta_i \\ w_j \\ \theta_j \end{Bmatrix} dx$$

- It is not necessary to use natural coordinates, because integrals may be calculated exactly.

VARIATIONAL METHODS (TIMOSHENKO BEAM ELEMENT)

- The deformation energy may be calculated in discrete format : $\delta U = \delta W$

$$\delta U = \int_0^{Le} EI \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \left(\frac{dN_i}{dx}\right)^2 & 0 & \frac{dN_i}{dx}\frac{dN_j}{dx} \\ 0 & 0 & 0 & 0 \\ 0 & \frac{dN_j}{dx}\frac{dN_i}{dx} & 0 & \left(\frac{dN_j}{dx}\right)^2 \end{bmatrix} dx \begin{Bmatrix} w_i \\ \theta_i \\ w_j \\ \theta_j \end{Bmatrix} + \int_0^{Le} GA^* \begin{bmatrix} \left(\frac{dN_i}{dx}\right)^2 & \frac{dN_i}{dx}(-N_i) & \frac{dN_i}{dx}\frac{dN_j}{dx} & \frac{dN_i}{dx}(-N_j) \\ (-N_i)\frac{dN_i}{dx} & (N_i)^2 & (-N_i)\frac{dN_j}{dx} & (-N_i)(-N_j) \\ \frac{dN_j}{dx}\frac{dN_i}{dx} & \frac{dN_j}{dx}(-N_i) & \left(\frac{dN_j}{dx}\right)^2 & \frac{dN_j}{dx}(-N_j) \\ (-N_j)\frac{dN_i}{dx} & (-N_j)(-N_i) & (-N_j)\frac{dN_j}{dx} & (N_j)^2 \end{bmatrix} dx \begin{Bmatrix} w_i \\ \theta_i \\ w_j \\ \theta_j \end{Bmatrix}$$

$$\delta U = EI \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{1}{L_e} & 0 & -\frac{1}{L_e} \\ 0 & 0 & 0 & 0 \\ 0 & -\frac{1}{L_e} & 0 & \frac{1}{L_e} \end{bmatrix} \begin{Bmatrix} w_i \\ \theta_i \\ w_j \\ \theta_j \end{Bmatrix} + GA^* \begin{bmatrix} \frac{1}{L_e} & \frac{1}{2} & -\frac{1}{L_e} & \frac{1}{2} \\ \frac{1}{2} & \frac{L_e}{3} & -\frac{1}{2} & \frac{L_e}{6} \\ -\frac{1}{L_e} & -\frac{1}{2} & \frac{1}{L_e} & -\frac{1}{2} \\ \frac{1}{2} & \frac{L_e}{6} & -\frac{1}{2} & \frac{L_e}{3} \end{bmatrix} \begin{Bmatrix} w_i \\ \theta_i \\ w_j \\ \theta_j \end{Bmatrix}$$

$$[K_e] = \begin{bmatrix} \frac{GA^*}{L_e} & \frac{GA^*}{2} & -\frac{GA^*}{L_e} & \frac{GA^*}{2} \\ \frac{GA^*}{2} & \frac{EI}{L_e} + \frac{GA^* L_e}{3} & -\frac{GA^*}{2} & -\frac{EI}{L_e} + \frac{GA^* L_e}{6} \\ -\frac{GA^*}{L_e} & -\frac{GA^*}{2} & \frac{GA^*}{L_e} & -\frac{GA^*}{2} \\ \frac{GA^*}{2} & -\frac{EI}{L_e} + \frac{GA^* L_e}{6} & -\frac{GA^*}{2} & \frac{EI}{L_e} + \frac{GA^* L_e}{3} \end{bmatrix}$$

VARIATIONAL METHODS (TIMOSHENKO BEAM ELEMENT)

- Same procedure as for the Bernoulli beam element provides the following stiffness matrix:

$$K_{Timoshenko} = kGA \begin{bmatrix} \frac{1}{h} & -\frac{1}{2} & -\frac{1}{h} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{h}{3}+\frac{EI}{kGAh} & \frac{1}{2} & \frac{h}{6}-\frac{EI}{kGAh} \\ -\frac{1}{h} & \frac{1}{2} & \frac{1}{h} & \frac{1}{2} \\ -\frac{1}{2} & \frac{h}{6}-\frac{EI}{kGAh} & \frac{1}{2} & \frac{h}{3}+\frac{EI}{kGAh} \end{bmatrix}$$

- To avoid shear locking, a mixed interpolation approach may be used, or instead a rigorous derive interpolating function:

$$K_{\substack{exact \\ Timoshenko}} = \frac{kGA}{\left(12+\frac{kGAh^2}{EI}\right)h} \begin{bmatrix} 12 & 6h & -12 & 6h \\ 6h & 4h^2\left(1+3\frac{EI}{kGA}\right) & -6h & 2h^2\left(1-6\frac{EI}{kGA}\right) \\ -12 & -6h & 12 & -6h \\ 6h & 2h^2\left(1-6\frac{EI}{kGA}\right) & -6h & 4h^2\left(1+3\frac{EI}{kGA}\right) \end{bmatrix}$$

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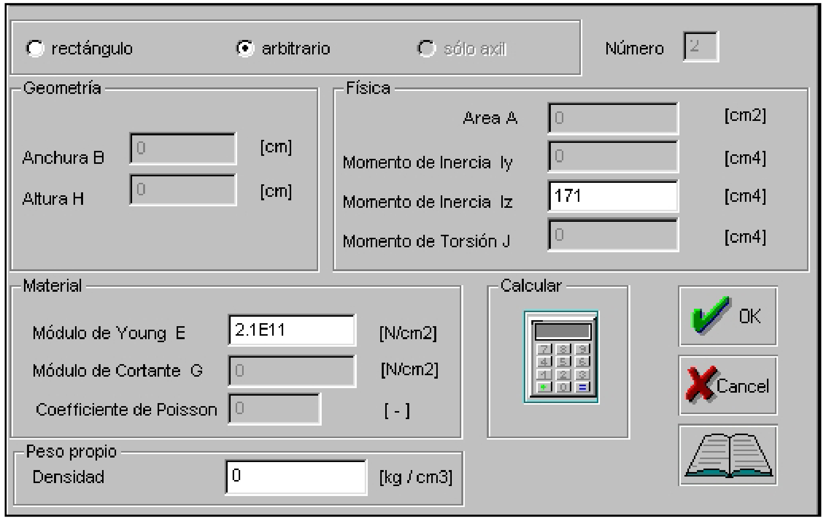
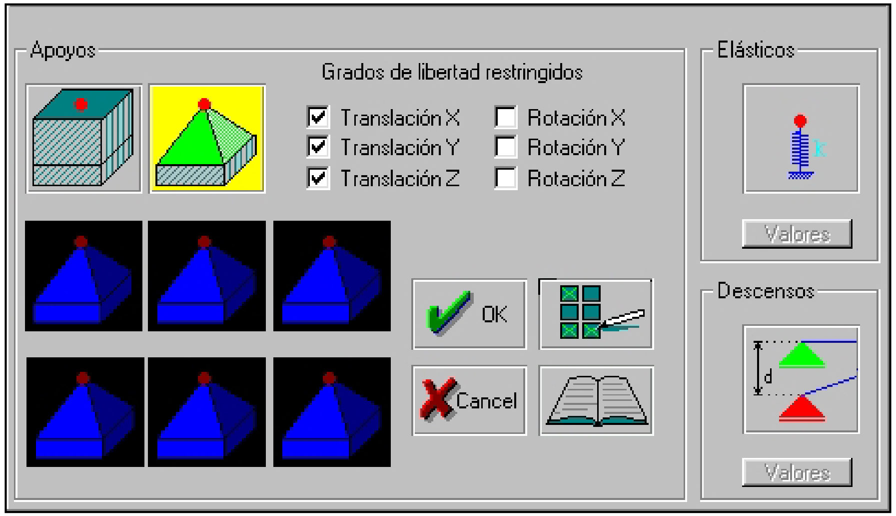
THEMATIC EXERCISE WITH ED-Elas2D

- http://www.cimne.com/tiendaCIMNE/MenuSoftEdu.asp (free demo version).
- Simply supported beam with two finite elements, submitted to mid span load.
  - Material definition;
  - Boundary conditions definition;





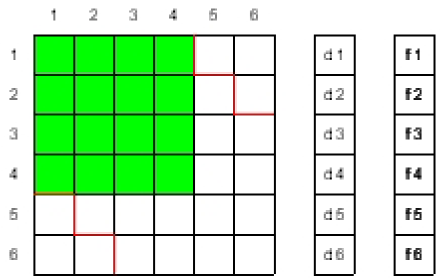
  - Stiffness for each element , in global coordinate system.

	1	2	3	4
1	4.31E+010	2.15E+010	-4.31E+010	2.15E+010
2	2.15E+010	1.44E+010	-2.15E+010	7.18E+009
3	-4.31E+010	-2.15E+010	4.31E+010	-2.15E+010
4	2.15E+010	7.18E+009	-2.15E+010	1.44E+010

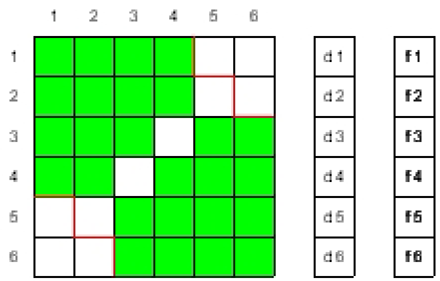
	1	2	3	4
1	4.31E+010	2.15E+010	-4.31E+010	2.15E+010
2	2.15E+010	1.44E+010	-2.15E+010	7.18E+009
3	-4.31E+010	-2.15E+010	4.31E+010	-2.15E+010
4	2.15E+010	7.18E+009	-2.15E+010	1.44E+010

THEMATIC EXERCISE WITH ED-Elas2D

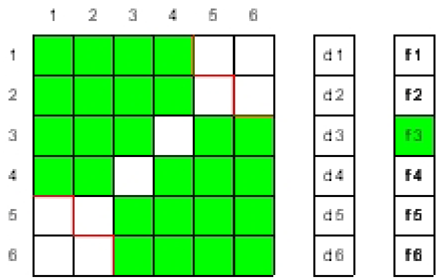
- Element 1



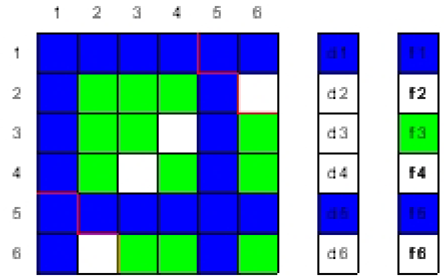
- Element 2



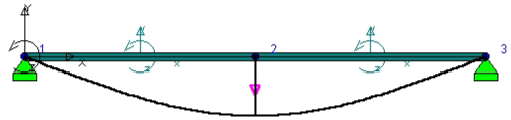
- Assembling



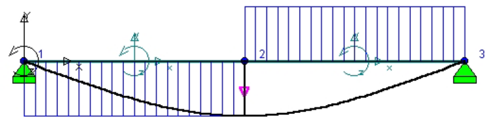
- Imposing boundary conditions:



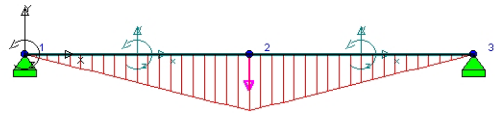
- Deformed shape:



- Transverse internal effort:



- Bending moment effort:



THERMAL ANALYSIS – 2D

- Thermal balance:
  - Heat flux in/out with respect to infinitesimal area;
  - Internal energy variation;
  - The law of Heat Conduction, also known as Fourier's law, states that the time rate of heat transfer through a material is proportional to the negative gradient.

$$\sum \phi_{in} - \sum \phi_{out} = \Delta U$$

$$\Delta U = \rho C (dz.dy.x) \frac{\partial T}{\partial t}$$

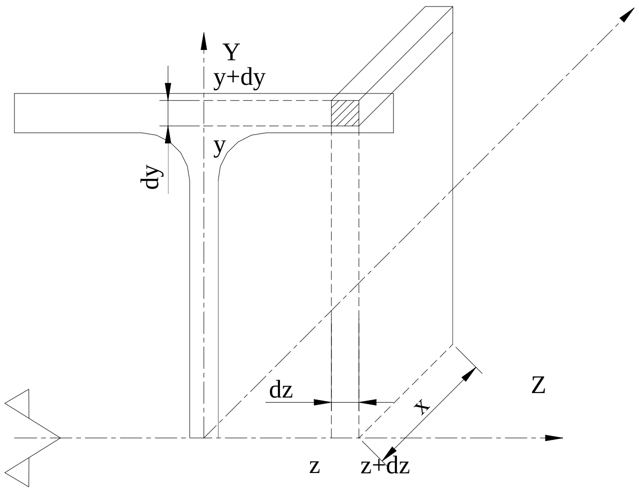
$$\Delta \phi_z = -\left(\lambda \frac{\partial T}{\partial z} \right)_z (dy.x) - \left[-\left(\lambda \frac{\partial T}{\partial z} \right)_{z+dz} (dy.x) \right]$$

$$\left(\lambda \frac{\partial T}{\partial z} \right)_{z+dz} = \left(\lambda \frac{\partial T}{\partial z} \right)_z + \frac{\partial}{\partial z} \left[\left(\lambda \frac{\partial T}{\partial z} \right)_z \right] dz$$

$$\Delta \phi_z = \frac{\partial}{\partial z} \left[\left(\lambda \frac{\partial T}{\partial z} \right)_z (dz.dy.x) \right]$$

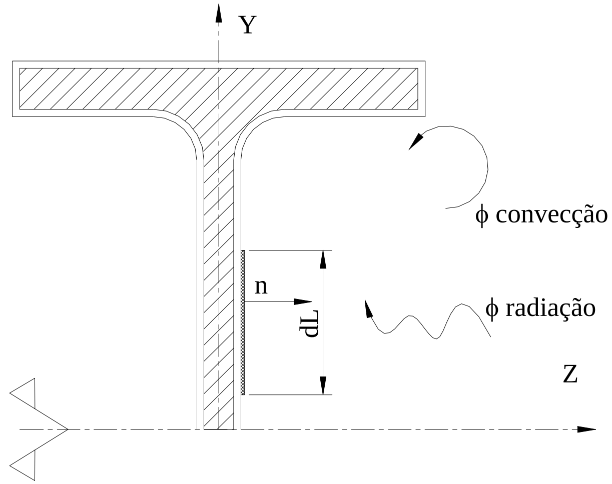
$$\Delta \phi_y = \frac{\partial}{\partial y} \left[\left(\lambda \frac{\partial T}{\partial y} \right)_y (dy.dz.x) \right]$$

$$\lambda \left(\frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} \right) = \rho C \frac{\partial T}{\partial t}$$



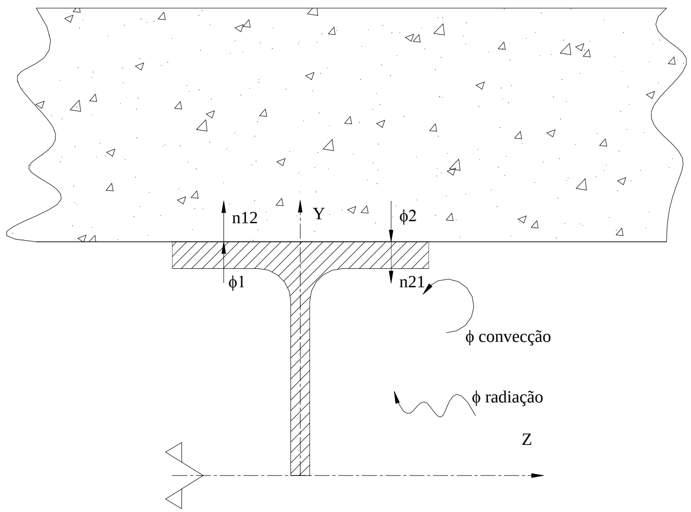
THERMAL ANALYSIS – 2D

- Boundary conditions:
  - Dirichlet: prescribed temperature:
  - Cauchy: heat flux (convection, radiation):



$$\left[\left(-\lambda\frac{\partial T}{\partial y}\right).\vec{j}+\left(-\lambda\frac{\partial T}{\partial z}\right).\vec{k}\right]\bullet\vec{n}.dL=-\phi\sigma\varepsilon\left(T_g^{\,4}-T_L^4\right)-\alpha\left(T_g-T_L\right)$$

  - Perfect contact between two adjacent elements / materials:



$$\left[\left(-\lambda\frac{\partial T_1}{\partial y}\right).nx+\left(-\lambda\frac{\partial T_1}{\partial z}\right).ny\right]=\left[\left(-\lambda\frac{\partial T_2}{\partial y}\right).nx+\left(-\lambda\frac{\partial T_2}{\partial z}\right).ny\right]$$

$$\int_V\left(\vec{A}\bullet\vec{n}\right)dV=\int_{V1}\left(\vec{A}\bullet\vec{n}_{12}\right)dV+\int_{V1}\left(\vec{A}\bullet\vec{n}_{21}\right)dV=0$$

THERMAL ANALYSIS – 2D

- Weighted Residual Method:
  - If the physical formulation of the problem is described as a differential equation, then the most popular solution method is the Method of Weighted Residuals
  - For the energy equilibrium equation (simplified format): $L(T^*)+U=0$
  - Assume T as an approximation to temperature. $L(T)+U\neq 0$
  - Assume ψ as weighting function. The number of weighting functions equals the number of unknown coefficients in the approximate solution. There are several choices for the weighting functions:
- Galerkin's method, the weighting functions are the same functions that were used in the approximating equation.

$$\int_V \psi(y,z).\left[L(T)+U\right].dv=\sum_{ele}\int_{Ve}\psi(y,z).\left[L(T)+U\right].dVe=0$$

THERMAL ANALYSIS – 2D

- Weak integral formulation:
  - Increases weigthing function differentiability;
  - Reduces Temperture differentiability

$$\sum_{ele}\int_{Ve}\psi(y,z).\left[\frac{\partial}{\partial y}.\left(\lambda\frac{\partial T}{\partial y}\right)+\frac{\partial}{\partial z}.\left(\lambda\frac{\partial T}{\partial z}\right)-\rho\,C\frac{\partial T}{\partial t}\right].dVe=0\Leftrightarrow$$

$$\sum_{ele}\int_{Ve}\left[\psi(y,z).div(\vec{V})\right]dVe-\sum_{ele}\int_{Ve}\psi(y,z)\rho\,C\frac{\partial T}{\partial t}dVe=0\Leftrightarrow$$

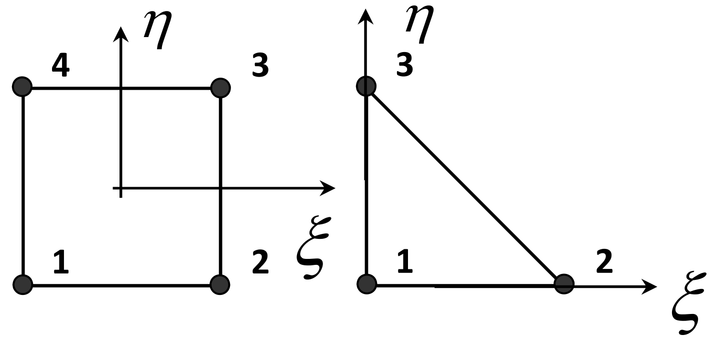
$$\sum_{ele}\int_{Ve}div\left(\psi\vec{V}\right)dVe-\sum_{ele}\int_{Ve}\vec{V}\bullet grad\left(\psi\right)dVe-\sum_{ele}\int_{Ve}\psi(y,z)\rho\,C\frac{\partial T}{\partial t}dVe=0\Leftrightarrow$$

$$\sum_{ele}\oint_{Se}\psi\vec{V}\bullet\vec{n}dSe-\sum_{ele}\int_{Ve}\vec{V}\bullet grad\left(\psi\right)dVe-\sum_{ele}\int_{Ve}\psi(y,z)\rho\,C\frac{\partial T}{\partial t}dVe=0\Leftrightarrow$$

$$\sum_{ele}\oint_{Se}\psi\left(\lambda\frac{\partial T}{\partial y}ny+\lambda\frac{\partial T}{\partial z}nz\right)dSe-\sum_{ele}\int_{Ve}\left\langle 0,\lambda\frac{\partial T}{\partial y},\lambda\frac{\partial T}{\partial z}\right\rangle\bullet\left\langle\frac{\partial\psi}{\partial x},\frac{\partial\psi}{\partial y},\frac{\partial\psi}{\partial z}\right\rangle dVe$$

$$-\sum_{ele}\int_{Ve}\psi(y,z)\rho\,C\frac{\partial T}{\partial t}dVe=0$$

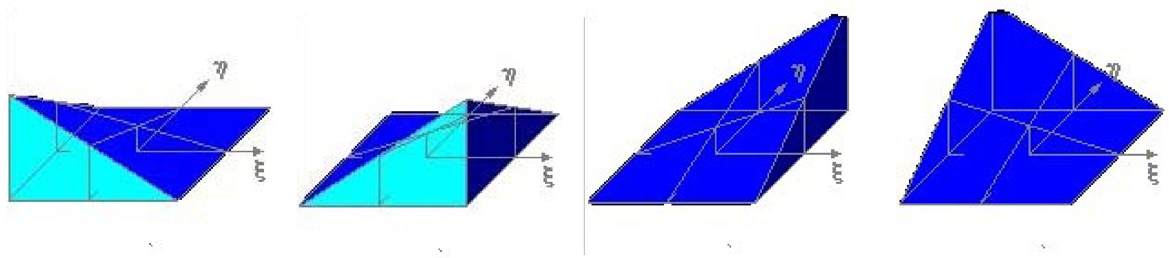
THERMAL ANALYSIS – 2D



- Temperture approximation:

$$T^{ele} = \sum_{i=1}^{nne} T_i N_i$$

- Geometry approximation



$$y = \sum_{i=1}^{nne} y_i N_i$$

$$z = \sum_{i=1}^{nne} z_i N_i$$

- Local coordinate system (natural coordinates) should be used to facilitate numerical integration.

$$\langle N \rangle = \frac{1}{4} \langle (1-\xi)(1-\eta) \quad (1+\xi)(1-\eta) \quad (1+\xi)(1+\eta) \quad (1-\xi)(1+\eta) \rangle$$

- Galerkine approximation:

$$\psi = \delta T = \sum_{i=1}^{nne} \langle N_i \rangle . \{\delta T_i\} = \sum_{i=1}^{nne} \langle \delta T_i \rangle . \{N_i\}$$

THERMAL ANALYSIS – 2D

- Solution for integrals, introducing Jacobian:
  - In volume:
  - In boundary:

$$\int_{Ve} f(y,z).dy.dz = \int_{Vr} f\left[y(\xi,\eta), z(\xi,\eta)\right]\det[J].d\xi d\eta$$

$$\int_{Se} f(X).dS = \int_{Sr} f\left[X(\xi,\eta)\right].\sqrt{\left(\frac{\partial y}{\partial S}\right)^2 + \left(\frac{\partial z}{\partial S}\right)^2}.dSr$$

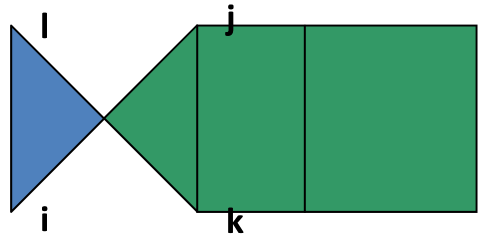
- Jacobian of coordinate transformation, from Natural to Cartesian:
  - Assuming the interpolating function to geometry.

$$\begin{Bmatrix} \frac{\partial}{\partial \xi} \\ \frac{\partial}{\partial \eta} \end{Bmatrix} = \begin{bmatrix} \frac{\partial z}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial z}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{bmatrix} \begin{Bmatrix} \frac{\partial}{\partial z} \\ \frac{\partial}{\partial y} \end{Bmatrix}$$

$$\frac{\partial z}{\partial \xi} = \sum_{i=1}^{nne} \frac{\partial N_i}{\partial \xi} \times z_i$$

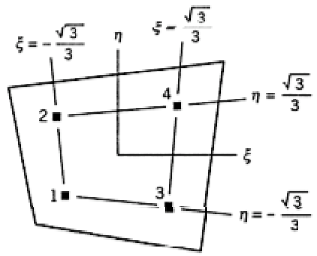
$$[J] = \begin{bmatrix} \sum_{i=1}^{nne} \frac{\partial N_i}{\partial \xi} \times z_i & \sum_{i=1}^{nne} \frac{\partial N_i}{\partial \xi} \times y_i \\ \sum_{i=1}^{nne} \frac{\partial N_i}{\partial \eta} \times z_i & \sum_{i=1}^{nne} \frac{\partial N_i}{\partial \eta} \times y_i \end{bmatrix}$$

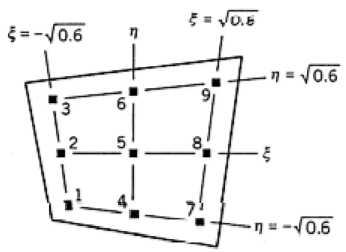
- For planar element, the Jacobian determinant is numerically equal to the area of the finite element.

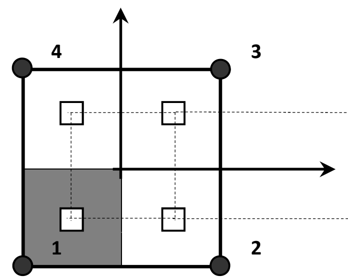


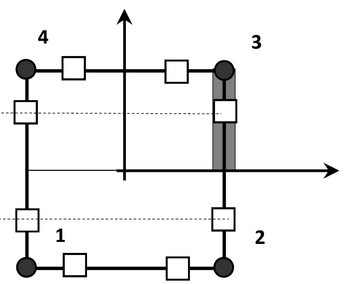
THERMAL ANALYSIS – 2D

- Numerical integration:
  - Gauss method: In numerical analysis, a quadrature rule is an approximation of the definite integral of a function, usually stated as a weighted sum of function values at specified points within the domain of integration (Gauss points).
    - “M” integration points are necessary to exactly integrate a polynomial of degree “2M-1’”
    - Less expensive
    - Exponential convergence, error proportional to $\left(1/2M\right)^{2M}$
  - Newton Cotes method: (Less used).
    - “M” integration points are necessary to exactly integrate a polynomial of degree “M-1’”
    - More expensive









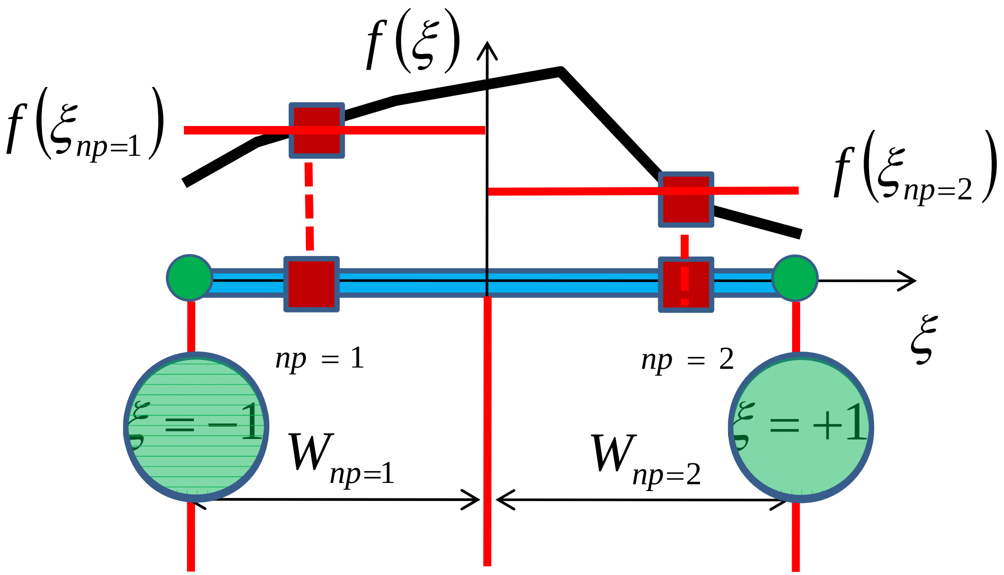
$$\int_{-1}^{+1}\int_{-1}^{+1} f\left(\xi,\eta\right).d\xi.d\eta = \sum_{igauss=1}^{npg}\sum_{jgauss=1}^{npg} w_i.w_j.f\left(\xi_i,\eta_j\right)$$

Johann Carl Friedrich Gauss, (1777–1855)
German mathematician and scientist

THERMAL ANALYSIS – 2D

- Numerical integration:
  - Example of integration in one dimension.

$$\int_{-1}^{+1} f(\xi) \approx \sum_{np=1}^{n} f(\xi_i) \times W_i$$



n	$\pm\xi_i$	W_i
1	0.0	2.0
2	0.5773502692	1.0
3	0.774596697 0.0	0.5555555556 0.8888888889
4	0.8611363116 0.3399810436	0.3478548451 0.6521451549
5	0.9061798459 0.5384693101 0.0	0.2369268851 0.4786286705 0.5688888889
6	0.9324695142 0.6612093865 0.2386191861	0.1713244924 0.3607615730 0.4679139346
7	0.9491079123 0.7415311856 0.4058451514 0.0	0.1294849662 0.2797053915 0.3818300505 0.4179591837
8	0.9602898565 0.7966664774 0.5255324099 0.1834346425	0.1012285363 0.2223810345 0.3137066459 0.3626837834

THERMAL ANALYSIS – 2D

- Solution:
  - System of algebraic equations.

$$
-\sum_{ele}\langle \delta T_i\rangle \int_{Vref} [B_{\xi\eta}]^t [J^{-1}]^t \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} [J^{-1}][B_{\xi\eta}].\det[J].dv.\{T_j\}
$$

$$
+\sum_{ele}\langle \delta T_i\rangle \oint_{Sref} \{N_i\}\langle N_j\rangle\{\alpha_j\}\langle N_k\rangle \sqrt{\left(\frac{\partial z}{\partial s}\right)^2 + \left(\frac{\partial y}{\partial s}\right)^2}.dS.\{T_g^k\}
$$

$$
-\sum_{ele}\langle \delta T_i\rangle \oint_{Sref} \{N_i\}\langle N_j\rangle\{\alpha_j\}\langle N_k\rangle \sqrt{\left(\frac{\partial z}{\partial s}\right)^2 + \left(\frac{\partial y}{\partial s}\right)^2}\, dS.\{T_k\}
$$

$$
+\sum_{ele}\langle \delta T_i\rangle \oint_{Sref} \{N_i\}\langle N_j\rangle\{\alpha_{rad}^j\}\langle N_k\rangle \sqrt{\left(\frac{\partial z}{\partial s}\right)^2 + \left(\frac{\partial y}{\partial s}\right)^2}.dS.\{T_g^k\}
$$

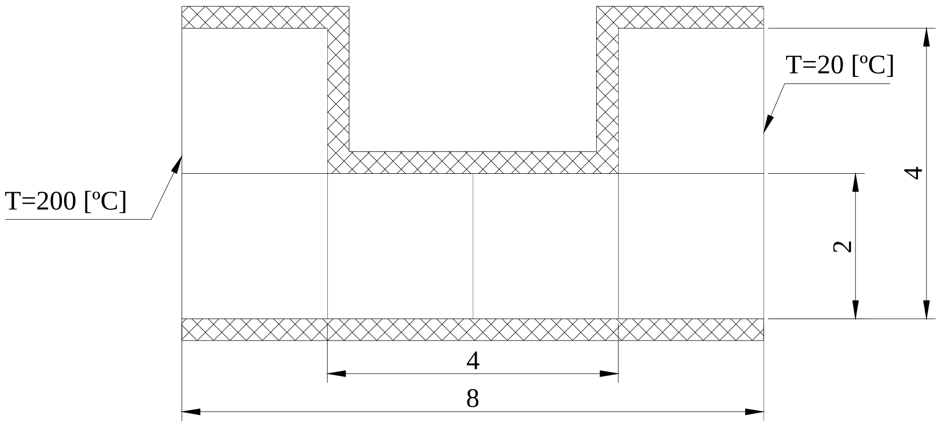
$$
-\sum_{ele}\langle \delta T_i\rangle \oint_{Sref} \{N_i\}\langle N_j\rangle\{\alpha_{rad}^j\}\langle N_k\rangle \sqrt{\left(\frac{\partial z}{\partial s}\right)^2 + \left(\frac{\partial y}{\partial s}\right)^2}\, dS.\{T_k\}
$$

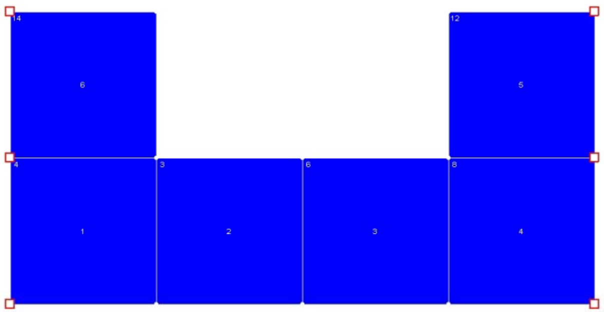
$$
= \sum_{ele}\langle \delta T_i\rangle \int_{Vref} \{N_i\}\rho\, C\langle N_j\rangle.\det[J].dV.\{\dot{T}_j\}
$$

$$
[RET]\{T_n\} + [SOL]\{U_p\} = [CAP][\dot{T}_n]
$$

THERMAL ANALYSIS – 2D – ED-Poiss

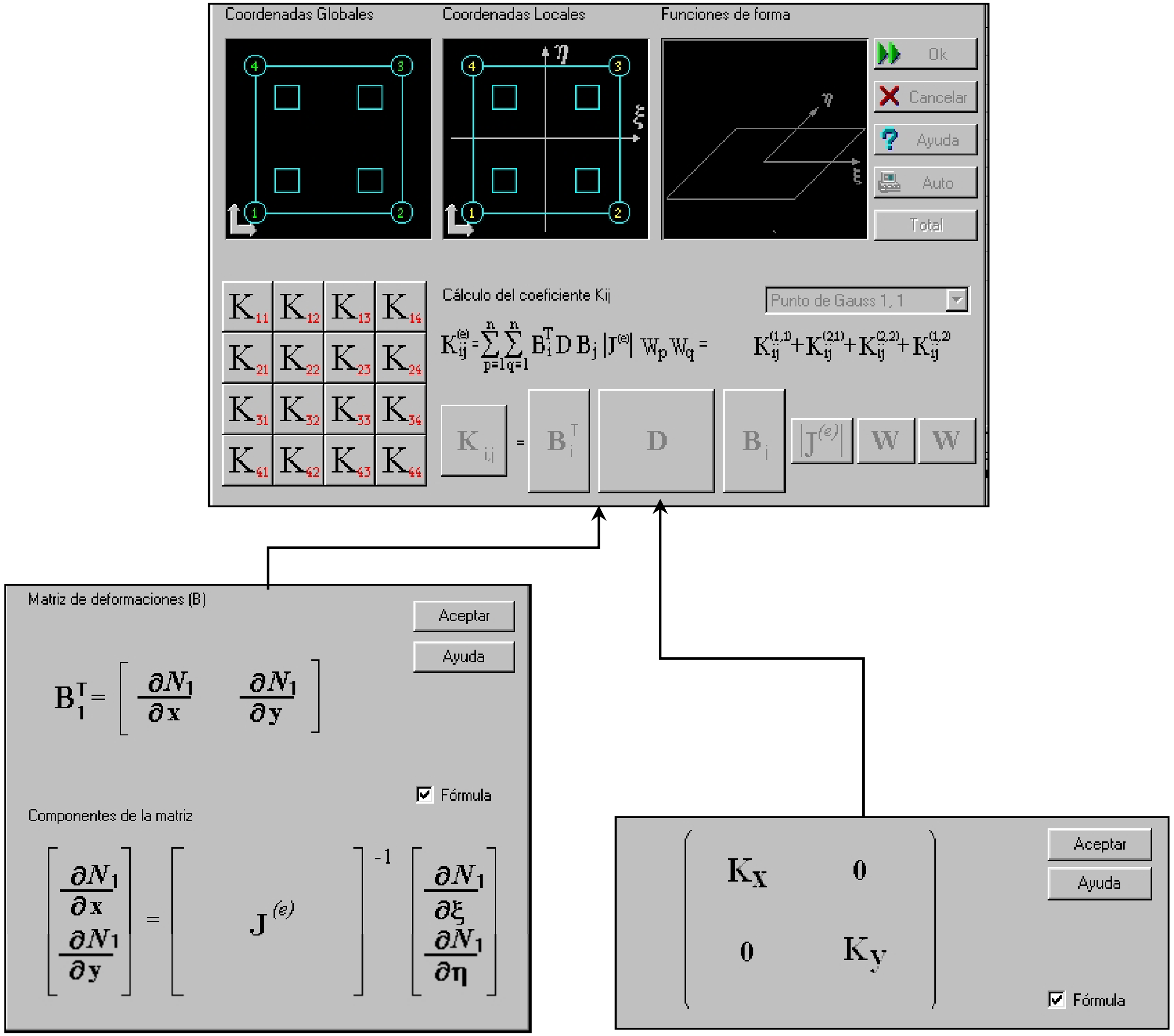
- Thermal analysis in two dimensions:
  - Imposed temperature for input and output.
  - Imposed null heat flux at the other boundary.
  - Using iso-parametric finite plane elements



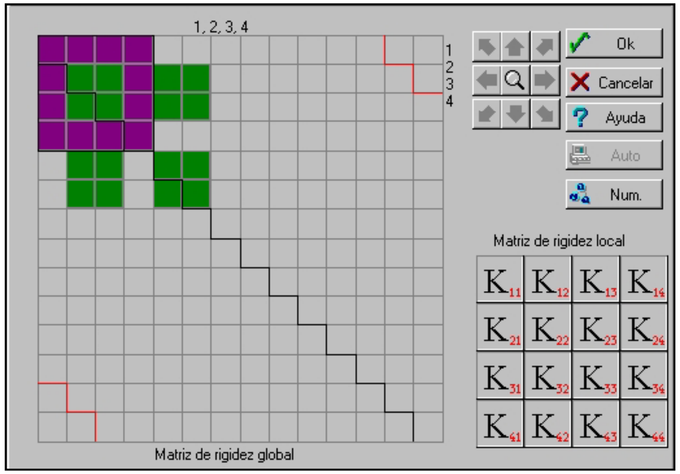


THERMAL ANALYSIS – 2D – ED-Poiss

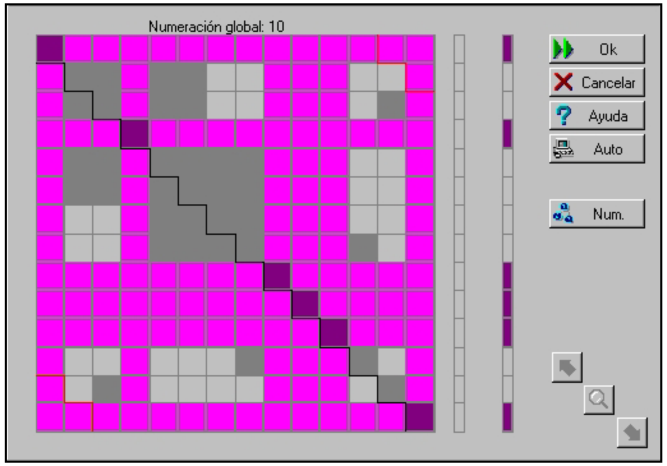
- Usign Educational ED Poisson



- Assembling matrix:
  - Two first elements:

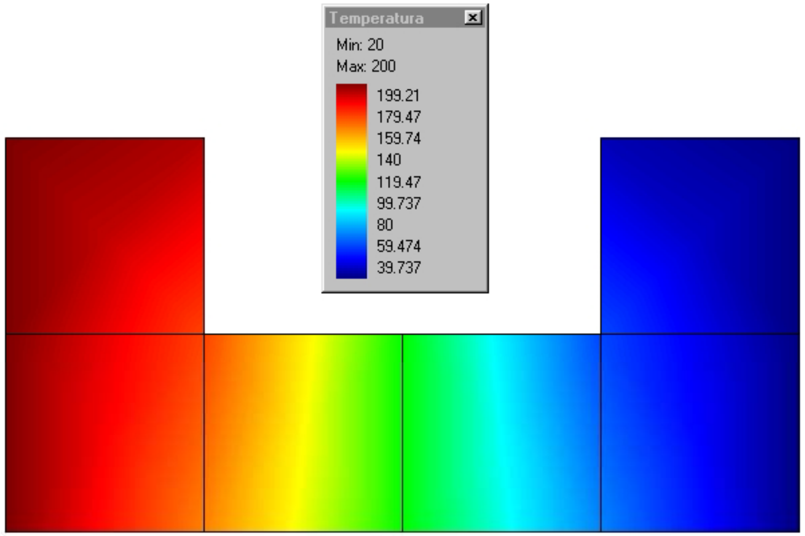


  - All elements with boundary conditions:

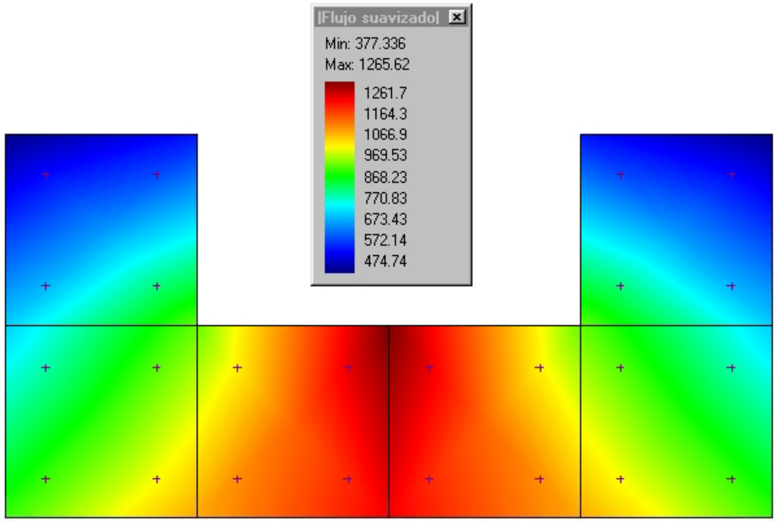


THERMAL ANALYSIS – 2D – ED-Poiss

- Post- Processing:
  - Heat

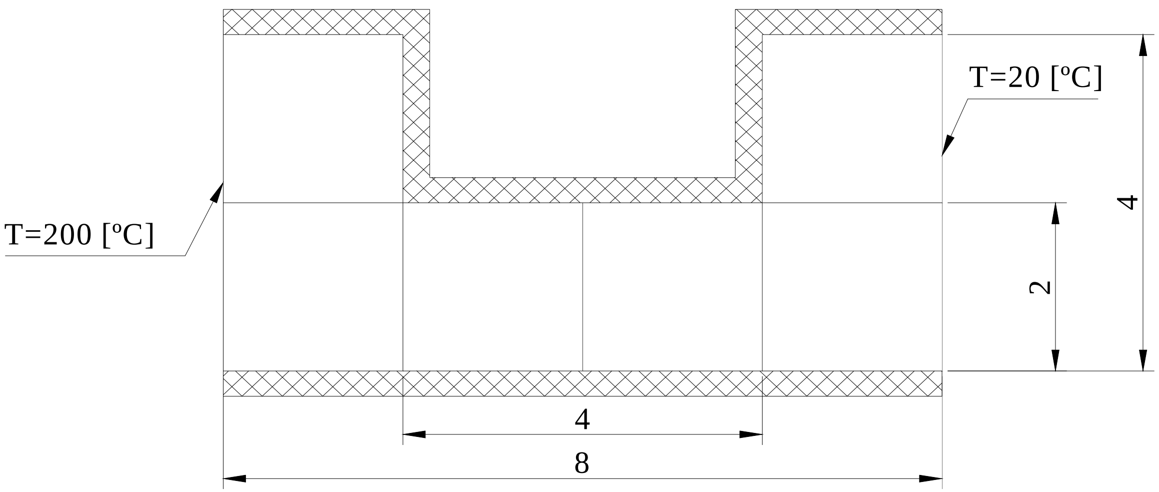


  - Flux



THERMAL ANALYSIS – 2D – ANSYS

- Thermal analysis:
  - Steady state conditions:
  - Thermal conductive material: Steel (Kxx=Kyy=45 [W/m2K])
  - Six finite elements: Plane55 (Isoparametric)
    - 4 nodes
    - 1 degree of freedom per node



THERMAL ANALYSIS – 2D – ANSYS

- Session file from ANSYS: Important to recover and manually change inputs.

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/BATCH
/COM,ANSYS RELEASE  7.0    UP20021010       22:30:20    10/09/2003
/input,menust,tmp,'',,,,,,,,,,,,,,,,1
/GRA,POWER
/GST,ON
/PLO,INFO,3
/GRO,CURL,ON
/REPLOT,RESIZE
!*
/NOPR
/PMETH,OFF,0
KEYW,PR_SET,1
KEYW,PR_STRUC,0
KEYW,PR_THERM,1
KEYW,PR_FLUID,0
KEYW,PR_ELMAG,0
KEYW,MAGNOD,0
KEYW,MAGEDG,0
KEYW,MAGHFE,0
KEYW,MAGELC,0
KEYW,PR_MULTI,0
KEYW,PR_CFD,0
/GO
!*
/COM,
/COM,Preferences for GUI filtering
/COM,  Thermal
!*
/PREP7
!*
!*
ET,1,PLANE55
!*
```

```
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MPTEMP,1,0
MPDATA,KXX,1,,45
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K,2,2,0,0,
K,3,4,0,0,
K,4,6,0,0,
K,5,8,0,0,
KGEN,2,P51X, , , ,2, ,5,0
/AUTO, 1
/REP
FLST,3,2,3,ORDE,2
FITEM,3,6
FITEM,3,-7
KGEN,2,P51X, , , ,2, ,5,0
FLST,3,2,3,ORDE,2
FITEM,3,9
FITEM,3,-10
KGEN,2,P51X, , , ,2, ,4,0
```

```
LSTR,      2,      7
LSTR,      7,     12
LSTR,      3,      8
LSTR,      4,      9
LSTR,      9,     13
LSTR,      5,     10
LSTR,     10,     14
LSTR,      1,      2
LSTR,      2,      3
LSTR,      3,      4
LSTR,      4,      5
LSTR,      6,      7
LSTR,      7,      8
LSTR,      8,      9
LSTR,      9,     10
LSTR,     13,     14
LSTR,     11,     12
FLST,2,4,4
FITEM,2,10
FITEM,2,3
FITEM,2,14
FITEM,2,1
AL,P51X
FLST,2,4,4
FITEM,2,11
FITEM,2,5
FITEM,2,15
FITEM,2,3
AL,P51X
....
```

```
LESIZE,ALL, , ,1, ,1, , ,1,
CM,_Y,AREA
ASEL, , , ,       1
CM,_Y1,AREA
CHKMSH,'AREA'
CMSEL,S,_Y
!*
MSHKEY,1
AMESH,_Y1
MSHKEY,0
!*
CMDELE,_Y
CMDELE,_Y1
CMDELE,_Y2
!*
/AUTO, 1
/REP
APLOT
FLST,5,5,5,ORDE,2
FITEM,5,2
FITEM,5,-6
CM,_Y,AREA
ASEL, , , ,P51X
CM,_Y1,AREA
CHKMSH,'AREA'
CMSEL,S,_Y
!*
MSHKEY,1
AMESH,_Y1
MSHKEY,0
!*
CMDELE,_Y
CMDELE,_Y1
CMDELE,_Y2
!*
/PNUM,KP,0
/PNUM,LINE,0
/PNUM,AREA,0
/PNUM,VOLU,0
/PNUM,NODE,1
/PNUM,TABN,0
/PNUM,SVAL,0
/NUMBER,0
....
```

```
/GO
D,P51X, ,200, , , ,TEMP, , , , ,
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FITEM,2,9
FITEM,2,-11
!*
/GO
D,P51X, ,20, , , ,TEMP, , , , ,
LSWRITE,1,
FINISH
```

```
/SOL
/STATUS,SOLU
SOLVE
FINISH
```

```
/POST1
/EFACE,1
!*
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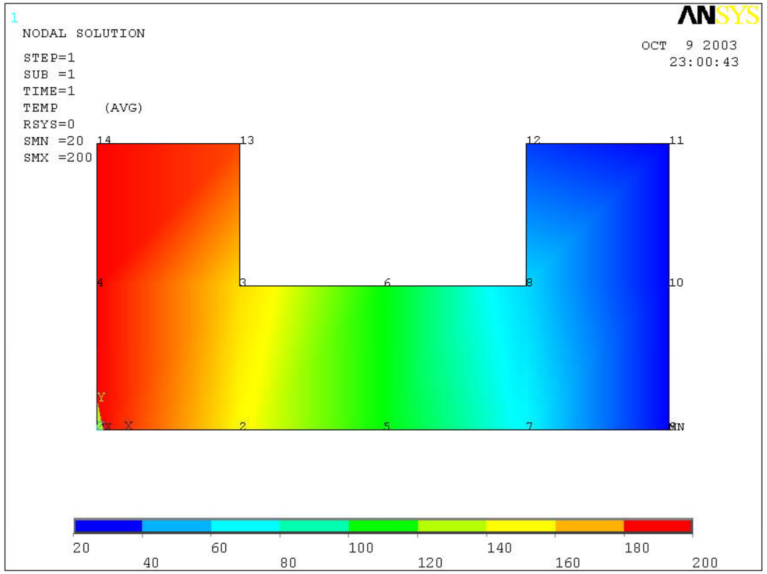
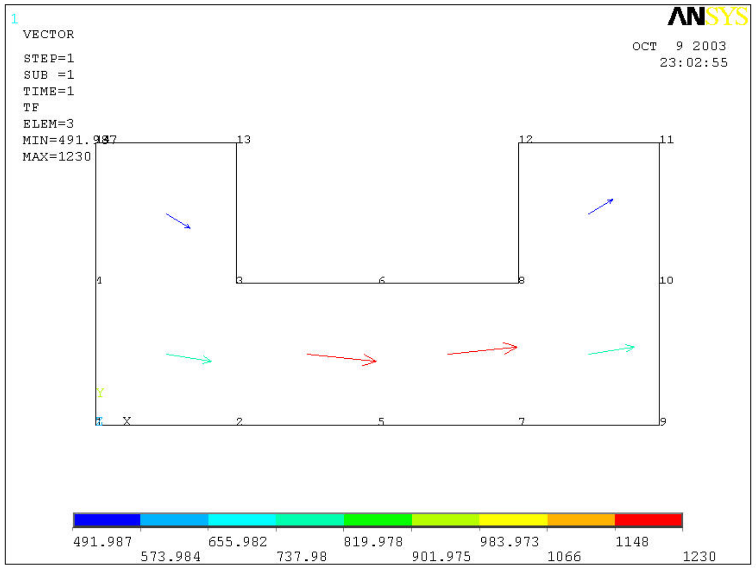
/EFACE,1
!*
PLNSOL,TF,SUM,0,
/EFACE,1
!*
PLNSOL,TG,SUM,0,
!*
/VSCALE,1,1,0
!
!*
PLVECT,TF, , , ,VECT,ELEM,ON,0
/EFACE,1
!*
PLNSOL,CONT,STAT,0,
/EFACE,1
!*
PLNSOL,TF,X,0,
/EFACE,1
!*
PLNSOL,TF,SUM,0,
/EFACE,1
```

THERMAL ANALYSIS – 2D – ANSYS

- Post- Processing:
  - Heat



  - Flux



INTERPOLATING FUNCTIONS 1D

- Linear functions (global coordinate system)
  - Determine temperature distribution in a one-dimensional fin, for the positions X=4 and X=8 [cm].
  - Assuming the mesh and temperature at nodes 2 and 3.
- Assume:
  - Linear behaviour.

$$T^e = c_1 + c_2 X$$

  - To know temperature at nodes:

$$T^e = \begin{bmatrix} N_i & N_j \end{bmatrix} \begin{Bmatrix} T_i \\ T_j \end{Bmatrix}$$

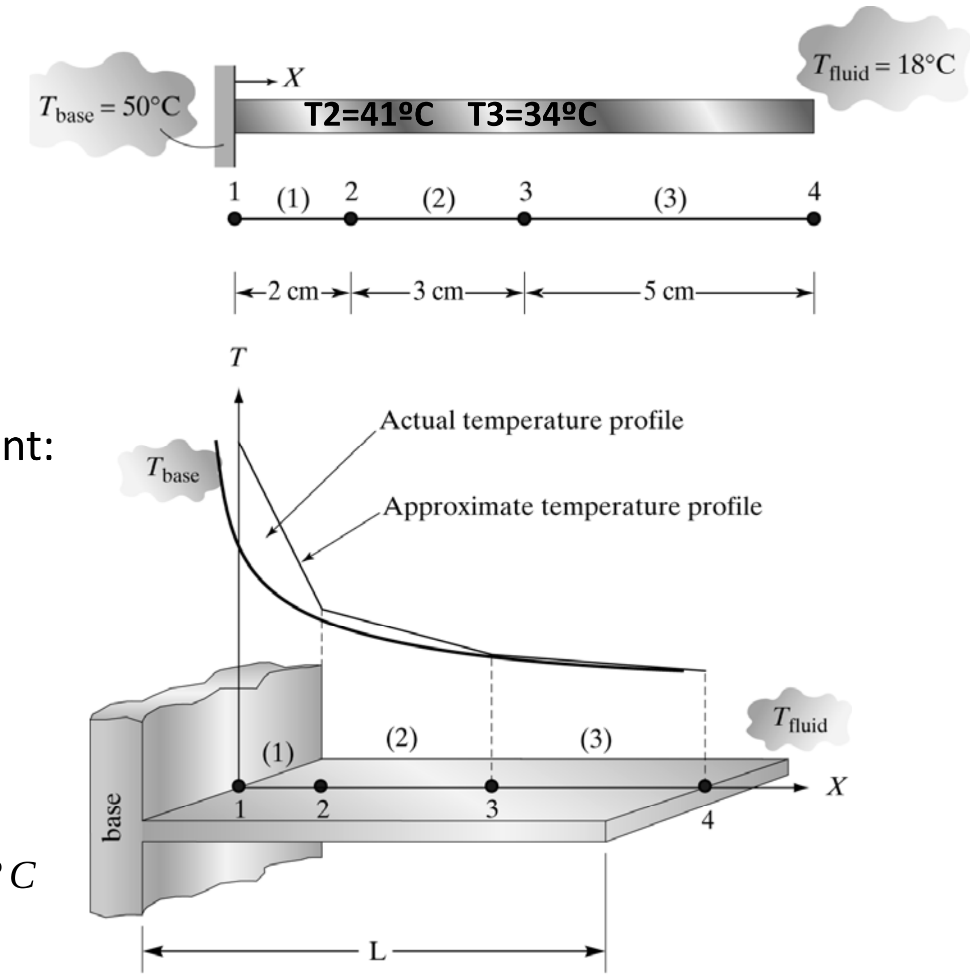
  - Temperature over each finite element:

$$T^e = \left(\frac{X_j - X}{X_j - X_i} \right) T_i + \left(\frac{X - X_i}{X_j - X_i} \right) T_j$$

  - Solution:

$$T^{X=4} = \left(\frac{5-4}{5-2} \right) 41 + \left(\frac{4-2}{5-2} \right) 34 = 36.3^{\circ} C$$

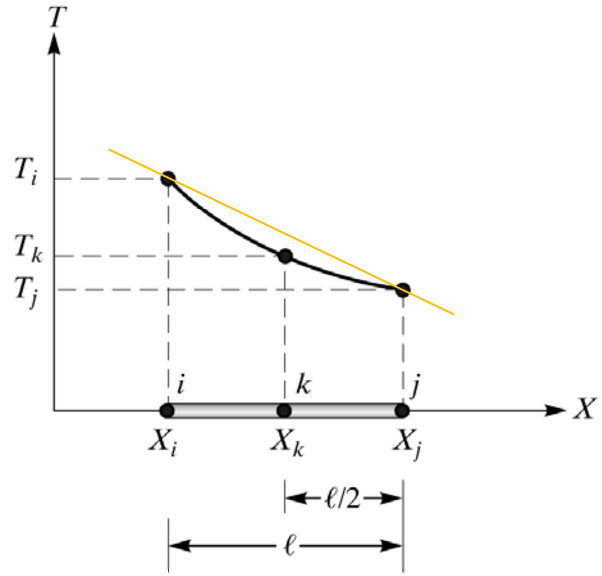
$$T^{X=8} = \left(\frac{10-8}{10-5} \right) 34 + \left(\frac{8-5}{10-5} \right) 18 = 24.4^{\circ} C$$



INTERPOLATING FUNCTIONS 1D

- Quadratic functions (global coordinate system)
  - Increase accuracy of solution.
- Assume :
  - Quadratic behaviour:

$$T^e = c_1 + c_2 X + c_3 X^2$$



  - To know temperature at nodes:

$$T = T_i \Rightarrow X = X_i \qquad T_i = c_1 + c_2 X_i + c_3 X_i^2$$

$$T = T_k \Rightarrow X = X_k \qquad T_k = c_1 + c_2 X_k + c_3 X_k^2$$

$$T = T_j \Rightarrow X = X_j \qquad T_j = c_1 + c_2 X_j + c_3 X_j^2$$

  - Temperature over each finite element:

$$T^e = N_i T_i + N_j T_j + N_k T_k$$

$$T^e = \begin{bmatrix} N_i & N_j & N_k \end{bmatrix} \begin{Bmatrix} T_i \\ T_j \\ T_k \end{Bmatrix}$$

$$N_i = \frac{2}{l^2}(X - X_j)(X - X_k)$$

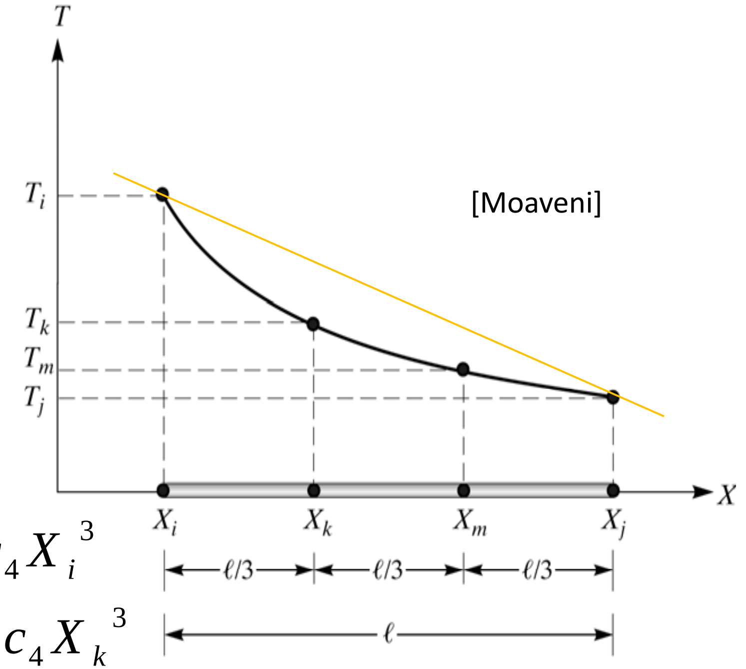
$$N_j = \frac{2}{l^2}(X - X_i)(X - X_k)$$

$$N_k = \frac{-4}{l^2}(X - X_i)(X - X_j)$$

INTERPOLATING FUNCTIONS 1D

- Cubic functions (global coordinate system)
  - Increase accuracy of solution.
- Assume :
  - Cubic behaviour:

$$T^e = c_1 + c_2 X + c_3 X^2 + c_4 X^3$$



  - To know temperature at nodes:

$$T = T_i \Rightarrow X = X_i \qquad T_i = c_1 + c_2 X_i + c_3 X_i^2 + c_4 X_i^3$$

$$T = T_k \Rightarrow X = X_k \qquad T_k = c_1 + c_2 X_k + c_3 X_k^2 + c_4 X_k^3$$

$$T = T_m \Rightarrow X = X_m \qquad T_m = c_1 + c_2 X_m + c_3 X_m^2 + c_4 X_m^3$$

$$T = T_j \Rightarrow X = X_j \qquad T_j = c_1 + c_2 X_j + c_3 X_j^2 + c_4 X_j^3$$

  - Temperature over each finite element:

$$T^e = N_i T_i + N_j T_j + N_k T_k + N_m T_m$$

$$T^e = \begin{bmatrix} N_i & N_j & N_k & N_m \end{bmatrix} \begin{Bmatrix} T_i \\ T_j \\ T_k \\ T_m \end{Bmatrix}$$

$$N_i = \frac{-9}{2l^3}(X - X_j)(X - X_k)(X - X_m)$$

$$N_j = \frac{9}{2l^3}(X - X_i)(X - X_k)(X - X_m)$$

$$N_k = \frac{27}{2l^3}(X - X_i)(X - X_j)(X - X_m)$$

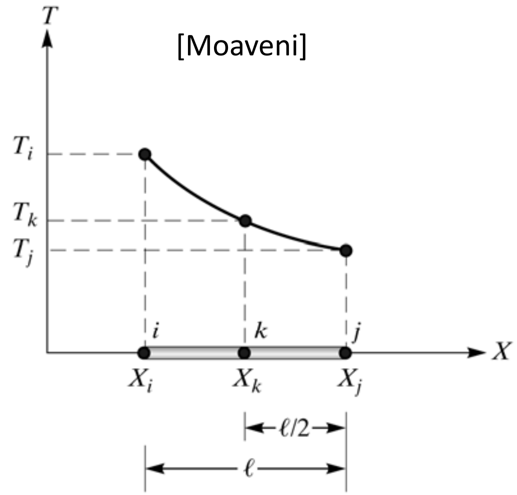
$$N_m = \frac{-27}{2l^3}(X - X_i)(X - X_j)(X - X_k)$$

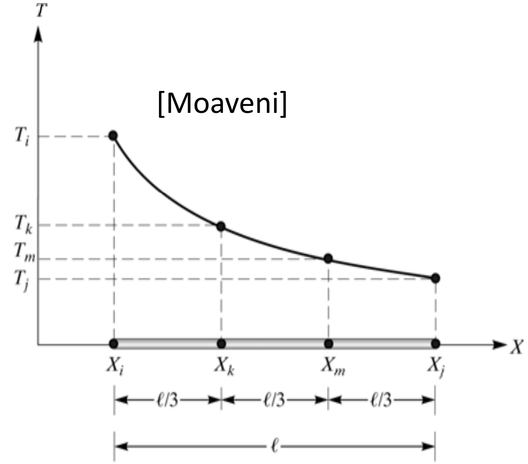
LAGRANGE INTERPOLATING FUNCTIONS 1D

- Lagrange interpolation functions (global coordinate system)
  - Advantage to increase accuracy of solution.
  - Advantage of no need to determine nodal parameters .
- Assume :
  - To generate shape functions of an (n-1) order polynomial.
    - If n=3, polynomial order=2.
    - If n=4, polynomial order=3.
  - Shape functions determined in terms of product of linear functions:

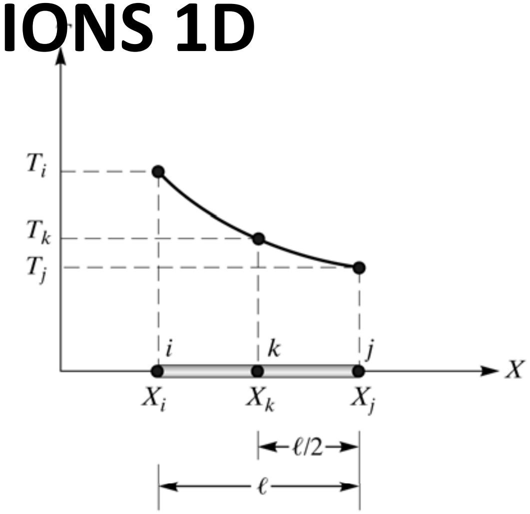
$$N_i(x) = \prod_{j=1 (j \neq i)}^{n} \frac{(x - x_j)}{(x_i - x_j)}$$

    - Interpolating function at node "i "takes value equal to 1 and 0 at other nodes.





LAGRANGE INTERPOLATING FUNCTIONS 1D



- Lagrange interpolation functions (global coordinate system):
  - Assume n=3;
  - polynomial order=2
- Node i

$$N_i = N_1 = \frac{(X - X_2)(X - X_3)}{(X_1 - X_2)(X_1 - X_3)} = \frac{(X - X_2)(X - X_3)}{\left(-\frac{l}{2}\right)(-l)} = \frac{2}{l^2}(X - X_2)(X - X_3)$$

- Node K

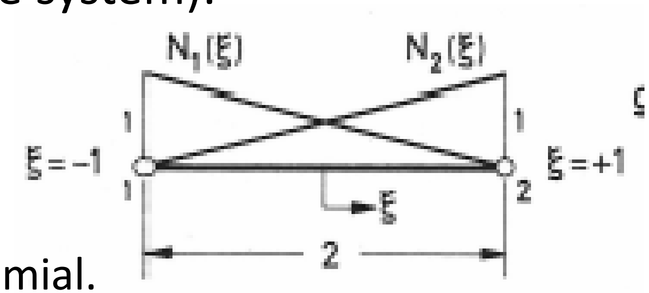
$$N_k = N_2 = \frac{(X - X_1)(X - X_3)}{(X_2 - X_1)(X_2 - X_3)} = \frac{(X - X_1)(X - X_3)}{\left(\frac{l}{2}\right)\left(-\frac{l}{2}\right)} = \frac{-4}{l^2}(X - X_1)(X - X_3)$$

- Node j

$$N_j = N_3 = \frac{(X - X_1)(X - X_2)}{(X_3 - X_1)(X_3 - X_2)} = \frac{(X - X_1)(X - X_2)}{(l)\left(\frac{l}{2}\right)} = \frac{2}{l^2}(X - X_1)(X - X_2)$$

LAGRANGE INTERPOLATING FUNCTIONS 1D

- Lagrange interpolation functions (natural coordinate system):
  - The same previous advantages.
  - One more advantage for numerical integration.

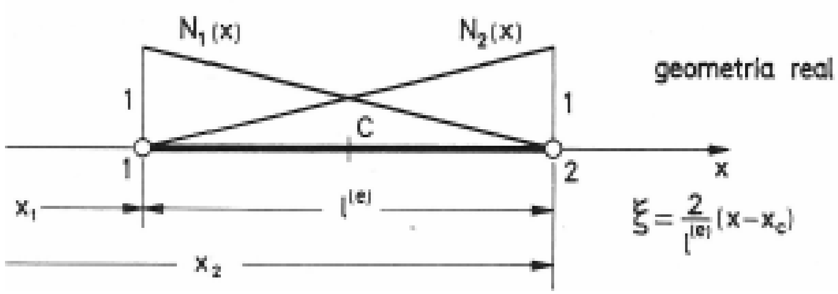


- Assume :
  - To generate shape functions of an (n-1) order polynomial.
    - If n=3, polynomial order=2.
    - If n=4, polynomial order=3.
  - Shape functions determined in terms of product of linear functions:

$$\xi = 2\,\frac{x - x_c}{L^{(e)}}$$

$$N_i(\xi) = \prod_{j=1 (j \neq i)}^{n} \frac{(\xi - \xi_j)}{(\xi_i - \xi_j)}$$

- Interpolating function at node “i “takes value equal to 1 and 0 at other nodes.
- Natural coordinates are local coordinates in a dimensionless form. “xc” represents the central element coordinate.
- Limits of integration from “-1” to “+1”.



LAGRANGE INTERPOLATING FUNCTIONS 1D

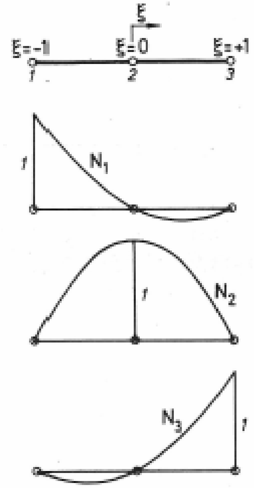
- In natural coordinates:
  - Interpolating functions for two nodes element.

$$u(\xi)=\langle N_1 \quad N_2 \rangle \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

$$N_1 = \frac{1}{2}(1-\xi)$$
$$N_2 = \frac{1}{2}(1+\xi)$$

  - Interpolating functions for three nodes element.

$$u(\xi)=\langle N_1 \quad N_2 \quad N_3 \rangle \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}$$

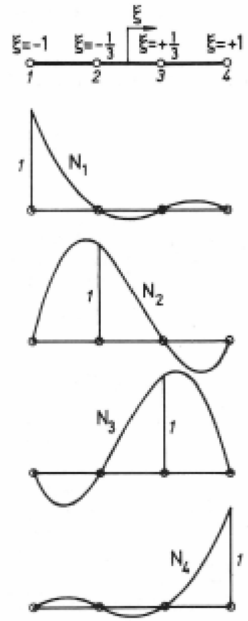


$$N_1 = \frac{1}{2}\xi(\xi-1)$$
$$N_2 = (1+\xi)(1-\xi)$$
$$N_3 = \frac{1}{2}\xi(1+\xi)$$

  - Interpolating funtions for four node element.

$$u(\xi)=\langle N_1 \quad N_2 \quad N_3 \quad N_4 \rangle \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix}$$

$$N_1 = -\frac{1}{16}+\frac{1}{16}\xi+\frac{9}{16}\xi^2-\frac{9}{16}\xi^3$$
$$N_2 = \frac{9}{16}-\frac{27}{16}\xi-\frac{9}{16}\xi^2+\frac{27}{16}\xi^3$$
$$N_3 = \frac{9}{16}+\frac{27}{16}\xi-\frac{9}{16}\xi^2-\frac{27}{16}\xi^3$$
$$N_4 = \frac{1}{16}-\frac{1}{16}\xi+\frac{9}{16}\xi^2+\frac{9}{16}\xi^3$$



ISOPARAMETRIC INTERPOLATION 1D

- For the case one dimensional finite element (bar)
  - Interpolating shape functions (geometry), are the same used for interpolating unknown function (displacement, temperature, etc.)
- Assume displacement field “u” as unknown:

$$u(\xi) = N_1(\xi)u_1 + N_2(\xi)u_2$$

- Strain displacement should be calculated with:

$$\varepsilon = \frac{du}{dx} = \frac{dN_1(\xi)}{dx}u_1 + \frac{dN_2(\xi)}{dx}u_2$$

- The space derivative of each interpolating function should be calculated according:

$$\frac{dN_1(\xi)}{dx} = \frac{dN_1(\xi)}{d\xi}\frac{d\xi}{dx} = \frac{-1}{2}\frac{d\xi}{dx}$$

$$\frac{dN_2(\xi)}{dx} = \frac{dN_2(\xi)}{d\xi}\frac{d\xi}{dx} = \frac{+1}{2}\frac{d\xi}{dx}$$

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ISOPARAMETRIC INTERPOLATION 1D

- To complete formulation:
  - Geometry should be approximated by the same funtions (isoparametric concept).

$$x(\xi) = N_1(\xi)\, x_1 + N_2(\xi)\, x_2$$

  - Differentiate between natural and global coordinates to obtain the Jacobian* determinant:

$$\frac{dx(\xi)}{d\xi} = \frac{dN_1(\xi)}{d\xi} x_1 + \frac{dN_2(\xi)}{d\xi} x_2 = \left\langle \frac{dN_1(\xi)}{d\xi} \quad \frac{dN_2(\xi)}{d\xi} \right\rangle \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix}$$

$$N_1 = \frac{1}{2}(1-\xi)$$

$$N_2 = \frac{1}{2}(1+\xi)$$

- Recall interpolating function derivatives and assuming Jacobian as a real number:

$$\frac{dN_i(\xi)}{dx} = \frac{dN_i(\xi)}{d\xi}\frac{d\xi}{dx} = \frac{dN_i(\xi)}{d\xi}\frac{1}{\frac{dx}{d\xi}} = \frac{1}{\frac{dx}{d\xi}}\frac{dN_i(\xi)}{d\xi} = \frac{1}{J}\frac{dN_i(\xi)}{d\xi}$$

- Strain approaches:

$$\varepsilon = \frac{du}{dx} = \frac{dN_1(\xi)}{dx}\frac{dx}{d\xi} u_1 + \frac{dN_2(\xi)}{dx}\frac{dx}{d\xi} u_2 = \frac{1}{\frac{dx}{d\xi}} \left\langle \frac{dN_1(\xi)}{d\xi} \quad \frac{dN_2(\xi)}{d\xi} \right\rangle \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

- Super-parametric interpolation:
  - degree for shape functions is higher than degree of unknown function.
- Sub-parametric interpolation:
  - degree for shape functions is smaller than degree of unknown function.

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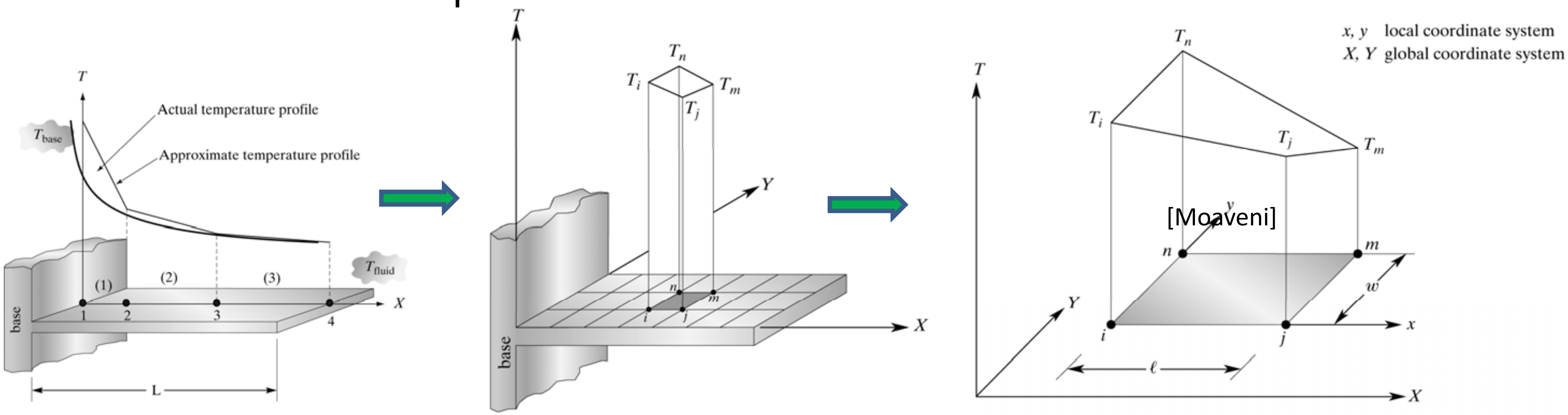
INTERPOLATING FUNCTIONS 2D

- Element with 1 DOF per node



- Consider a linear approximation over the rectangular finite element

$$T^e = b_1 + b_2 x + b_3 y + b_4 xy = \langle P_i \rangle \{ b_i \} \qquad \langle P \rangle = \langle 1 \quad x \quad y \quad x\,y \rangle$$

- Substitute the nodal parameters in to the assumed 1st order displacement field.
- The interpolation functions (Ni, Nj) will appear (in global coordinates).

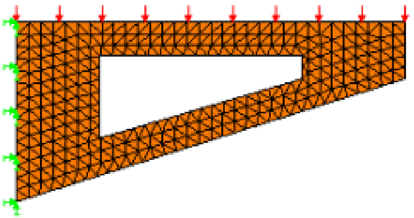
$$T^e = \begin{bmatrix} N_i & N_j & N_m & N_n \end{bmatrix} \begin{Bmatrix} T_i \\ T_j \\ T_m \\ T_n \end{Bmatrix}$$

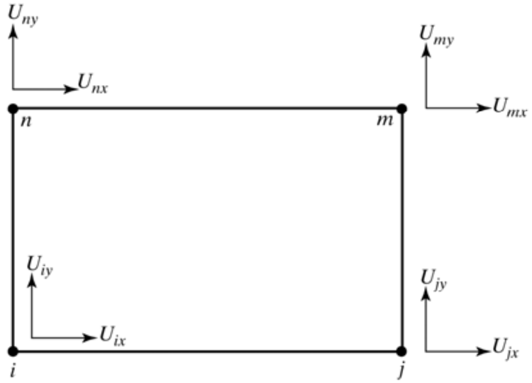
$$[N_i] = \left(1 - \frac{x}{l}\right)\left(1 - \frac{y}{w}\right) \quad [N_j] = \frac{x}{l}\left(1 - \frac{y}{w}\right)$$

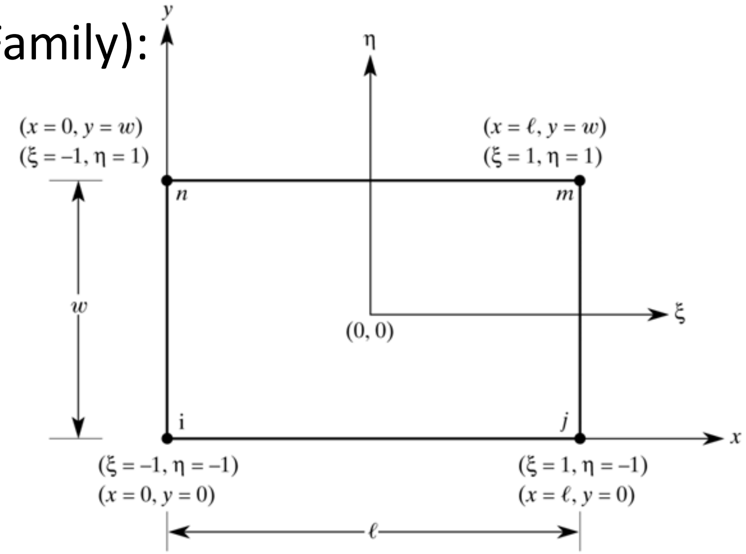
$$[N_m] = \frac{xy}{lw} \quad [N_n] = \frac{y}{w}\left(1 - \frac{x}{l}\right)$$

INTERPOLATING FUNCTIONS 2D

- Linear Element with 2 DOF per node (Lagrange Family):



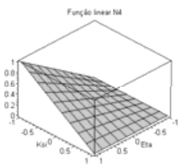




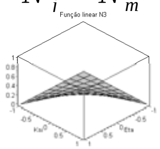
- Consider a linear approximation for x and y displacement (u,v) over the rectangular finite element.

$$u^e = \langle 1 \quad \xi \quad \eta \quad \xi\eta \rangle \langle b_1 \quad b_2 \quad b_3 \quad b_4 \rangle^t = \langle P_i \rangle \{ b_i \}$$

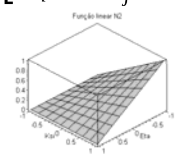
- Substitute the nodal parameters in to the assumed 1st order displacement field. The interpolation functions (Ni, Nj) will appear (in local coordinates).

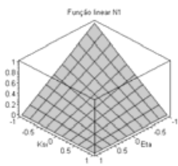


$$u^e = \begin{bmatrix} N_i & N_i & N_m & N_n \end{bmatrix} \begin{Bmatrix} U_{ix} \\ U_{jx} \\ U_{mx} \\ U_{nx} \end{Bmatrix}$$



$$v^e = \begin{bmatrix} N_i & N_j & N_m & N_n \end{bmatrix} \begin{Bmatrix} U_{iy} \\ U_{jy} \\ U_{my} \\ U_{ny} \end{Bmatrix}$$

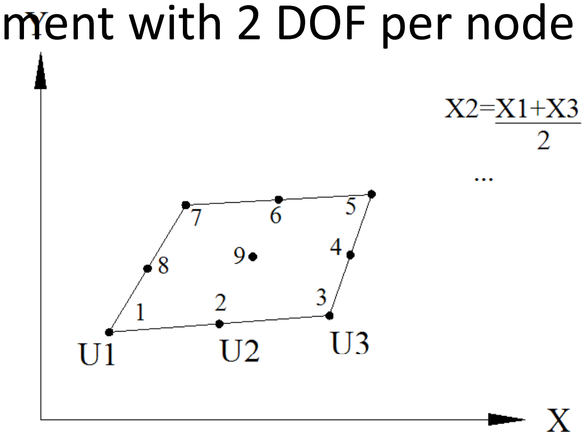


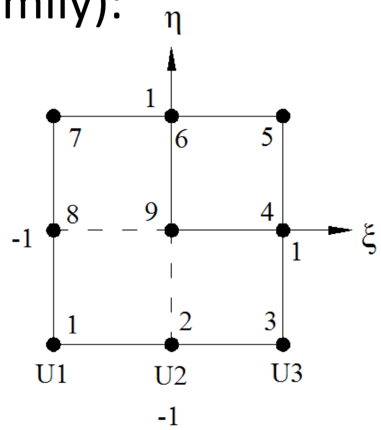


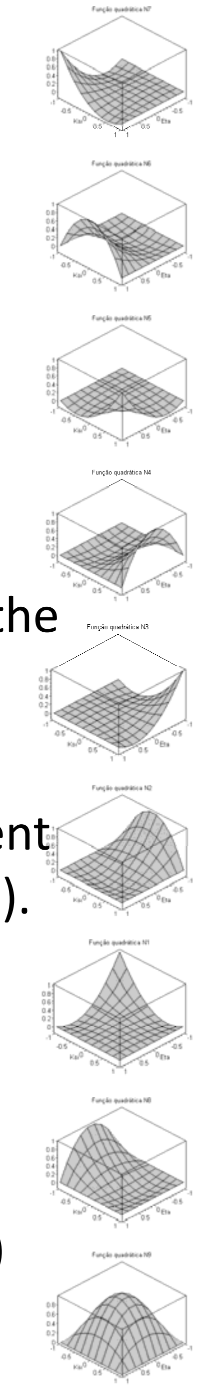
$$N_i = \frac{1}{4}(1-\xi)(1-\eta) \quad N_j = \frac{1}{4}(1+\xi)(1-\eta) \quad N_m = \frac{1}{4}(1+\xi)(1+\eta) \quad N_n = \frac{1}{4}(1-\xi)(1+\eta)$$

INTERPOLATING FUNCTIONS 2D

- Quadratic element with 2 DOF per node (Lagrange Family):







- Consider a quadratic approximation for x and y displacement (u,v) over the rectangular finite element.

$$u^e = \langle 1 \quad \xi \quad \eta \quad \xi^2 \quad \xi\eta \quad \eta^2 \quad \xi^2\eta \quad \xi\eta^2 \quad \xi^2\eta^2 \rangle \langle b_1 \quad b_2 \quad \dots \quad b_9 \rangle^t = \langle P_i \rangle \{b_i\}$$

- Substitute the nodal parameters in to the assumed 2nd order displacement field. The interpolation functions (Ni, Nj) will appear (in local coordinates).

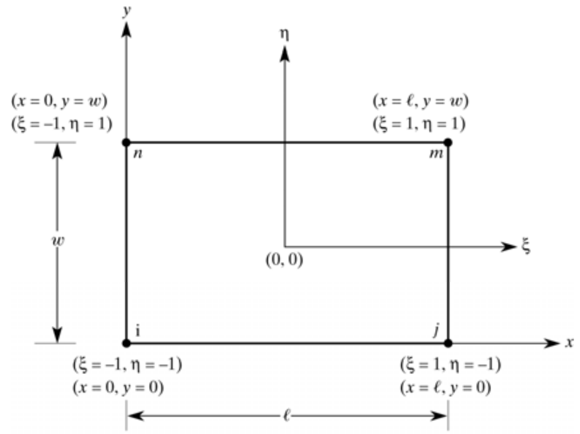
$$u^e = \langle N_i \rangle \{U_i\} \qquad u^e = \langle N_i \rangle \{V_i\}$$

$$N_1 = \frac{1}{4}\xi\eta(1-\xi)(1-\eta) \qquad N_2 = -\frac{1}{2}\eta(1-\xi^2)(1-\eta) \qquad N_3 = -\frac{1}{4}\xi\eta(1+\xi)(1-\eta) \qquad N_4 = \frac{1}{2}\xi(1+\xi)(1-\eta^2)$$

$$N_5 = \frac{1}{4}\xi\eta(1+\xi)(1+\eta) \qquad N_6 = \frac{1}{2}\eta(1-\xi^2)(1+\eta) \qquad N_7 = -\frac{1}{4}\xi\eta(1-\xi)(1+\eta) \qquad N_8 = -\frac{1}{2}\xi(1-\xi)(1-\eta^2)$$

$$N_9 = (1-\xi^2)(1-\eta^2)$$

FINITE PLANE ELEMENT



- Assume displacement field $\{u\} = \begin{Bmatrix} u \\ v \end{Bmatrix}$

- Perform strain calculation in plane:

$$\{\varepsilon\} = \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{bmatrix} \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \end{bmatrix} \begin{Bmatrix} u \\ v \end{Bmatrix} = [L]\{u\} = [L][N]\{u\} = [B]\{u\}$$

- Determine important matrices for stiffness matrice [K]: B=LN

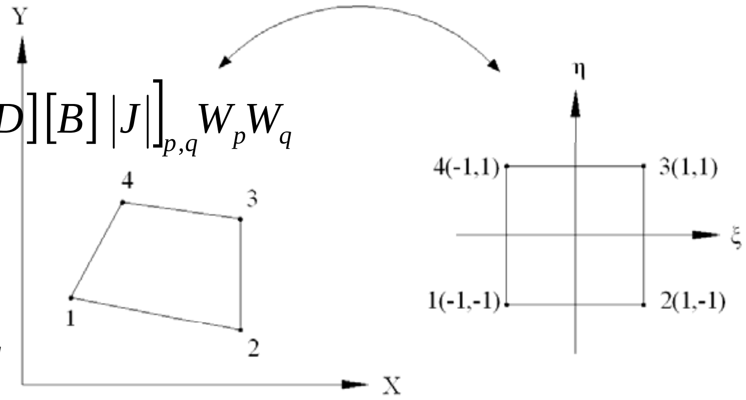
$$[B] = \begin{bmatrix} \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \end{bmatrix} \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 & N_n & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 & 0 & N_n \end{bmatrix}$$

$$[B] = \begin{bmatrix} \frac{\partial N_1}{\partial x} & 0 & \frac{\partial N_2}{\partial x} & 0 & \frac{\partial N_3}{\partial x} & 0 & \frac{\partial N_n}{\partial x} & 0 \\ 0 & \frac{\partial N_1}{\partial y} & 0 & \frac{\partial N_2}{\partial y} & 0 & \frac{\partial N_3}{\partial y} & 0 & \frac{\partial N_n}{\partial y} \\ \frac{\partial N_1}{\partial y} & \frac{\partial N_1}{\partial x} & \frac{\partial N_2}{\partial y} & \frac{\partial N_2}{\partial x} & \frac{\partial N_3}{\partial y} & \frac{\partial N_3}{\partial x} & \frac{\partial N_n}{\partial y} & \frac{\partial N_n}{\partial x} \end{bmatrix}$$

FINITE PLANE ELEMENT

- Stiffness matrice in natural coordinates:

$$K = \int_{\Omega} [B]^T [D][B]\, d\Omega = \int_{-1}^{+1}\int_{-1}^{+1} [B]^T [D][B] |J| d\xi d\eta = \sum_{p=1}^{n_p}\sum_{q=1}^{n_q} \left[[B]^T [D][B]\, |J| \right]_{p,q} W_p W_q$$



- Load vector in natural coordinates:

$$\{f\} = \int_{\Omega} [N]^T b\, d\Omega = \int_{-1}^{+1}\int_{-1}^{+1} [N]^T b |J| d\xi d\eta = \sum_{p=1}^{n_p}\sum_{q=1}^{n_q} \left([N]^T b |J| \right)_{p,q} W_p W_q$$

- Due to variables substitution, do not forget to use Jacobian

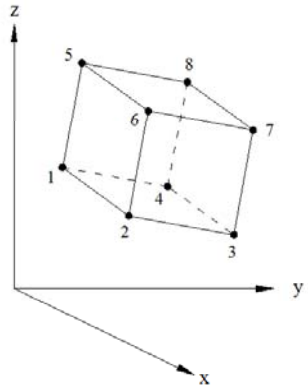
$$d\Omega = |J| d\xi d\eta$$

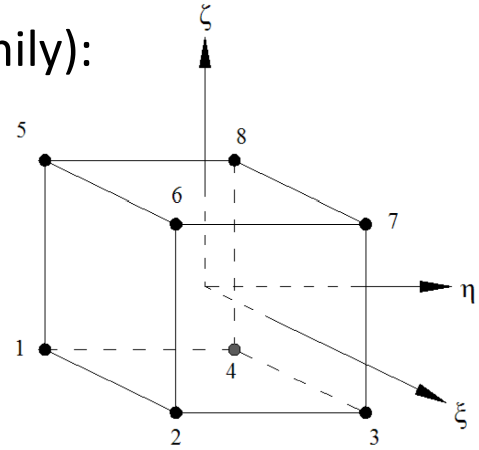
$$\boldsymbol{J}(\xi,\eta) = \begin{bmatrix} \dfrac{\partial x}{\partial \xi} & \dfrac{\partial y}{\partial \xi} \\ \dfrac{\partial x}{\partial \eta} & \dfrac{\partial y}{\partial \eta} \end{bmatrix}.$$

- Special attention is due to numerical integration over reference element (natural coordinates).
  - "Reduced" versus "Full" integration.
  - Full integration: Quadrature scheme sufficient to provide exact integrals of all terms of the stiffness matrix if the element is geometrically undistorted.
  - Reduced integration: An integration scheme of lower order than required by "full" integration.

INTERPOLATING FUNCTIONS 3D

- Linear Element with 8 nodes, hexahedron (Lagrange Family):





- Consider a linear approximation for each degree of freedom:

$$\langle P \rangle = \langle 1 \quad \xi \quad \eta \quad \zeta \quad \xi\eta \quad \eta\zeta \quad \zeta\xi \quad \xi\eta\zeta \rangle$$

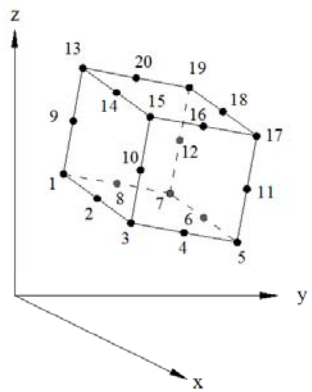
- Interpolating functions:

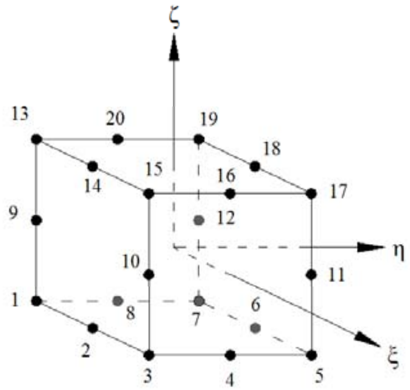
$$N_1 = \frac{1}{8}(1-\xi)(1-\eta)(1-\zeta) \qquad N_2 = \frac{1}{8}(1+\xi)(1-\eta)(1-\zeta) \qquad N_3 = \frac{1}{8}(1+\xi)(1+\eta)(1-\zeta) \qquad N_4 = \frac{1}{8}(1-\xi)(1+\eta)(1-\zeta)$$

$$N_5 = \frac{1}{8}(1-\xi)(1-\eta)(1+\zeta) \qquad N_6 = \frac{1}{8}(1+\xi)(1-\eta)(1+\zeta) \qquad N_7 = \frac{1}{8}(1+\xi)(1+\eta)(1+\zeta) \qquad N_8 = \frac{1}{8}(1-\xi)(1+\eta)(1+\zeta)$$

INTERPOLATING FUNCTIONS 3D

- Quadratic element, incomplete, with 20 nodes, hexahedron (Lagrange family):





- Consider a quadratic, incomplete approximation for each degree of freedom:

$$\langle P \rangle = \langle 1\ \xi\ \eta\ \zeta;\ \xi^2\ \xi\eta\ \eta^2\ \eta\zeta\ \zeta^2\ \xi\zeta;\ \xi^2\ \eta\ \xi\eta^2\ \eta^2\ \zeta\ \eta\zeta^2\ \xi\zeta^2\ \xi^2\ \zeta\ \xi\eta\zeta;\xi^2\ \eta\zeta\ \xi\eta^2\ \zeta\ \xi\eta\zeta^2 \rangle$$

- Interpolating functions:
  - For vertex nodes: $N_i(\xi,\eta,\zeta) = \frac{1}{8}(1+\xi\xi_i)(1+\eta\eta_i)(1+\zeta\zeta_i)(\xi\xi_i+\eta\eta_i+\zeta\zeta_i-2)$
    - Node numbers: 1,3,5,7,13,15,17,19
  - Every mid side node parallel to first natural coordinate: $N_i(\xi,\eta,\zeta) = \frac{1}{4}(1-\xi^2)(1+\eta_i\eta)(1+\zeta_i\zeta)$
    - Node numbers: 2,6,14,18
  - Every mid side node parallel to second natural coordinate: $N_i(\xi,\eta,\zeta) = \frac{1}{4}(1-\eta^2)(1+\zeta_i\zeta)(1+\xi_i\xi)$
    - Node numbers: 4,8,16,20
  - Every node parallel to third natural coordinate: $N_i(\xi,\eta,\zeta) = \frac{1}{4}(1-\zeta^2)(1+\eta_i\eta)(1+\xi_i\xi)$
    - Node numbers: 9,10,11,12

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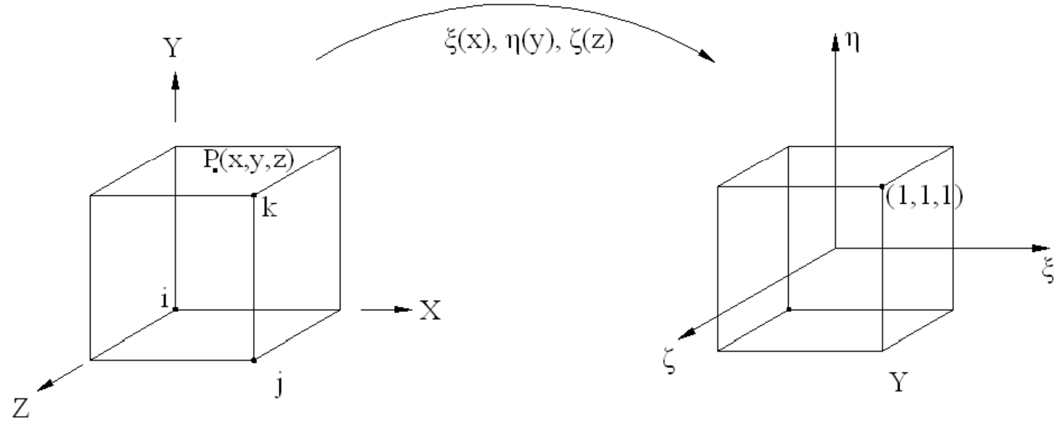
FINITE SOLID ELEMENT

- Numerical integration

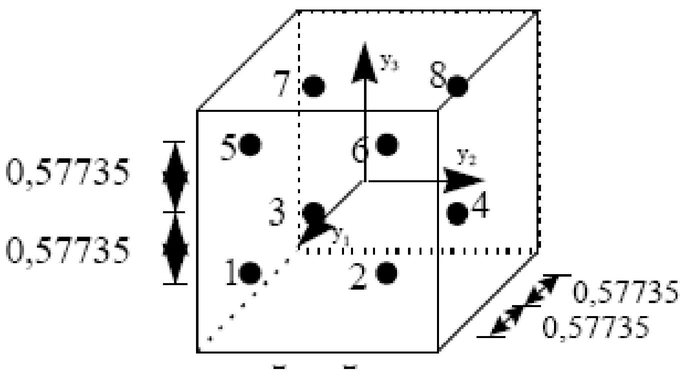
$$I=\int_{-1}^{+1}\int_{-1}^{+1}\int_{-1}^{+1} f(\xi,\eta,\zeta)\,d\xi\,d\eta\,d\zeta =$$

$$=\sum_{i=1}^{n_1}\sum_{j=1}^{n_2}\sum_{k=1}^{n_3} W_i W_j W_k\, f(\xi_i,\eta_j,\zeta_k)$$



$$\{f\}=\int_\Omega [N]^T b\, d\Omega=\int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1}[N]^T b|J|\,d\xi d\eta d\zeta=\sum_{p=1}^{n}\sum_{q=1}^{n}\sum_{r=1}^{n}\left([N]^T b|J|\right)_{p,q,r} W_p W_q W_r$$

$$K=\int_\Omega [B]^T[D][B]\, d\Omega=\int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1}[B]^T[D][B]|J|\,d\xi d\eta d\zeta=\sum_{p=1}^{n}\sum_{q=1}^{n}\sum_{r=1}^{n}\left[[B]^T[D][B]|J|\right]_{p,q,r} W_p W_q W_r$$



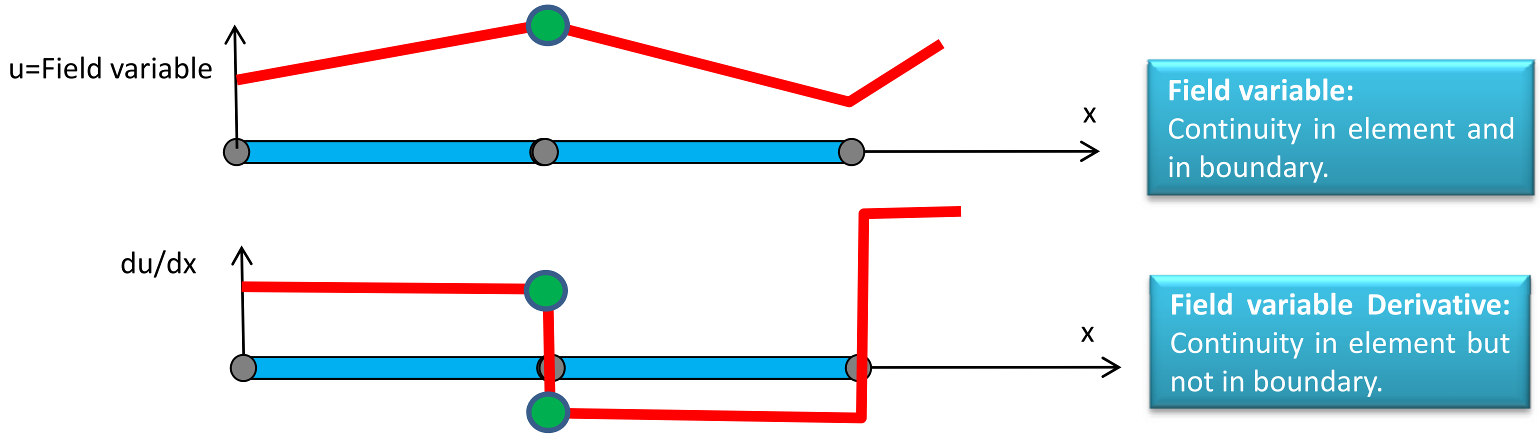
FINITE ELEMENT CONTINUITY

- Continuity is important from the physical point of view (prevent gaps in displacement, for example) and important from the mathematical point of view.
- The finite element solution will converge to the exact solution as the number of elements increases, provided that two conditions are satisfied:
  - Compatibility: Cn-1, Continuity exists at the element interface;
  - Completeness: Cn continuity of the field variable within element.
    - “n” is the highest order derivative that appears in the element interpolating functions.

FINITE ELEMENT CONTINUITY 1D

- For one dimension finite elements:

Element	Degree of polynomial	Continuity	Number of nodes	DOF
Lagrange	1	C_0	2	2
Lagrange	2	C_0	3	3
Lagrange	3	C_0	4	4
Lagrange	n-1	C_0	n	n
Hermite	3	C_1	2	4
Hermite	5	C_2	2	6
Lagrange - Hermiite	4	C_1	3	5



FINITE ELEMENT CONTINUITY 2D

- For two dimensions finite elements:

Element quadrilateral	Degree of polynomial	Continuity	Number of nodes	DOF
Lagrange	1	C_0	4	4
Lagrange	2	C_0	9	9
Lagrange	3	C_0	16	16
Lagrange incomplete	2	C_0	8	8
Lagrange incomplete	3	C_1	12	12
Hermite	3	Semi C_1	4	12
Hermiite (quadrilateral)	3	C_1	4	16

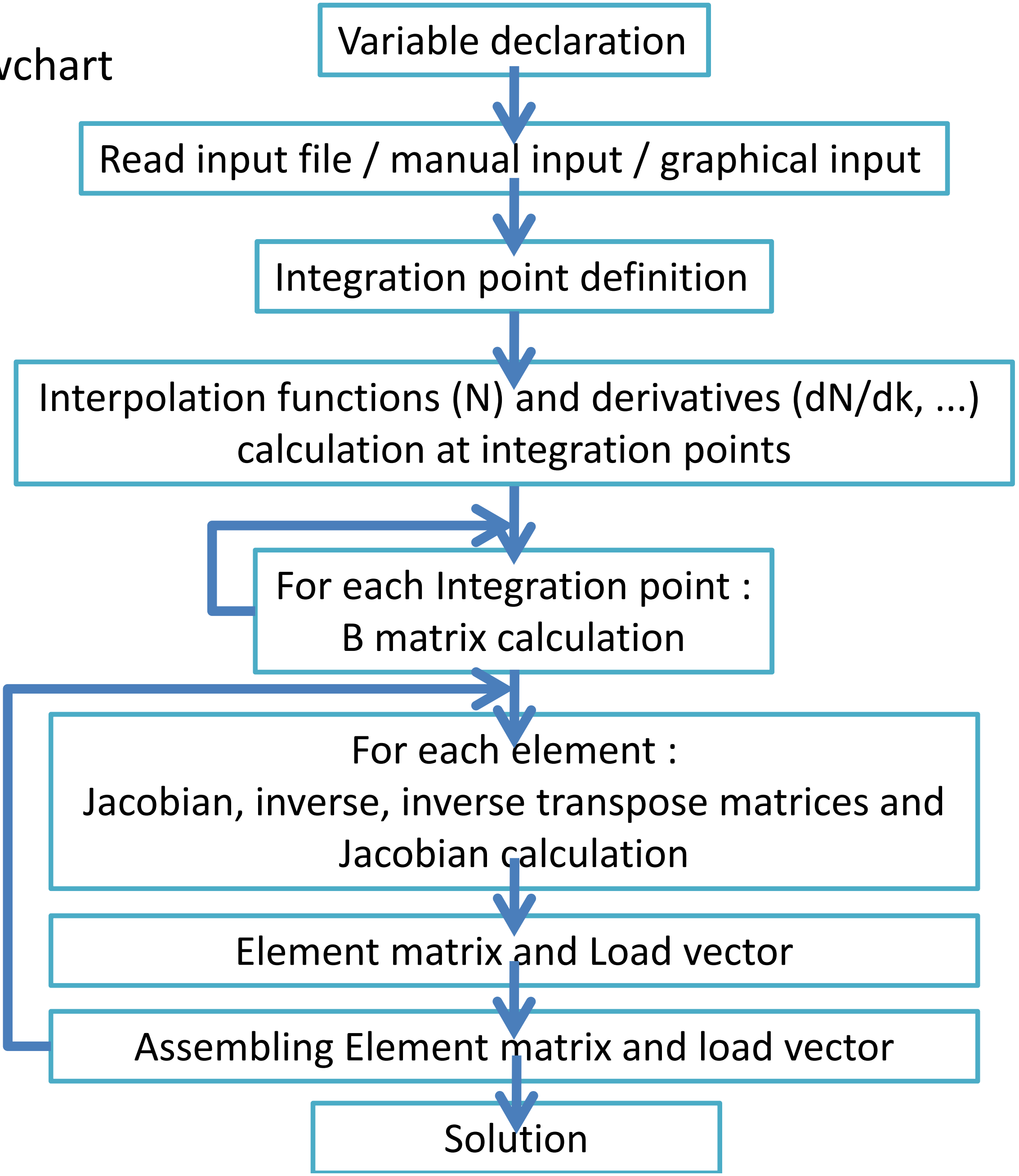
FINITE ELEMENT CONTINUITY 2D

- For three dimensions finite elements:

Element tetraedrum	Degree of polynomial	Continuity	Number of nodes	DOF
Lagrange	1	C_0	4	4
Lagrange	2	C_0	10	10
Lagrange	3	C_0	20	20
Element hexaedrum	Degree of polynomial	Continuity	Number of nodes	DOF
Lagrange	1	C_0	8	8
Lagrange	2	C_0	27	27
Hermiite	3	Semi C_1	8	32

FINITE ELEMENT GENERAL FLOWCHART

- Sample flowchart



FINITE ELEMENT (FORTRAN)

- Read input file / manual input / graphical input (Use the functionality of sub-routine). Use free or fixed format.
  - Read node coordinates, connectivity matrix
  - Input:
    - NDIM: problem dimension (ex:1,2,3)
    - NNEL: Number of nodes for each element
    - NR: logical unit for reading data
    - NP: Logical unit for printing
  - Output:
    - NNT: total number of nodes
    - NELT: Total number of elements
    - VCORG: nodal coordinates
    - KCONEC: connectivity table
  - Program Sample:

```
SUBROUTINE GRID(NDIM,NNEL,NR,MP,NNT,MELT,VCORG,KCONEC)
IMPLICITE REAL*8(A-H,O-Z)
DIMENSION VCORG(NDIM,1),KCONEC(NNEL,1)
READ(MR,*) NNT,NELT
WRITE(MP,*) NNT,NELT
DO 10 IN=1,NNT
        READ (MR,*) (VCORG(I,IN), I=1,NDIM)
10 WRITE (MP,*) IN,(VCORG(I,IN), I=1,NDIM)
DO 20 IE=1,NELT
        READ (MR,*) (KCONEC(I,IE), I=1,NNEL)
20 WRITE (MP,*) IE,(KCONE(I,IE), I=1,NNEL)
RETURN
END
```

FINITE ELEMENT (FORTRAN)

- Automatic calculation of interpolating functions and derivatives (Use free or fixed format).
  - Example for quadrilateral 2D, 4 node finite element.
  - Input:
    - VKPG: Coordinates of each integration point
    - IPG: Number of integration points
  - Output:
    - VNI: funtions N, dN/dksi and dN/deta
  - Program Sample:

```
SUBROUTINE NIQ(VKPG,IPG,VNI)
IMPLICITE REAL*8(A-H,O-Z)
DIMENSION VKPG(1), VNI(1)
•CYCLE FOR GAUSS POINTS:
•II=0
•IJ=0
•DO 10 IG=1,IPG
        •XG=VKPG(II+1)
        •YC=VKPG(IJ+2)
•* INTERPOLATIONG FUNCTIONS
        •VNI(II+1)=0.25*(1-XG)*(1-YG)
        •VNI(II+2)=0.25*(1+XG)*(1-YG)
        •VNI(II+3)=0.25*(1+XG)*(1+YG)
        •VNI(II+4)=0.25*(1-XG)*(1+YG)
•* INTERPOLATIONG FUNCTION DERIVATIVES TO KSI:
        •VNI(II+5)=-0.25*(1-YG)
        •VNI(II+6)=0.25*(1-YG)
        •VNI(II+7)=0.25*(1+YG)
        •VNI(II+8)=-0.25*(1+YG)
•* INTERPOLATIONG FUNCTION DERIVATIVES TO ETA:
        •VNI(II+9)=-0.25*(1-XG)
        •VNI(II+10)=-0.25*(1+YG)
        •VNI(II+11)=0.25*(1+XG)
        •VNI(II+12)=-0.25*(1-XG)
•II=II+12
•IJ=IJ+2
•10 CONTINUE
RETURN
END
```

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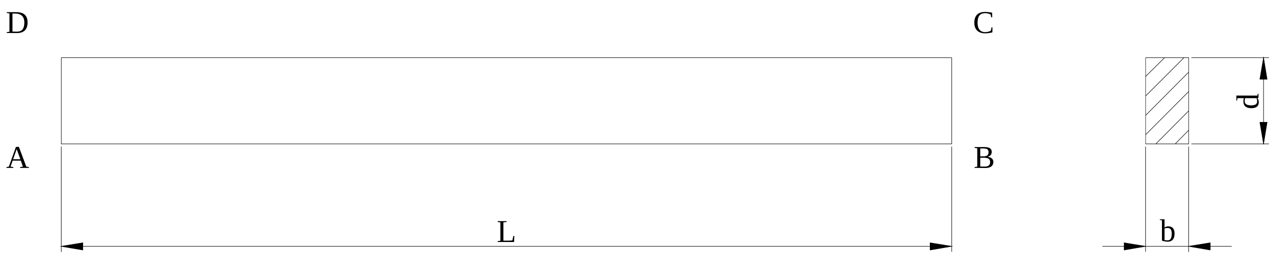
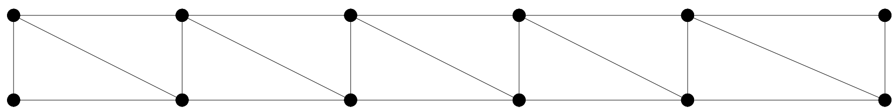
2D - ELASTIC PLANE STRESS ANALYSIS - CASE 5

- Consider a beam subjected to shear load, defined by its medium plane.
  - Consider material with elastic modulus E=200 [GPa] and Poisson coefficient 0.3.
  - Consider defined geometry.
    - L=1.0 [m], b=0.025 [m], d=0.1 [m].
  - Consider boundary conditions:
    - Left: fixed in x and y directions.
    - Right: Vertical load at point B, 1000 [N].
  - Test both finite triangular and quadrangular elements for:
    - Vertical displacement at point B;
    - Shear stress at mid span, cross section path;
    - Longitudinal stress at top fibre, line DC.





2D - ELASTIC PLANE STRESS ANALYSIS - CASE 5

- Theoretical results:
  - Vertical displacement due to bending:
    - P represents load and I represents 2nd order moment of area.

$$w|_{bending}(x = L) = \frac{PL^3}{3EI}$$

  - Vertical displacement due to shear:
    - For slender beams this contribution may be disregarded.
    - For stocky beams this contribution is important:
      - G represents the elastic shear modulus;
      - F is the shear section factor, in this case equals 1,2.

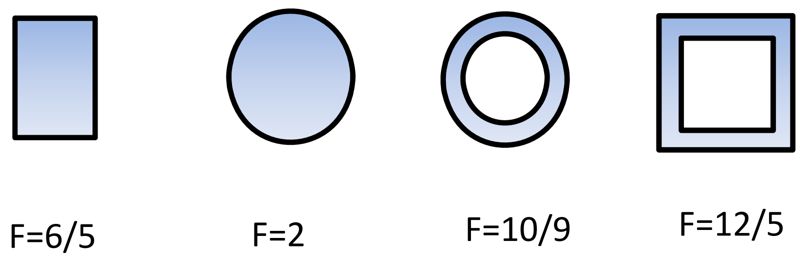
$$w|_{shear} = F\frac{PL}{AG}$$

  - Bending stress:

$$\sigma = \frac{\pm My}{I}$$

  - Shear Stress:
    - Q represents the 1^{st} order moment of area, regarding sub-section level where stress is calculated.

$$\tau = \frac{PQ}{Ib}$$



SIMPLE FINITE ELEMENT ANALYSIS

- Linear and Static Analysis
  - the most common and the most simplified analysis of structures is based on assumptions:
- Static:
  - Loading is so slow that dynamic effects can be neglected
- Linear :
  - Material obeys Hooke's law.
  - External forces are conservative.
  - Supports remain unchanged during loading
  - Deformations are so small that change of the structure configuration is negligible.
- Consequences:
  - displacements and stresses are proportional to loads, principle of superposition holds.
  - A set of linear algebraic equations for computation of displacements is used.

$$[K]\{d\} = \{F\}$$

ADVANCED FINITE ELEMENT ANALYSIS

- Sources of structural nonlinearities can be classified as:
  - **Material:** material behaves nonlinearly and linear Hooke's law cannot be used. More sophisticated material models should be then used instead .
    - Nonlinear elastic (Mooney-Rivlin's model for materials like rubber)
    - elastoplastic (Huber-von Mises for metals, Drucker-Prager model to simulate the behaviour of granular soil materials such as sand and gravel).
    - Etc.
  - **Geometry:** changes of the structure shape (or configuration changes) cannot be neglected and its deformed configuration should be considered.
  - **Boundary nonlinearities** - displacement dependent boundary conditions.
    - The most frequent boundary nonlinearities are encountered in contact problems.
- Consequences:
  - Instead of set of linear algebraic equations, a set of nonlinear algebraic equations are achieved.

$$[R]\{d\} = \{F\}$$

  - The principle of superposition cannot be applied.
  - The results of several load cases cannot be combined. Only one load case can be handled at a time.
  - Results of the nonlinear analysis cannot be scaled.

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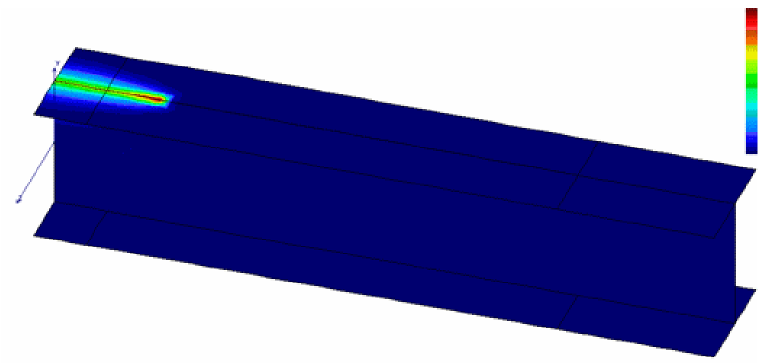
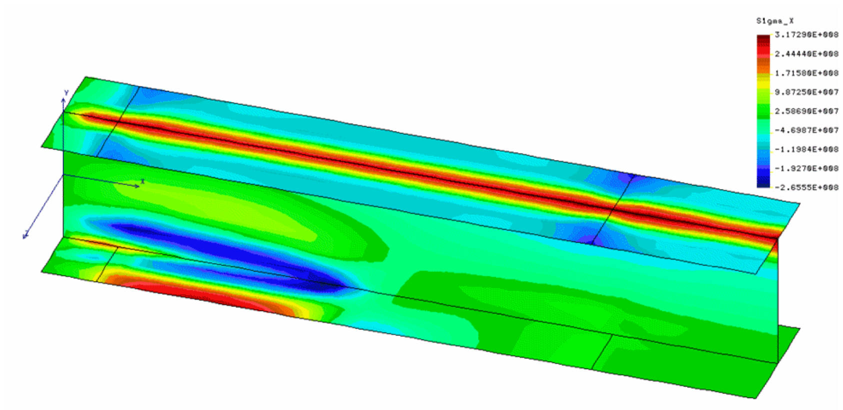
ADVANCED FINITE ELEMENT ANALYSIS

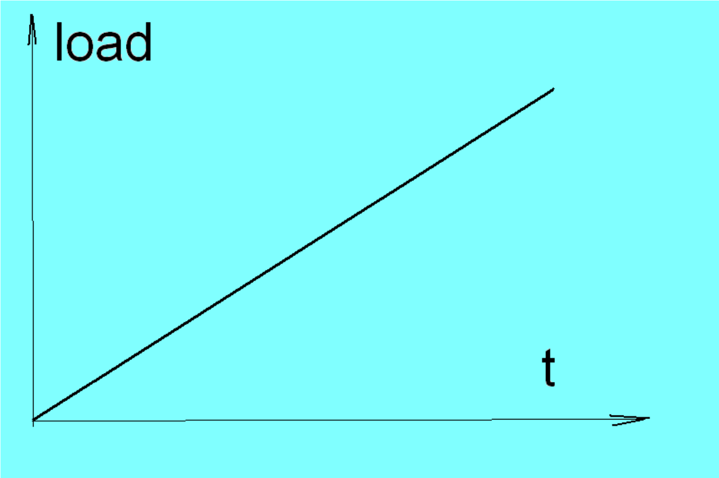
- Consequences (continued):
  - The loading history may be important, especially, plastic deformations depend on a manner of loading. This is a reason for dividing loads into small increments in nonlinear FE analysis
  - The structural behaviour can be markedly non-proportional to the applied load. The initial state of stress (residual stresses from heat treatment or manufacturing welding etc.) may be important.



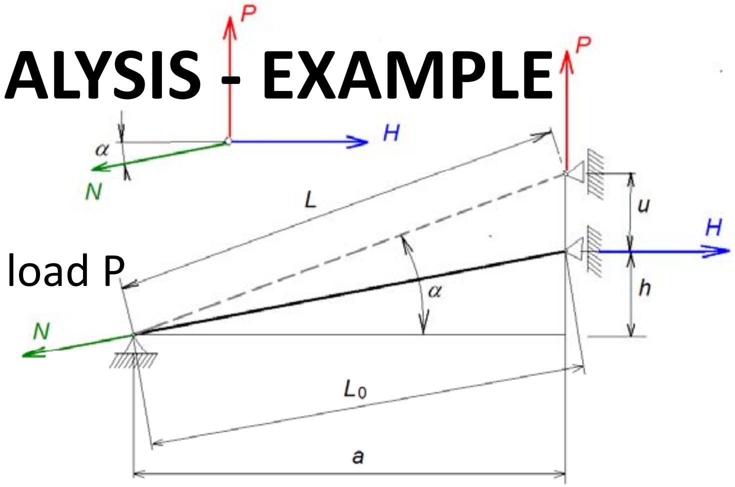


ADVANCED FINITE ELEMENT ANALYSIS

- To reflect loading history :
  - Loads are associated with “pseudo time curves”.
  - the “time” variable represents a “pseudo time”, which denotes the intensity of the applied loads at certain step.
- For nonlinear dynamic analysis and nonlinear static analysis with time-dependent material properties:
  - the “time” represents the real time associated with the loads’ application.



ADVANCED FINITE ELEMENT ANALYSIS - EXAMPLE



- Geometrically nonlinear finite element analysis:
  - Example : linearly elastic truss subjected to vertical load P
    - Undeformed configuration:
      - vertical position of right extremity = h
    - Deformed configuration:
      - vertical position of right extremity = h+u
  - Static equilibrium:
    - (1) Nodal equilibrium in vertical direction: $$\sum F_{i,y} = 0 \quad \Leftrightarrow \quad N \sin\alpha - P = 0$$
    - (2) Equilibrium at deformed configuration: $$\sin\alpha = \frac{h+u}{L}$$
  - From (1) and (2): $$N\,\frac{h+u}{L} - P = 0$$
  - N should be considered as axial effort; $$N = E\ A_0\ \varepsilon$$
  - Being the cross section of truss defined by: A_0
  - Strain (1^{st} order approximation), engineering strain) may de determined: $$\varepsilon = \left(L - L_0\right)/L_0$$
  - Initial and current length of the truss are: $$L_0 = \sqrt{a^2 + h^2} \qquad L = \sqrt{a^2 + (h+u)^2}$$

ADVANCED FINITE ELEMENT ANALYSIS - EXAMPLE

- New measure of strain (Green strain tensor):
  - it is convenient to introduce this definition (2nd order approximation).

$$\varepsilon_G = \frac{L^2 - L_0^2}{2L_0^2} \qquad \varepsilon_G = \frac{a^2 + \left(h^2 + 2hu + u^2\right) - \left(a^2 + h^2\right)}{2L_0^2} = \frac{h}{L_0}\frac{u}{L_0} + \frac{1}{2}\left(\frac{u}{L_0}\right)^2$$

- The 1st order may be related with 2nd order of approximation:

$$\varepsilon_G = \varepsilon + \frac{1}{2}\varepsilon^2$$

- The stress-strain relation may be written if the following format:

$$\sigma = E\,\frac{\varepsilon}{\varepsilon_G}\,\varepsilon_G = E\,\frac{\varepsilon}{\varepsilon + \frac{1}{2}\varepsilon^2}\,\varepsilon_G = \frac{E}{1 + \frac{1}{2}\varepsilon}\,\varepsilon_G = E^*\,\varepsilon_G$$

- Where E* is the new modulus of elasticity:
  - not constant and depends on strain.

$$E^* = \frac{E}{1 + \frac{1}{2}\varepsilon}$$

ADVANCED FINITE ELEMENT ANALYSIS - EXAMPLE

- The new modulus of elasticity E*:
  - If strain is small (e.g. less than 2%) differences are negligible, see table.

$\Delta L / L_0$	ε	ε_G	E (MPa)	E^*(MPa)	$E\,\varepsilon$ (MPa)	$E^*\varepsilon_G$(MPa)
0,0000	0,0000	0,000000	21 000	21 000	0	0
0,0050	0,0050	0,005013	21 000	20 948	1050	1050
0,0100	0,0100	0,010050	21 000	20 896	210	210
0,0150	0,0150	0,015113	21 000	20 844	315	315
0,0200	0,0200	0,020200	21 000	20 792	420	420

- Assuming that strain is small, AXIAL effort equals to:

$$N = E\,A_0\,\varepsilon = E^*\,A_0\,\varepsilon_G \approx E\,A_0\varepsilon_G$$

- After substituting into previous equation, the condition of equilibrium appears to demonstrate a non linear relation between load *P* and displacement *u* .

$$N\frac{h+u}{L} - P = 0 \quad \Leftrightarrow \quad \frac{EA_0}{2L_0^3}\left[u^3 + 3hu^2 + 2h^2u\right] = P$$

ADVANCED FINITE ELEMENT ANALYSIS

- Generally, using FEM a set of nonlinear algebraic equations for unknown nodal displacements may be determined.

$$[R]\{d\} = \{F\}$$

- Assuming **infinitesimal increments** for internal and external forces:
  - Incremental displacement dd.
  - Incremental force dF.

$$[R]\{d + dd\} = \{F + dF\}$$

- Assuming the first order approximation for the internal energy.
  - The concept of tangent stiffness matrix become important.

$$[R]\{d + dd\} = [R]\{d\} + \frac{\partial [R]}{\partial d} dd$$

$$= [R]\{d\} + [K_T] dd$$

- A new relation between incremental displacement and incremental force:

$$[K_T]\{dd\} = \{dF\}$$

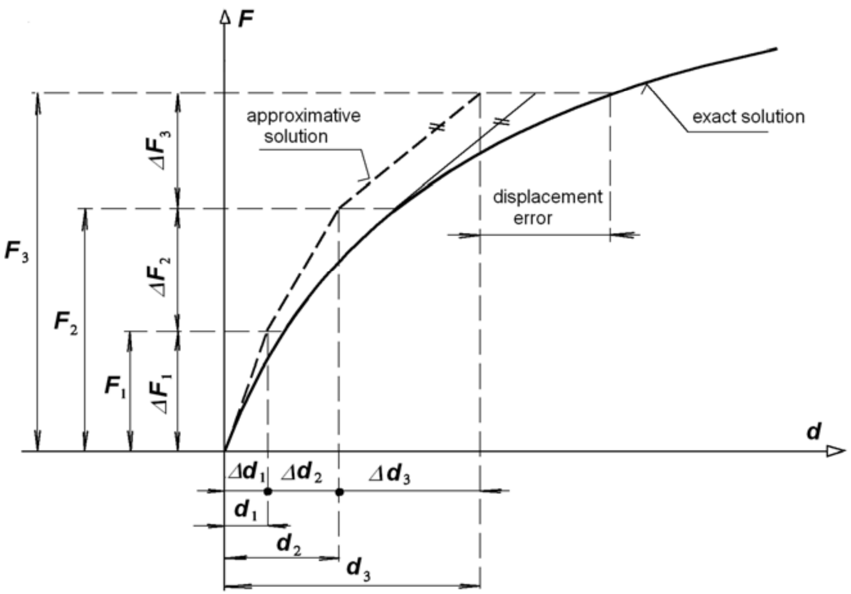
ADVANCED FINITE ELEMENT ANALYSIS

- Incremental method
  - The load is divided into a set of small increments $\Delta \boldsymbol{F}_{i}$.
  - Increments of displacements are calculated from the set of linear simultaneous equations :

$$[K_T]_{i-1}\{\Delta d\}_i = \{\Delta F\}_i$$

  - where $\boldsymbol{K}_{T(i-1)}$ is the tangent stiffness matrix computed form displacements $\boldsymbol{d}_{(i-1)}$ obtained in previous incremental step.
  - Nodal displacements after incremental load of $\Delta \boldsymbol{F}_i$ may be computed from:

$$\{d\}_i = \{\Delta d\}_{i-1}\{\Delta d\}_i$$



ADVANCED FINITE ELEMENT ANALYSIS

- Iterative method (Newton-Raphson)
  - Consider that $\boldsymbol{d}_i$ is an estimation of nodal displacement. As it is only an estimation, the condition of equilibrium would not be satisfied:

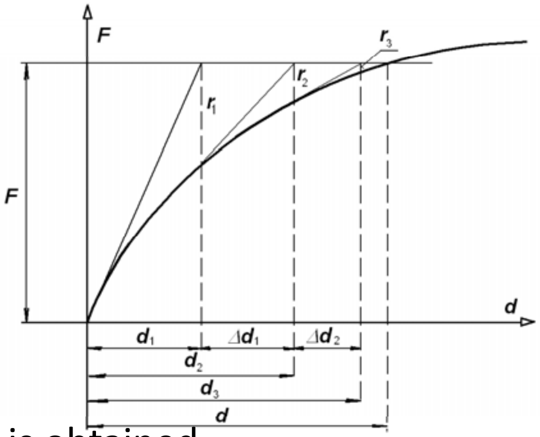
$$[R]\{d\}_i \neq \{F\}$$

  - This means that condition of equilibrium of internal and external nodal forces are not satisfied and there are unbalanced forces at nodes (residuals).

$$\{r\}_i = [R]\{d\}_i - \{F\}$$

  - Correction of nodal displacements can be then obtained from the set of linear algebraic equations:

$$[K_T]_i\{\Delta d\}_i = \{r\}_i$$



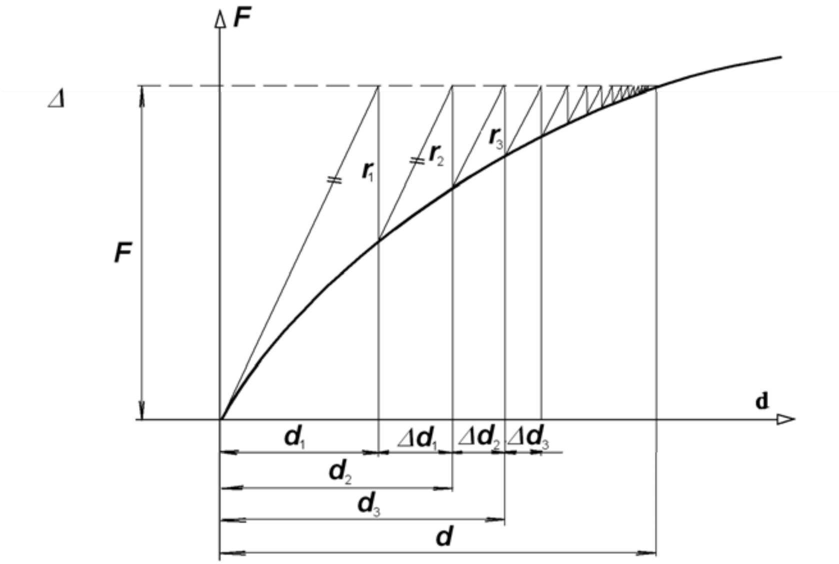
  - and new, corrected estimation of nodal displacements is:

$$\{d\}_i^{n+1} = \{d\}_i^{n}\{\Delta d\}_i^{n}$$

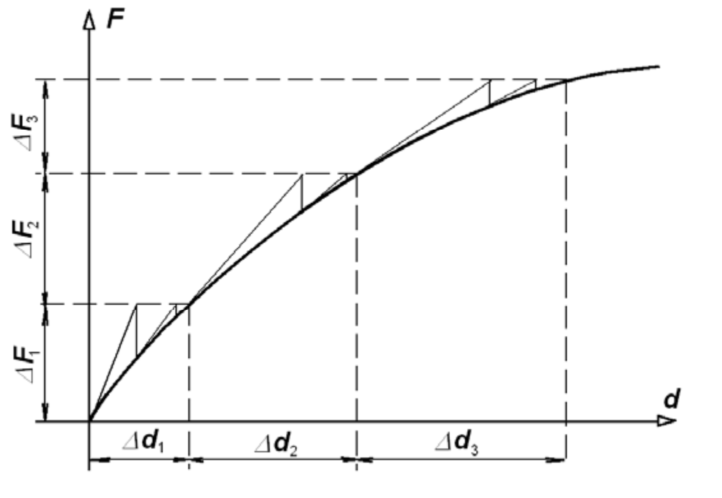
    - The iterative procedure (n) is repeated until accurate solution is obtained.
    - The first estimation is obtained from linear analysis.

ADVANCED FINITE ELEMENT ANALYSIS

- Iterative method (Modified Newton-Raphson)



- Combination of Newton-Raphson and incremental methods



ADVANCED FINITE ELEMENT ANALYSIS

- Material nonlinearities: Nonlinear elasticity models
  - For any nonlinear elastic material model, it is possible to define relation between stress and strain increments as :

$$\{\Delta\sigma\} = [D_T]\{\Delta\varepsilon\}$$

  - Matrix $\boldsymbol{D}_T$ is function of strain field $\boldsymbol{\varepsilon}$. Consequently, a set of equilibrium equations we receive in FEM is nonlinear and must be solved by use of any method described previously.
- Material nonlinearities: Elastoplastic material models
  - The total strains are decomposed into elastic and plastic parts

$$\{\varepsilon\} = \{\varepsilon^e\} + \{\varepsilon^p\}$$

  - The yield criterion defines whether plastic deformation will occur.
  - The plastic behaviour of a material after achieving plastic deformations is defined by so-called *flow rule* in which, the rate and the direction of plastic strains is related to the stress state and the stress rate. This relation can be expressed as:

$$\{d\varepsilon^p\} = d\lambda \frac{\partial Q}{\partial \sigma}$$

ADVANCED FINITE ELEMENT ANALYSIS

- Material nonlinearities: Elastoplastic material models
  - Constitutive equation can be formulated as:

$$\{\Delta\sigma\} = [D_T]\{\Delta\varepsilon\}$$

  - The tangential material matrix $\boldsymbol{D}_T$ is used to form a tangential stiffness matrix $\boldsymbol{K}_T$.
  - When the tangential stiffness matrix is defined, the displacement increment is obtained for a known load increment

$$[K_T]_{i-1}\{\Delta d\}_i = \{\Delta F\}_i$$

  - As load and displacement increments are final, not infinitesimal, displacements obtained by solution of this set of linear algebraic equation will be an approximate solution. That means, conditions of equilibrium of internal and external nodal forces will not be satisfied and iterative process is necessary.
  - The solution problem - not only equilibrium equations but also constitutive equations of material must be satisfied. This means that within the each equilibrium iteration step check of stress state and iterations to find elastic and plastic part of strains at every integration point must be included.
  - The iteration process continues until both, equilibrium conditions and constitutive equations are satisfied simultaneously.
  - The converged solution at the end of load increment is then used at the start of new load increment.

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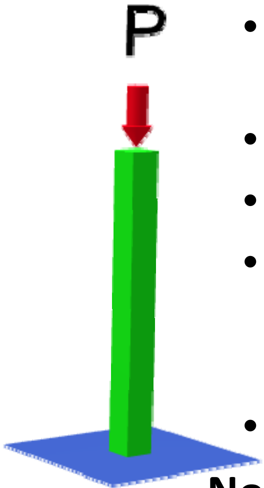
ADVANCED FINITE ELEMENT ANALYSIS (BUCKLING)

- Buckling loads represent critical loads where certain types of structures become unstable. Each load has an associated buckled mode shape; this is the shape that the structure assumes in a buckled condition. There are two primary means to perform a buckling analysis:
  - **Eigen-value**:



    - Eigen-value buckling analysis predicts the theoretical buckling strength of an ideal elastic structure.
    - It computes the structural eigen-values for the given system loading and constraints.
    - This is known as classical Euler buckling analysis.
    - However, in real-life, structural imperfections and nonlinearities prevent most real-world structures from reaching their eigen-value predicted buckling strength; ie. it over-predicts the expected buckling loads.
    - This method is not recommended for accurate, real-world buckling prediction analysis.
  - **Nonlinear:**
    - Nonlinear buckling analysis is more accurate than eigen-value analysis because it employs non-linear, large-deflection, static analysis to predict buckling loads.
    - Mode of calculation: gradually increases the applied load until a load level is found whereby the structure becomes unstable. Suddenly a very small increase in the load will cause very large deflections).
    - The real non-linear nature of this analysis involves the modeling of geometric imperfections or load perturbations and material nonlinearities.

2D – BUCKLING ANALYSIS - CASE 6

- Member Design – Columns (Eurocode):
  - Design concerned with compression members (eg pin-ended struts) subject to :
    - axial compression only ;
    - no bending.
  - In practice real columns are subject to:
    - eccentricities of axial loads;
    - transverse forces.
  - stocky columns:
    - very low slenderness are unaffected by overall buckling.
    - The compressive strength of stocky columns is dictated by the cross-section, being a function of the section classification.
    - Class 1,2,3: cross-sections are unaffected by local buckling:
      - design compression resistance Nc.Rd equals the plastic resistance Npl.Rd
      - $N_{c.Rd} = Af_y / \gamma_{M0}$
    - Class 4: local buckling prevents the attainment of the squash load.
      - design compression resistance limited to local buckling resistance,
      - $N_{c.Rd} = N_{o.Rd} = A_{eff} f_y / \gamma_{M1}$
      - A_{eff} is the area of the effective cross-section
  - slender columns:
    - Presents a quasi elastic buckling behaviour.

2D – BUCKLING ANALYSIS - CASE 6

- Member Design – Columns (Eurocode):
  - A compression member should be verified against buckling as follows:

$$\frac{N_{Ed}}{N_{b,Rd}} \leq 1,0$$

    - NEd is the design value of the compression force;
    - Nb,Rd is the design buckling resistance of the compression member.
  - Design buckling resistance of the compression member (Nb,Rd)
    - Class 1,2,3 cross sections:

$$N_{b,Rd} = \frac{\chi A f_y}{\gamma_{M1}} \qquad \bar{\lambda} = \left[\frac{A f_y}{N_{cr}}\right]^{0,5}$$

    - Class 4 cross section:

$$N_{b,Rd} = \frac{\chi A_{eff} f_y}{\gamma_{M1}} \qquad \bar{\lambda} = \left[\frac{A_{eff} f_y}{N_{cr}}\right]^{0,5}$$

$$\chi = \frac{1}{\phi + [\phi^2 - \bar{\lambda}^2]^{0,5}} \leq 1$$

$$\phi = 0,5[1 + \alpha(\bar{\lambda} - 0,2) + \bar{\lambda}^2]$$

    - Dimensional slenderness: $\bar{\lambda}$
    - Imperfection factor: α
    - Ncr: is the elastic critical force for the relevant buckling mode.

Table 6.1: Imperfection factors for buckling curves

Buckling curve	a_0	a	b	c	d
Imperfection factor α	0,13	0,21	0,34	0,49	0,76

2D – BUCKLING ANALYSIS - CASE 6

- Member Design – Columns (elastic theory):
  - Euler critical load
    - Equilibrium equation:

$$EI\frac{d^2y}{dx^2} = M(x)$$

    - Bending moment:

$$M(x) = -P.y(x)$$

    - Substitute into the ODE:

$$P/EI = k^2$$

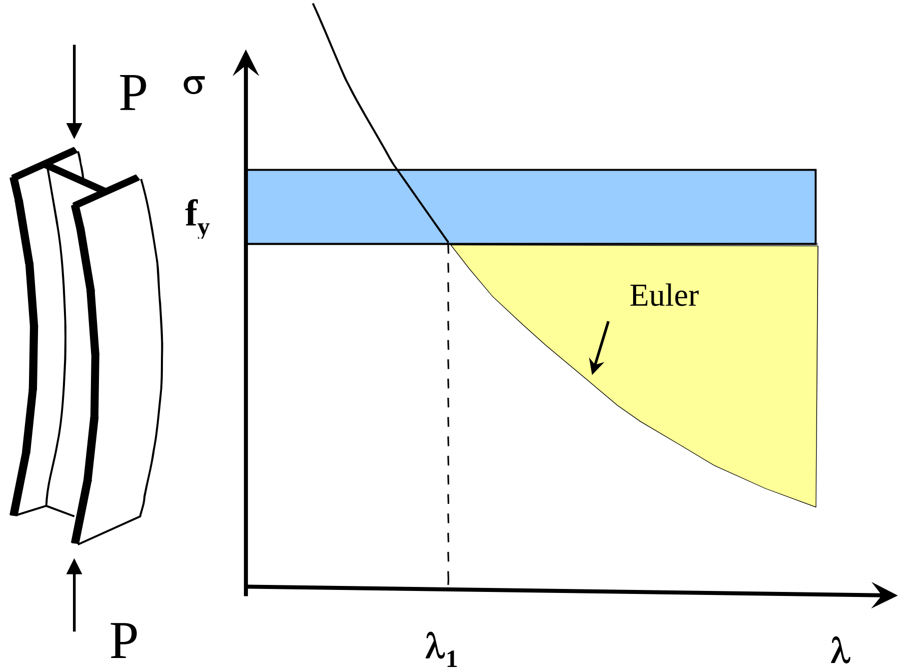
    - Get the new homogeneous ODE:

$$\frac{d^2y}{dx^2} + k^2 y = 0$$

    - Assume the following solution:

$$y(x) = C_1\sin(kx) + C_2\cos(kx)$$

    - Get the critical load and critical stress:
      - λ represents column slenderness,
      - L represents the length and r the radius of gyration.
      - N=1 represents the first mode.



$$P_{cr} = \frac{n^2\pi^2 EI}{L^2}$$

$$\sigma_{cr} = \frac{P_{cr}}{A} = \frac{\pi^2 E}{(L/r)^2} = \frac{\pi^2 E}{\lambda^2}$$

2D – BUCKLING ANALYSIS - CASE 6

- Non-dimensional buckling curve:
  - –inelastic buckling occurs before the Euler buckling load due to imperfections (initial out-of-straightness, residual stresses, eccentricity of axial applied loads, strain-hardening).
  - –lower bound curve were obtained from a statistical analysis of test results.
  - –Test results: More than 1000 for different sections (I,H,T,C,O,etc.). Range of slenderness ratios between 55 and 160.
  - –Supported by analysis.

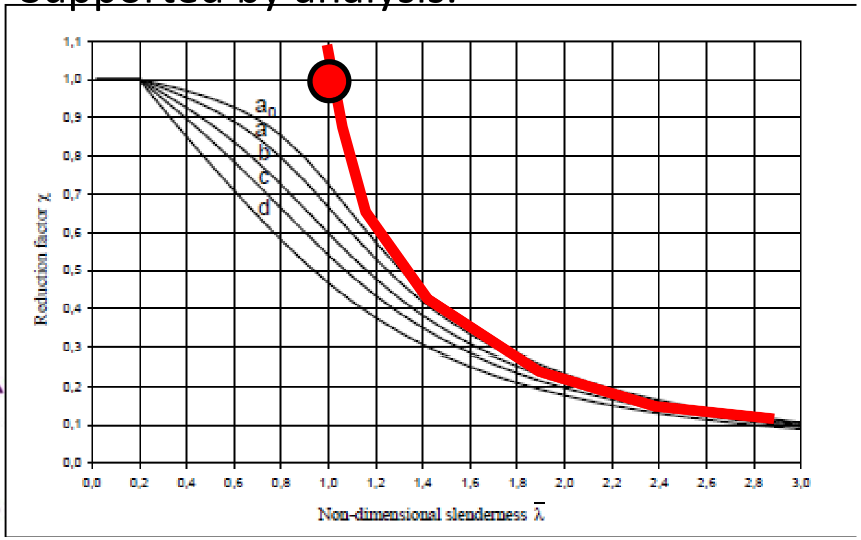
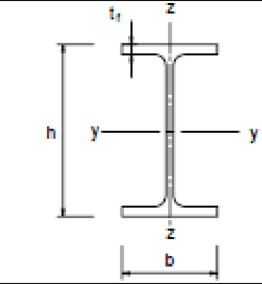
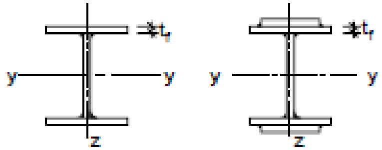
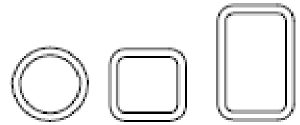
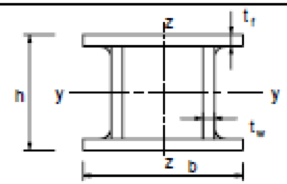
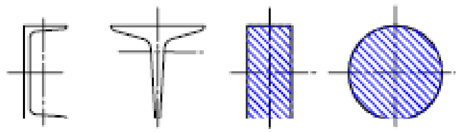
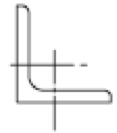


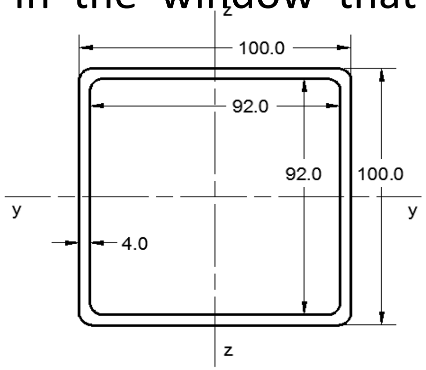
Table 6.2: Selection of buckling curve for a cross-section

Cross section		Limits	Buckling about axis	Buckling curve S 235 S 275 S 355 S 420	Buckling curve S 460
Rolled sections 	h/b > 1,2	$t_f \le 40$ mm	y – y z – z	a b	a_0 a_0
	h/b > 1,2	40 mm < $t_f \le 100$	y – y z – z	b c	a a
	h/b ≤ 1,2	$t_f \le 100$ mm	y – y z – z	b c	a a
	h/b ≤ 1,2	$t_f > 100$ mm	y – y z – z	d d	c c
Welded I-sections 		$t_f \le 40$ mm	y – y z – z	b c	b c
		$t_f > 40$ mm	y – y z – z	c d	c d
Hollow sections 		hot finished	any	a	a_0
		cold formed	any	c	c
Welded box sections 		generally (except as below)	any	b	b
		thick welds: $a > 0{,}5t_f$ $b/t_f < 30$ $h/t_w < 30$	any	c	c
U-, T- and solid sections 			any	c	c
L-sections 			any	b	b

2D – BUCKLING ANALYSIS - CASE 6

Property	Value	Unit
Izz	2.34E-6	m4
A	1.53E-3	m2
Wel	4.68E-5	m3
Wpl	5.49E-5	m3

- Eigen-value ANSYS analysis to determine Elastic curve.
  - Open preprocessor menu /PREP7
  - Define Keypoints Preprocessor > Modeling > Create > Keypoints > In Active CS ...
  - Create Lines Preprocessor > Modeling > Create > Lines > Lines > In Active Coord
  - Element Type > Add/Edit/Delete (For this problem we will use the **BEAM3**).
  - Real Constants... > Add... In the 'Real Constants for BEAM3' window, enter the information for squared hollow section profile 100x100x4, using the following geometric properties:
    - Cross-sectional area AREA, moment of inertia IZZ, Beam Height HEIGHT:
  - Material Models > Structural > Linear > Elastic > Isotropic In the window that appears, enter the following geometric properties for steel:
    - Young's modulus EX: 2.1e11 [N/m2], Poisson's Ratio PRXY: 0.3



  - Meshing > Size Cntrls > ManualSize > Lines > All Lines...
  - Meshing > Mesh > Lines > click 'Pick All'
  - Solution Phase: Assigning Loads and Solving
  - Define Analysis Type Solution > Analysis Type > New Analysis > Static ANTYPE,0
    - To perform an eigenvalue buckling analysis, prestress effects must be activated.
  - Apply Constraints Solution > Define Loads > Apply > Structural > Displacement.

2D – BUCKLING ANALYSIS - CASE 6

- Eigen-value ANSYS analysis to determine Elastic curve (cont.)
  - Define Loads > Apply > Structural > Force/Moment > On Keypoints
    - The eignen-value solver uses a unit force to determine the necessary buckling load.
    - Apply a vertical (FY) point load of -1 N to the top of the beam.
  - Solve > Current LS
  - Exit the Solution processor .
    - Normally at this point you enter the post-processing phase. However, with a buckling analysis you must re-enter the solution phase and specify the buckling analysis. Be sure to close the solution menu and re-enter it or the buckling analysis may not function properly.
  - Solution > Analysis Type > New Analysis > Eigen Buckling ANTYPE,1
  - Solution > Analysis Type > Analysis Options
    - Complete the window which appears, as shown below. Select 'Block Lanczos' as an extraction method and extract 1 mode.
    - The 'Block Lanczos' method is used for large symmetric eigen-value problems and uses the sparse matrix solver.
    - The 'Subspace' method could also be used, however it tends to converge slower as it is a more robust solver.
    - In more complex analyses the Block Lanczos method may not be adequate and the Subspace method would have to be used.

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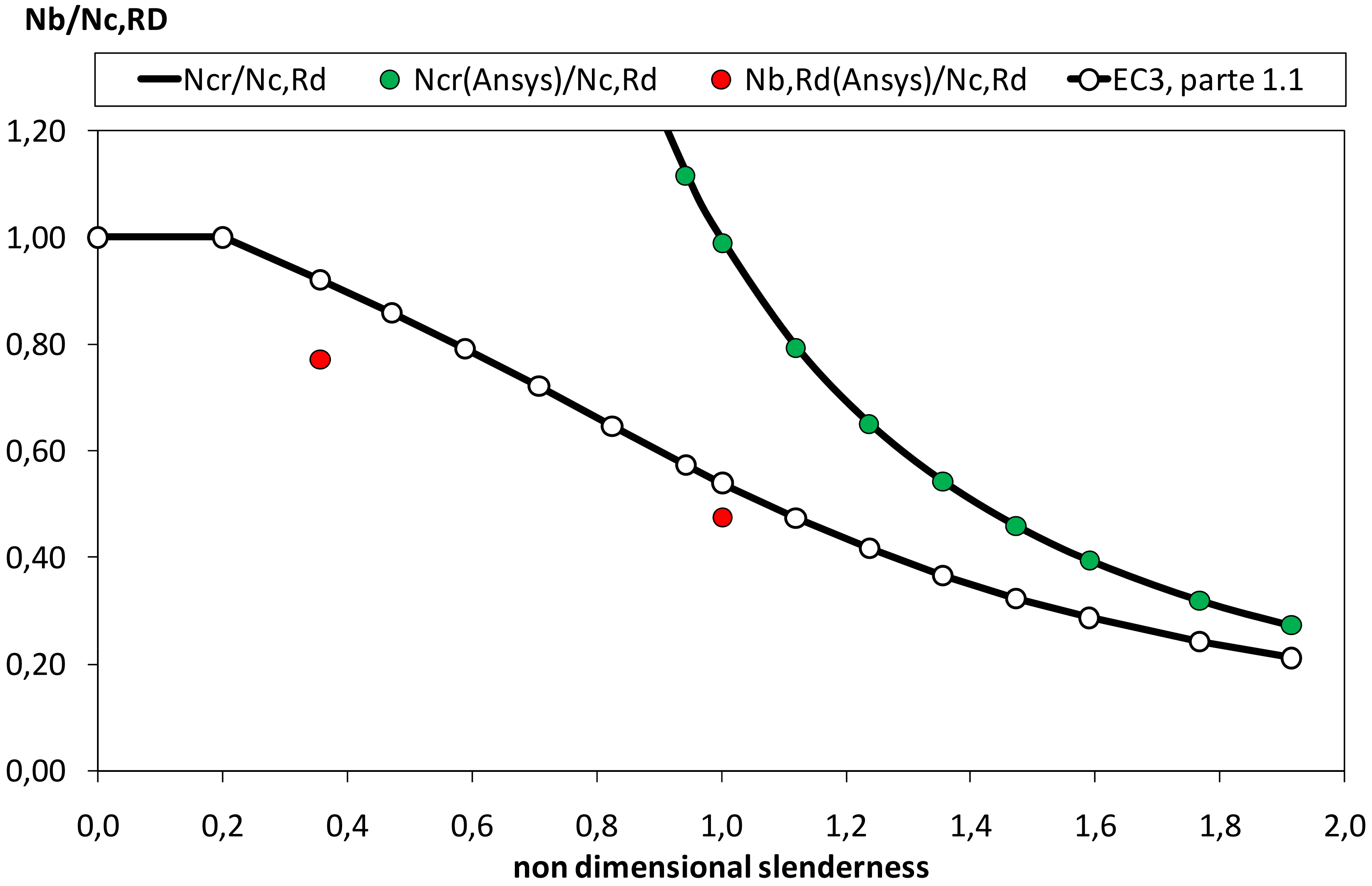
2D – BUCKLING ANALYSIS - CASE 6

- Results for elastic and non-linear plastic analysis for SHS100x100x4



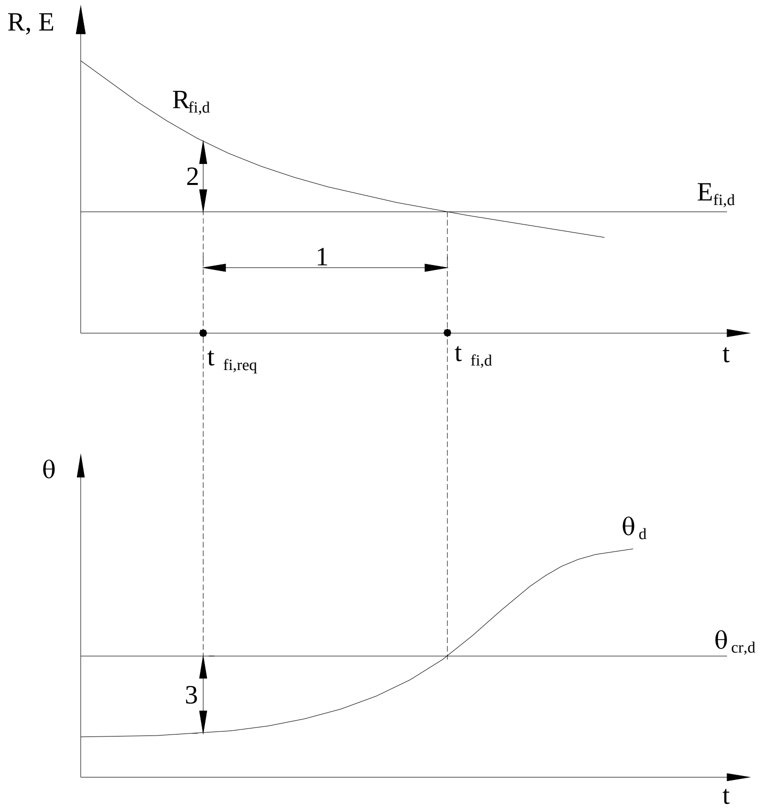
FIRE ANALYSIS

- Fire analysis is an uncouple thermal and mechanical analysis based on:

STRUCTURAL DESIGN PROCEDURE			TABULATED DATA (LEVEL 1)	SIMPLE CALCULATION MODEL (LEVEL 2)	ADVANCE CALCULATION MODEL (LEVEL 3)
PRESCRIPTIVE BASED RULES	Member	Calculation of mechanical actions and boundaries	YES	YES	YES
	Part of structure		NO	YES (If available)	YES
	Entire structure	Selection of mechanical actions	NO	NO	YES
PERFORMANCE BASED RULES	Member	Calculation of mechanical actions and boundaries	NO	YES (If available)	YES
	Part of structure		NO	NO	YES
	Entire structure	Selection of mechanical actions	NO	NO	YES

FIRE DESIGN PROCEDURES

- Alternative limit states:
  - Time domain; $t_{fi,d} \geq t_{fi,requ}$
  - Strength domain; $R_{fi,d,t} \geq E_{fi,d,t}$
  - Temperature domain. $\theta_d \leq \theta_{cr,d}$



$E_{fi,d,t}$ - The design effect of actions for the fire situation, determined in accordance with ENV 1991-2-2, including the effects of thermal expansions and deformations;

$R_{fi,d,t}$ - The corresponding design resistance in the fire situation.

$t_{fi,d}$ - Time design value for the limited design state.

$t_{fi,requ}$ - Required regulation time at fire conditions.

$\theta_{cr,d}$ - Critical temperature design value.

θ_d -Design steel temperature.

VERIFICATION METHODS FOR FIRE ANALYSIS

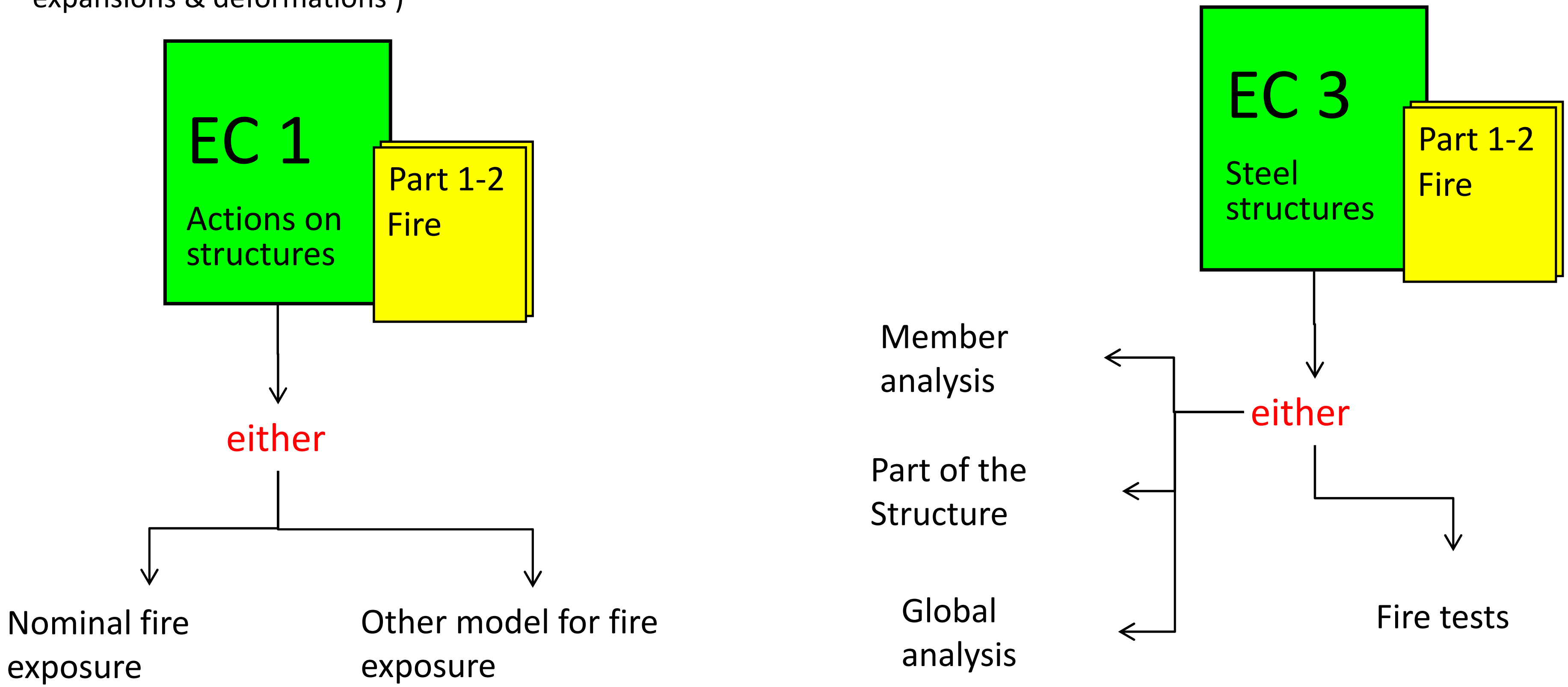
- Strength domain:

Design actions

$$E_{fi,d} \leq R_{fi,d,t}$$

Design resistance

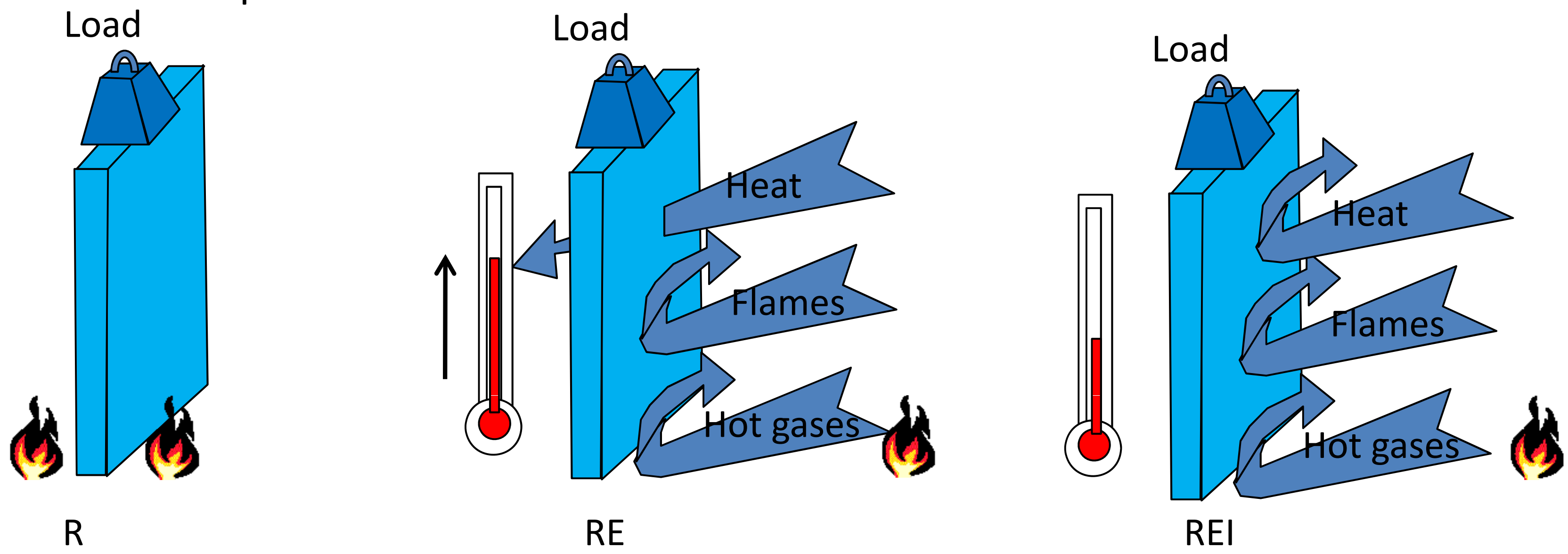
(including the effects of thermal expansions & deformations)



The rules given in the Code are valid only for the standard fire exposure.

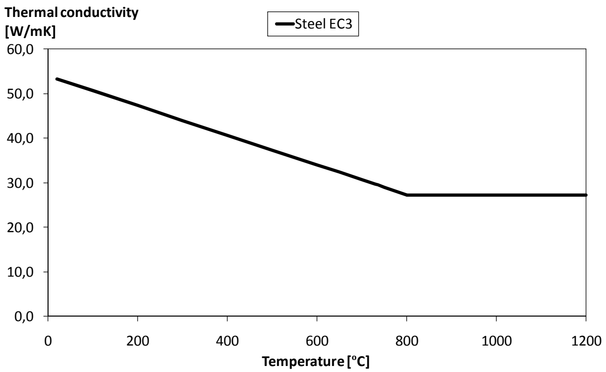
FIRE RESISTANCE RATING

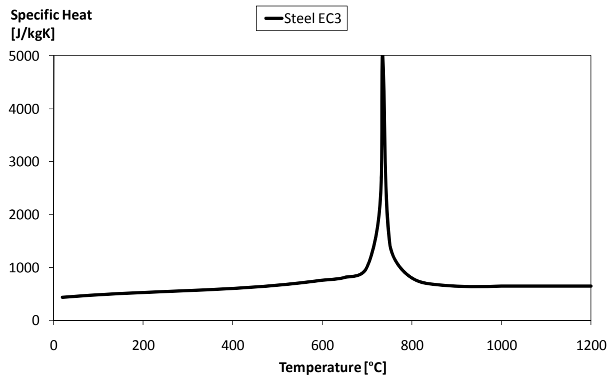
- Structural and separating elements may require:
  - **R criteria**: Load bearing function is maintained during the required time of fire exposure (mechanical resistance). With the hydrocarbon fire exposure curve the same criteria should apply, however the reference to this specific curve should be identified by the letters "HC";
  - **E criteria**: integrity is the ability of a separating element of building construction, when exposed to fire on one side, to prevent the passage through it of flames and hot gases and to prevent the occurrence of flames on the unexposed sides;
  - **I criteria**: insulation is the ability of a separating element of building construction when exposed to fire on one side, to restrict the temperature rise of the unexposed face below specified levels.

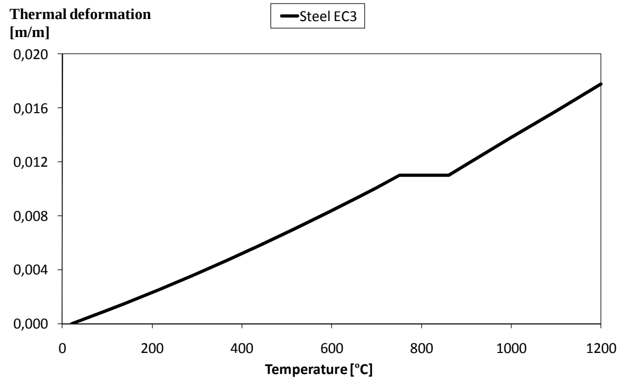


MATERIAL BEHAVIOUR AT ELEVATED TEMPERATURE

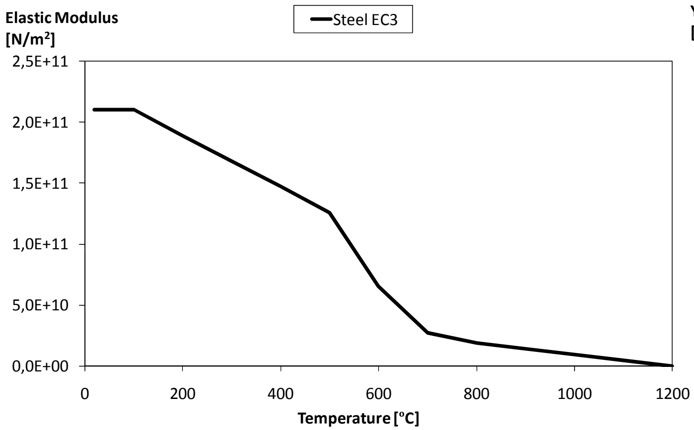
- Steel:
  - Thermal properties:
    - Evidence of crystal modification (allotropic transformation, phase γ (austenitic);
    - The thermal expansion coefficient may de determined by Thermal deformation derivative.

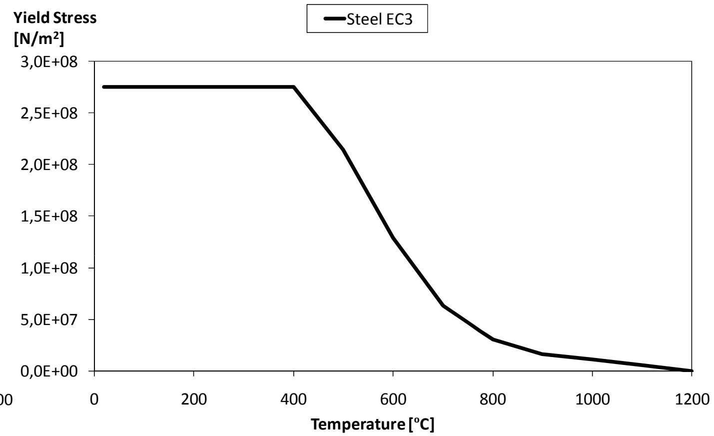






  - Mechanical properties:
    - Elastic modulus reduces after 100 [ºC];
    - Yield strength reduces after 400 [ºC].





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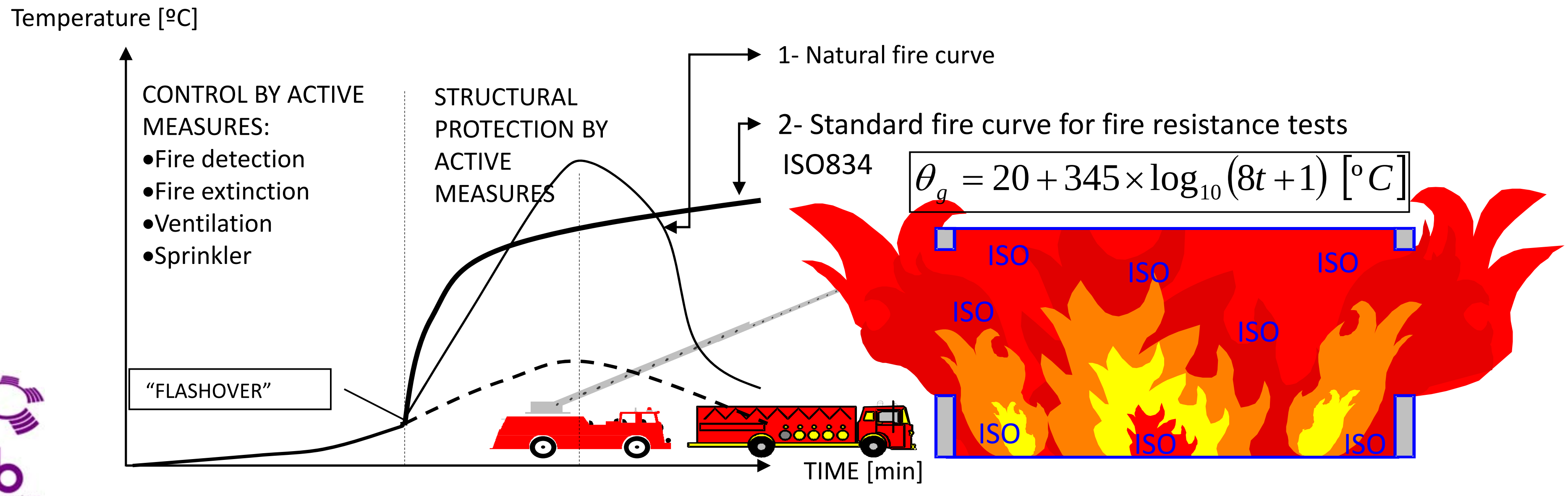
THERMAL LOADING ISO834

- Notes about ISO fire curve
  - Has to be considered in the WHOLE compartment, even if the compartment is huge.
  - Does not consider the PRE-FLASHOVER PHASE.
  - Does not depend on FIRE LOAD and VENTILATION CONDITIONS.
  - Never goes down.
- Heat flux:
  - Convection;
  - Radiation.

$$\dot{h}_{net} = \gamma_{n,r}\dot{h}_{net,r} + \gamma_{n,c}\dot{h}_{net,c} \left[W/m^2\right]$$

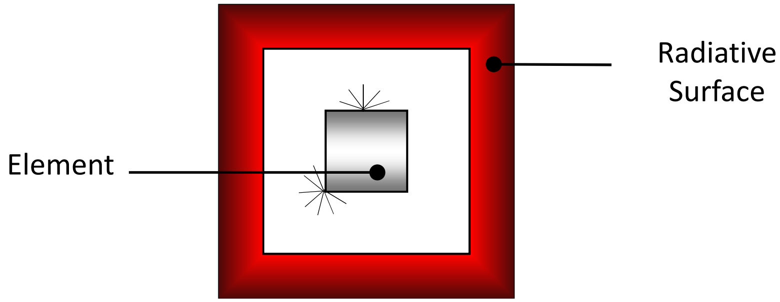
$$\dot{h}_{net,c} = \alpha_c\left(\theta_g - \theta_m\right) \qquad \left[W/m^2\right]$$

$$\dot{h}_{net,r} = \phi.\varepsilon_m.\varepsilon_f \times 5{,}67\times10^{-8}\left[\left(\theta_g + 273\right)^4 - \left(\theta_m + 273\right)^4\right] \; \left[W/m^2\right]$$

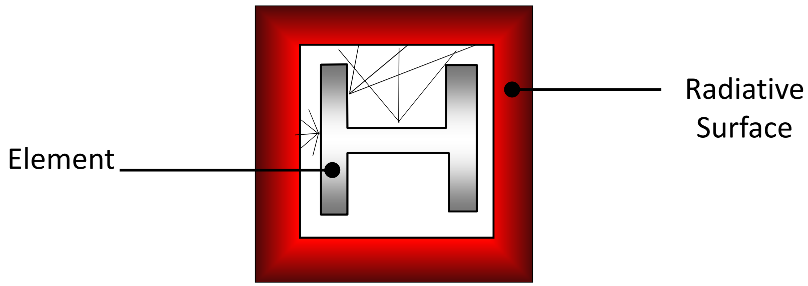


THERMAL LOADING ISO834

- View factor /Shape factor:
  - For convex element surface, each element point if exclusively influenced by the enclosure temperature.
  - The view factor should be considered equal to 1.

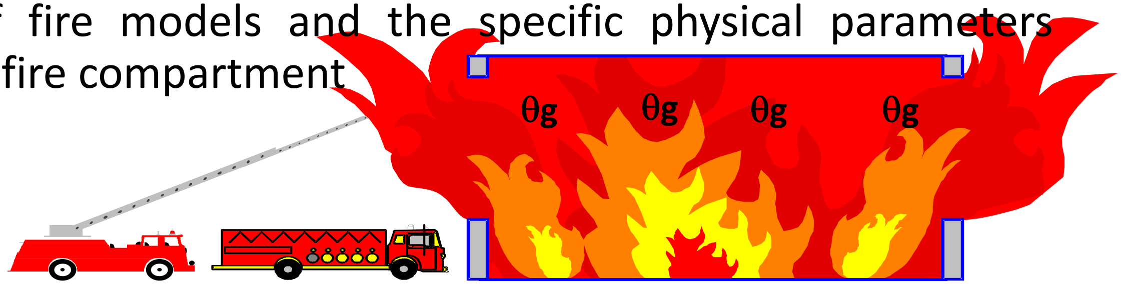


  - For concave element surface, each element point may be protected from enclosure temperature and may be dependent on its own temperature.
  - The view factor may be computed for each element surface.

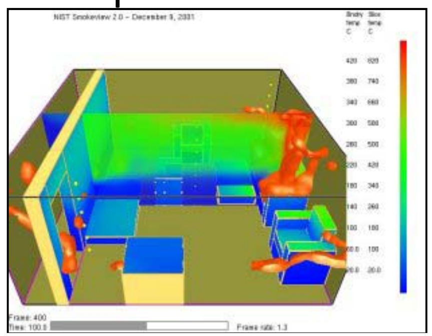


THERMAL LOADING ALTERNATIVES

- Nominal curves:
  - conventional curves, adopted for classification or verification of fire resistance, e.g. the standard temperature-time curve, external fire curve, hydrocarbon fire curve.
- Time equivalent:
  - Depends on the design fire load density and other factors.
- Parametric fires:
  - determined on the basis of fire models and the specific physical parameters defining the conditions in the fire compartment

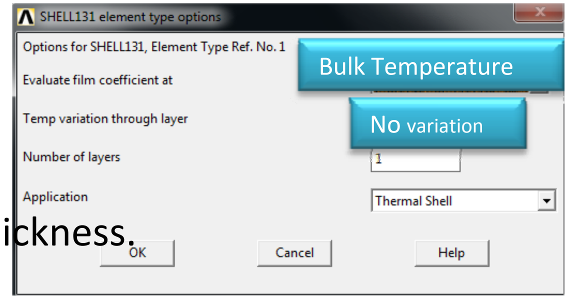


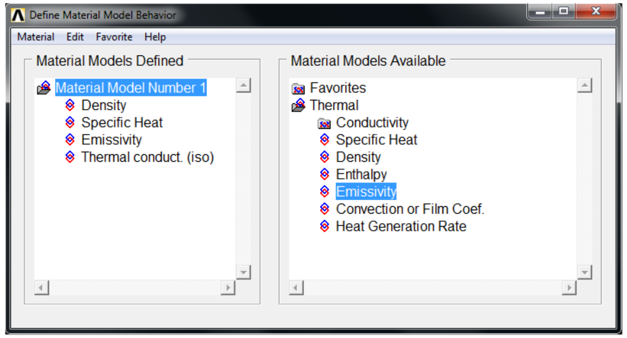
- 1 zone model
  - Assumes uniform, time dependent temperature distribution in the compartment
- 2 zone model
  - Assumes an upper layer with time dependent thickness and with time dependent uniform temperature, as well as a lower layer with a time dependent uniform and lower temperature.
- Computational Fluid Dynamics

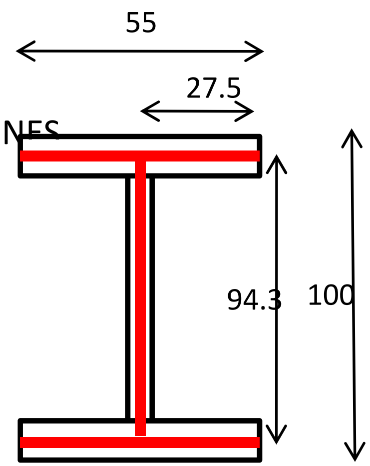


FIRE ANALYSIS USING ANSYS - THERMAL

- Use Ansys software to analyse thermal behaviour of a steel IPE 100 profile.
  - Preferences, Thermal Analysis.
  - Element Type: Shell 131, Options see figure.
    - Keyoption (3)=1, first to allow introducing thickness;
    - Keyoption(3)= 2, to define “no temperature” variation across thickness.
    - Keyoption (4)=1, defines the number of layers.
  - No real constants are defined. Instead sections are defined.
  - Material Properties:
    - Non linear behaviour (temperature dependent).
  - Modelling:
    - Create initial section at X=0, by definition of 7 Key points.
    - Create line for extrusion modelling. Define KP 8.
    - Create 5 lines at the initial section.
    - Create areas by extrusion of lines:
      - MODELING+OPERATE+EXTRUDE+LINES+ALONG LINES
    - Delete extra lines, using:
      - NUMBERING CONTROLS+MERGE ITENS+NODES
    - Meshing areas, using:
      - MESHING+MESH ATTRIBUTES+PICKED AREAS
      - MESHING+SIZE CONTROL+MANUAL SIZE+LINES
      - MESHING+MESH+AREAS+MAPPED+3or4 SIDE.



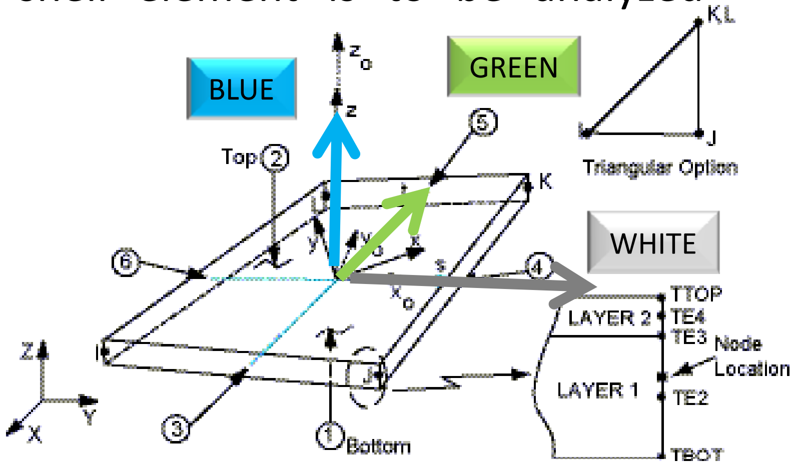




KP	X	Y	Z
1	0	0	0
2	0	0	0.04715
3	0	-0.0275	0.04715
4	0	0.0275	0.04715
5	0	0	-0.04715
6	0	--0.0275	-0.04715
7	0	-0.0275	-0.04715
8	0.5	0	0

FIRE ANALYSIS USING ANSYS - THERMAL

- SHELL131:
  - Is a 3-D layered shell element having in-plane and through-thickness thermal conduction capability.
  - The element has four nodes with up to 32 temperature degrees of freedom at each node.
  - The conducting shell element is applicable to a 3-D, steady-state or transient thermal analysis .
  - SHELL131 generates temperatures that can be passed to structural shell elements in order to apply thermo mechanical behaviour.
  - If the model containing the conducting shell element is to be analyzed structurally, SHELL181 is a good choice.



x_o = element x-axis if ESYS is not supplied.

x = element x-axis if ESYS is supplied.

FIRE ANALYSIS USING ANSYS - THERMAL

- Use Ansys software to analyse thermal behaviour of a steel IPE 100 profile.
  - Define Table for ISO834, using GUI interface:
    - PARAMETERS+ARRAY PARAMETER+DEFINE OR EDIT
    - ADD + Parameter Name=ISO834+ TABLE+ number rows= “n=181”data points to define curve.
  - Alternatively define Table for ISO834, using command line, and “past text in Windows format” command.
    - *DIM,ISO834,TABLE,181,1,1,,,
    - *SET,ISO834(1,0,1),0,1,1,,,
    - *SET,ISO834(1,1,1),20,1,1,,,
    - …
    - *SET,ISO834(181,0,1),10800,1,1,,,
    - *SET,ISO834(181,1,1),1109.7,1,1,,,

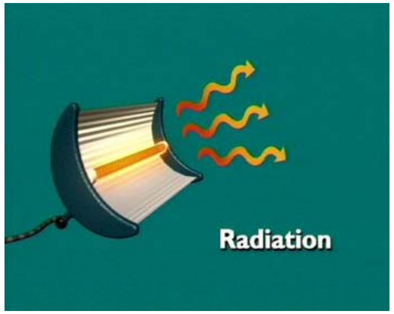
$$\theta_g = 20 + 345 \times \log_{10}(8t+1) \; [^{\circ}C]$$

FIRE ANALYSIS USING ANSYS - THERMAL

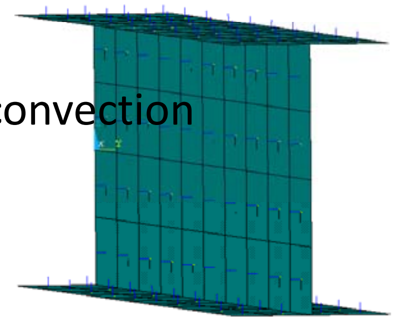
- Use Ansys software to analyse thermal behaviour of a steel IPE 100 profile.
  - Define extra node to apply environment fire condition. Node 777

Node	X	Y	Z
777	0.250	0.100	0

    - LOADS+DEFINE LOADS+THERNAL+TEMPERATURE+ON NODES+EXISTING TABLE +ISO834
  - Define options for radiation heat transfer and enclosure ID. Is the only form of heat transfer that can occur in the absence of any form of medium. Thermal radiation is a direct result of the movements of atoms and molecules (charged particles). Their movements result in the emission of electromagnetic radiation.
    - RADIATION OPTIONS+SOLUTION OTPTIONS
      - Stefan Boltzmann constant: 5.67E-8 [W/m2K4].
      - Temperature OFFSET: 273.15
      - Space Options: Space NODE + Value : 777
      - Enclosure Options: DEFINE + “1”.

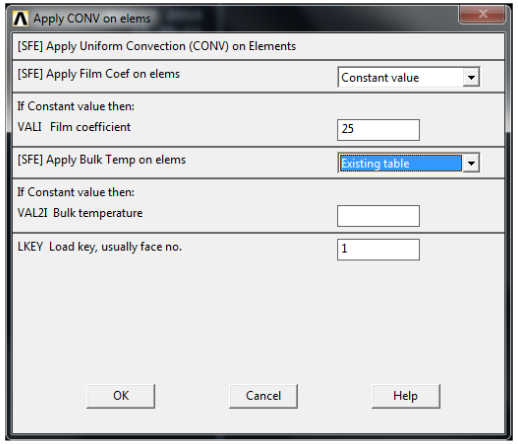


  - Define LIMIT CONDITIONS:
    - Time : Initial Temperature:
      - LOADS+DEFINE LOADS+APPLY+INITIAL CONDITIONS+APPLY
      - DEFINE INITIAL CONDITIONS ON NODES+ALL DOFS+20 [ºC].
  - Apply visibility for shell normal direction, to identify faces where convection boundary conditions should be applied.
    - PLOT CONTROLS+ SYMBOLS
      - Other symbols: element coordinate system ON

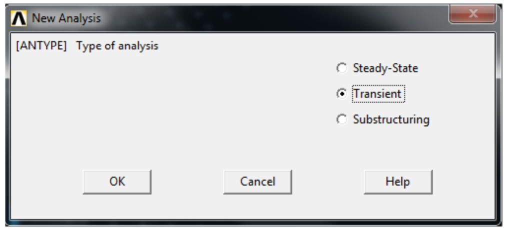


FIRE ANALYSIS USING ANSYS - THERMAL

- Use Ansys software to analyse thermal behaviour of a steel IPE 100 profile.
  - Apply convection boundary condition:
    - LOADS+DEFINE LOADS+APLLY+THERMAL
      - CONVECTION+ON ELEMENTS+UNIFORM+ SELECT ELEMENTS+25 (CONSTANT VALUE)+BULK TEMP. ON ELEMENTS + EXIST. TABLE (ISO834)
  - Apply radiation from node to element surfaces:
    - ANSYS 12 does not support GUI boundary conditions, reason why student may introduce boundary conditions by command.
    - Radiation from node space to finite shell element is applied by radiosity solver method, using RDSF surface load label.
    - Finite shell element only supports top and bottom surface load for radiation:
      - SFE, element number, element face, RDSF, label for property, property value. Property may be: emissivity=1, enclosure =2.
      - SFE,ALL,1,RDSF,1,1
      - SFE,ALL,1,RDSF,2,1
      - SFE,ALL,2,RDSF,1,1
      - SFE,ALL,2,RDSF,2,1



  - Define Analysis Type :
    - LOADS+ANALYSIS TYPE+NEW ANALYSIS + Transient.
    - Solution Method: FULL.

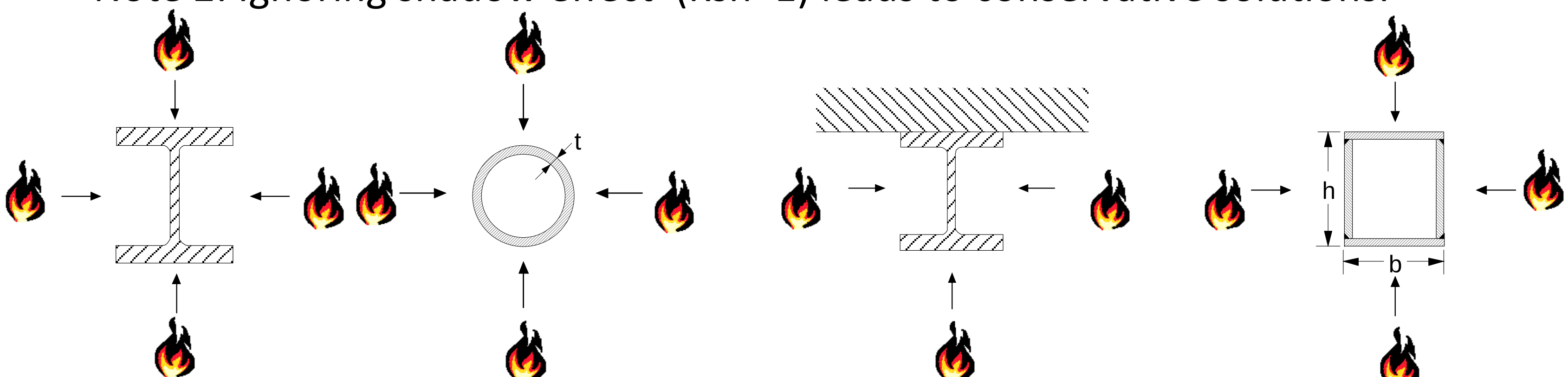


THERMAL ANALYSIS – SIMPLIFIED METHOD EC3

- For unprotected steel structures (simple calculation method):
  - Recalling the hypothesis of non massive elements, the increase of temperature $\Delta\theta a,t$ in an unprotected steel member during a time interval Δt <5[s] may be determined from:

$$\Delta\theta_{a,t} = K_{sh} \frac{A_m / V}{c_a \rho_a} \dot{h}_{net,d} \Delta t \qquad \dot{h}_{net,d} = \Phi \, \sigma \, \varepsilon_m \, \varepsilon_f \left(T_g^{\,4} - T_L^{4} \right) + \alpha \left(T_g - T_L \right) \qquad k_{sh} = 0.9 \times \frac{[Am/V]_{box}}{[Am/V]}$$

    - Where K_{sh} is a correction factor for the shadow effect.
    - $[Am/V]_{box}$ is the box value of the section factor
    - A_m/V is the section factor for unprotected steel members.
  - Note 1: for cross sections with convex shape (rectangular or hollow sections), Ksh equals unity.
  - Note 2: Ignoring shadow effect (Ksh=1) leads to conservative solutions.



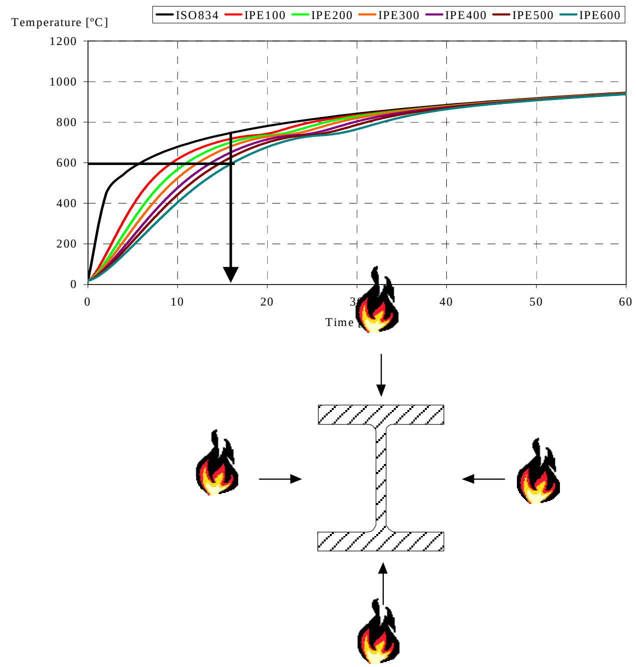
$$\frac{A_m}{V} = \frac{\text{perimeter}}{\text{cross-section area}} \qquad Am / V = 1 / t \qquad \frac{A_m}{V} = \frac{\text{surface exposed to fire}}{\text{cross-section area}} \qquad \frac{A_m}{V} = \frac{2(b+h)}{\text{cross-section area}}$$

THERMAL ANALYSIS – SIMPLIFIED METHOD EC3

- Solution without Ksh by a simple program (ex: fortran programming language)

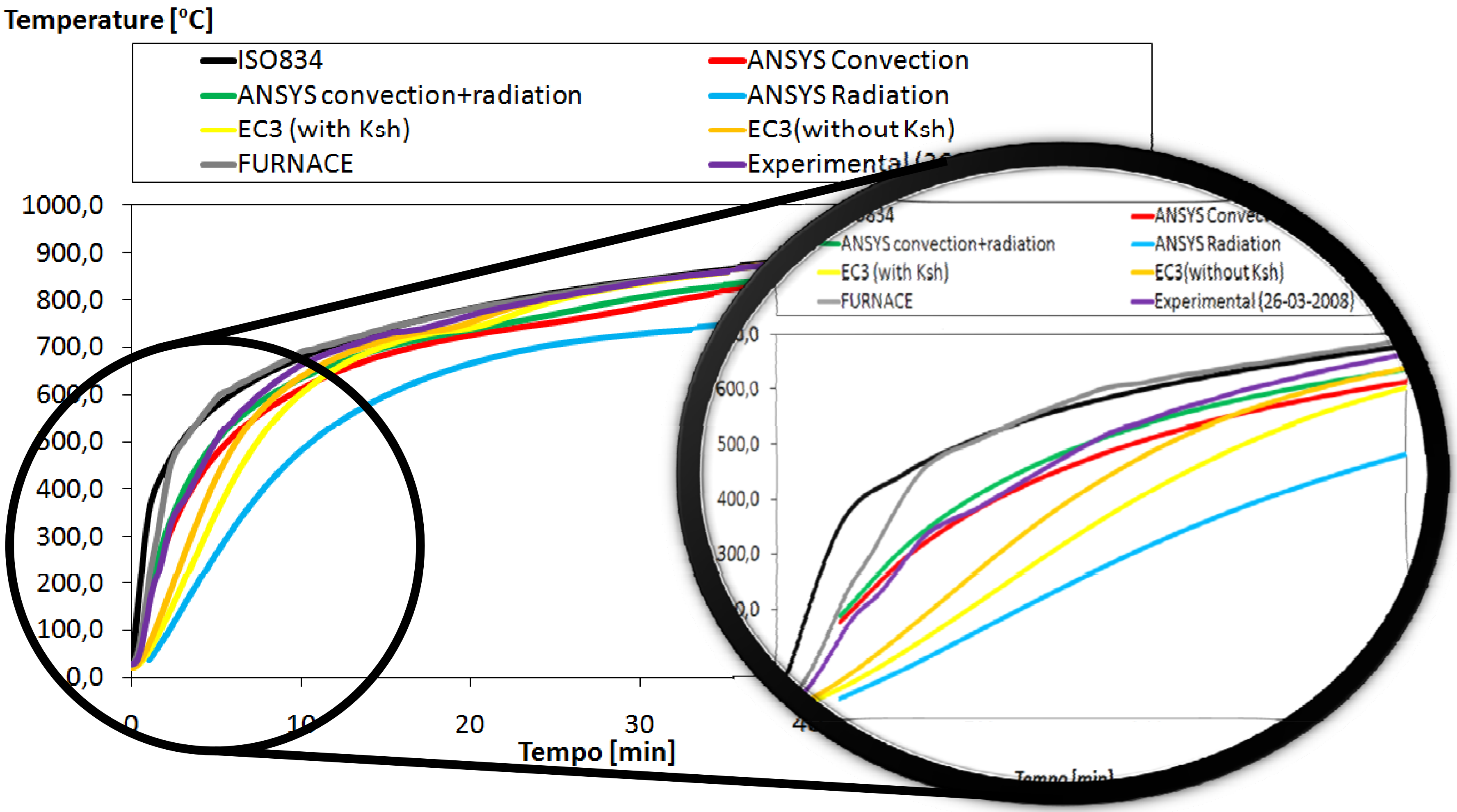
```
*****tn=0, tetan=273.15, deltat=5, sec=387 (IPE100)
do 10 i=1,itfinal+1
   tetav=tetan
   tv=tn
   tetag=(20+345*log10(8*tv/60+1))+273.15
   flux=0.5*0.0000000567*((tetag)**4-(tetav)**4)+alfa*(tetag-tetav)
   cap=ca(tetav)
   tetan=tetav+(sec*flux*deltat)/(ro*cap)
   tn=tv+deltat
   write(1,*)tv, flux, tetan
10 continue
stop
end

funtion ca (tetav)
   if (tetav.le.873.15) then
                ca=425+0.773*(tetav-273.15)-0.00169*(tetav-273.15)**2+ .0.00000222*(tetav-273.15)**3
   else if (tetav.gt.873.15.and.tetav.le.1008.15) then
                ca=666+(13002/(738-(tetav-273.15)))
   else
                ca=545+(17820/((tetav-273.15)-731))
   endif
return
end
```



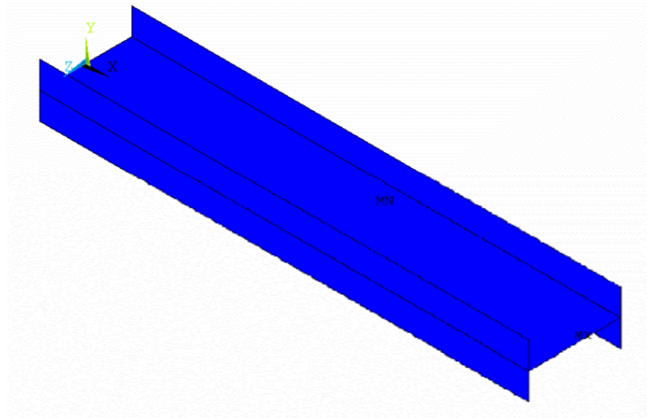
FIRE ANALYSIS USING ANSYS - THERMAL

- Comparison results for thermal analysis:
  - Effect of radiation and convection.
  - Comparison with experimental results.

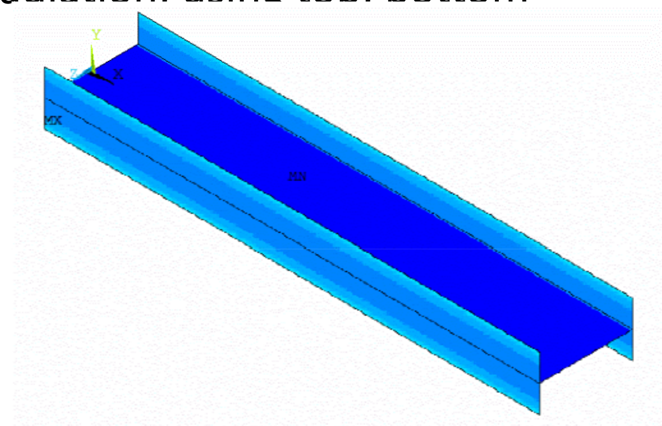


FIRE ANALYSIS USING ANSYS - THERMAL

- Animation for radiation load:
  - Uniform heating.
  - Heat flux arrives from enclosure, directly to top and bottom surface elements.

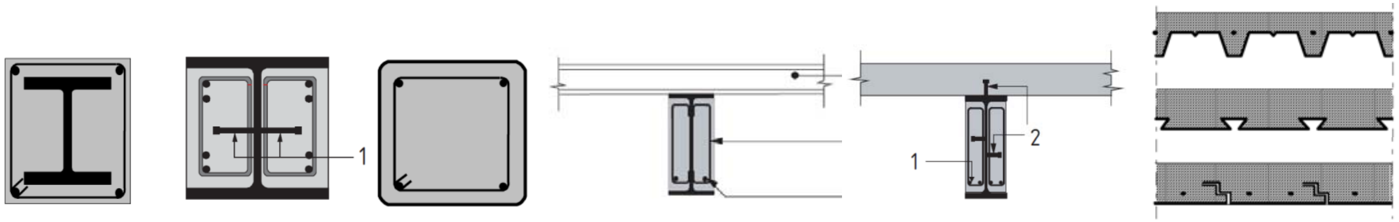


- Animation for fire thermal load(radiation+convection):
  - Non-uniform heating.
  - Heat flux arrives from enclosure, by convection and radiation, using top, bottom and lateral surface elements.



INTRODUCTION TO COMPOSITE STEEL AND CONCRETE

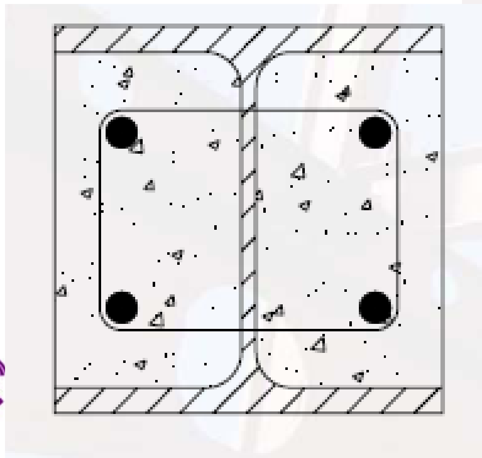
- Structural elements normally used:
  - Beams.
  - Columns.
  - Slabs.

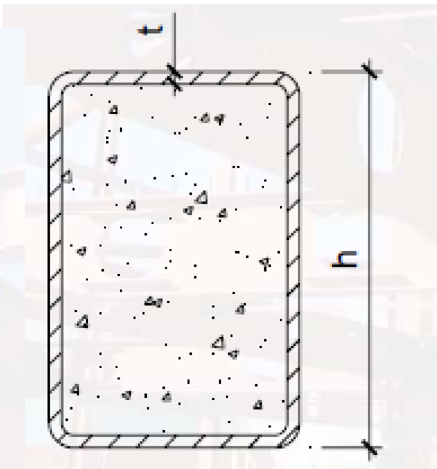


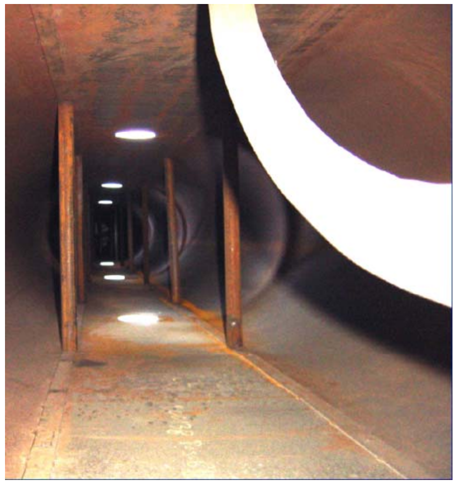
- Objective:
  - Synergy of mixing two different types of materials, increasing composite material properties.
  - Minimize the effect of weakness for each material.
- Positive aspects of Steel:
  - High strength to compression and tension.
  - Small ratio weight strength in comparison with concrete.
  - High quality material reliability, which may lead to reduce safety factors.
  - No intensive workmanship
- Negative aspects of steel:
  - Stability problems when submitted to compression, with non-profit high compressive strength.
  - Corrosion problems, if not protected.
  - Strength reduction at elevated temperatures.
  - High cost due to market values, transport,etc.

INTRODUCTION TO COMPOSITE STEEL AND CONCRETE

- Positive aspects of Concrete:
  - High strength in compression.
  - Facility to be cast in irregular geometry forms.
  - Low cost.
- Negative aspects of Concrete:
  - Low strength in tension.
  - High ratio weight strength.
  - Necessity to have formwork (“cofragem”) and centering (“cimbres”).
  - Intensive workmanship.







INTRODUCTION TO COMPOSITE STEEL AND CONCRETE

- Steel function in composite construction elements (pipes filled with concrete):
  - May be used to mould concrete;
  - May be used to confine concrete;
  - Increase bending resistance;
  - Also contributes to the load bearing resistance in compression .
- Steel function in composite construction elements(encased or partial encased):
  - Contribute to section resistance due to bending, compression and shear.

- Concrete function in composite construction elements (pipes filled with concrete):
  - Ensures the major contribution to the compression load bearing resistance.
  - Helps to sustain instability effects.
  - Helps to increase fire resistance.
- Concrete function in composite construction elements(encased or partial encased):
  - Contribute to section resistance due to compression.
  - Helps to protect steel from corrosion.
  - Increase fire resistance.

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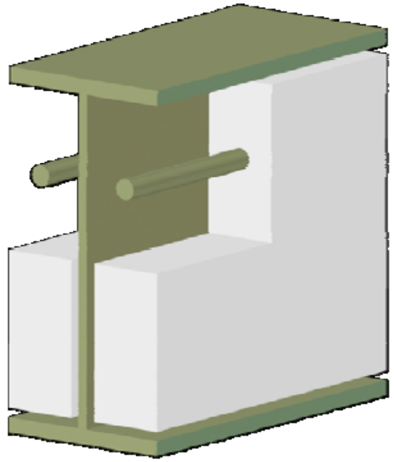
INTRODUCTION TO PARTIALLY-ENCASED SECTIONS



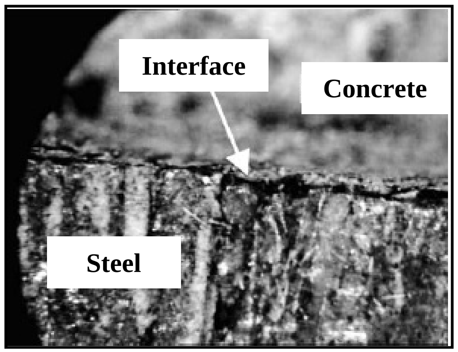
- Partially-encased beams are composite members:
  - Presents two or more different materials;
  - Different types of construction and design solutions;
  - Used with reinforcement stirrups and rebars;
  - Possible structural link to slab.
- Section resistance up to Ultimate Limit State (plastic section):
  - Disregard the behaviour of concrete in tension.
  - Assume to consider the design compression strength of concrete equal to 85% of cylindrical compressive stress (βR).
  - Assume to consider the plastic behaviour of steel in compression and in tension (Mpl,a).
  - Assume to consider the resistance of rebars with area equal to Ar and yield stress equal to "fyr".

$$M_{pl} = M_{pl,a} - \frac{2 f_{ya} t_w \left(0.5 h_1 - e_{pl}\right)^2}{2} + \beta_R 2 b_1 e_{pl}\left(0.5 e_{pl} + 0.5 h_1 - e_{pl}\right) + A_r \left(f_{yr} - \beta_R\right)\left(h - 2 e_r\right)$$

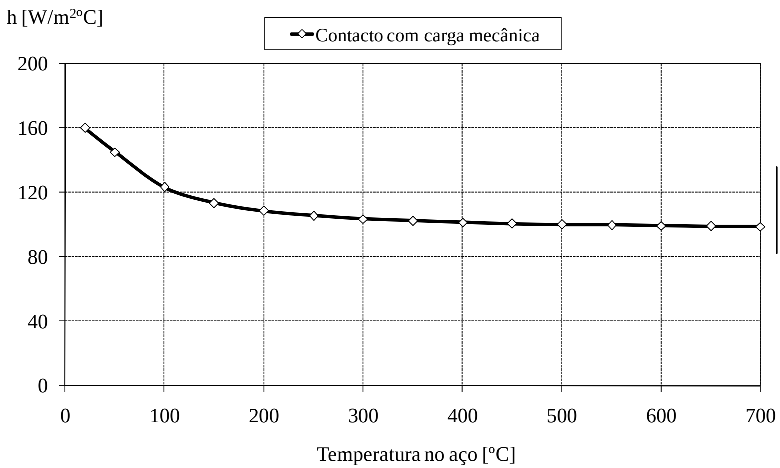
$$M_{pl} = M_{pl,a} + \beta_R 2 b_1 e_{pl}\left(0.5 e_{pl} + 0.5 h_1 - e_{pl}\right) + A_r \left(f_{yr} - \beta_R\right)\left(h - 2 e_r\right)$$

BONDING OF PARTIAL ENCASED ELEMENT

- For thermal analysis:
  - Partially encased elements may be subjected to fire conditions (IPE100 hot rolled-C20/25-Reinf. Steel with 8 [mm] diameter rebar):
    - ISO 834 fire nominal curve applied to the external surfaces of the 3D model (four sides).
    - Radiative and convective heat flux between fire environment and partially encased elements.
  - Non linear unsteady state thermal analysis will be applied, based on incremental time procedure.
  - Heat flux between concrete and steel will be controlled by conductance. Perfect contact could be also considered. This property is defined as the ratio of Heal flux to temperature variation between both materials.



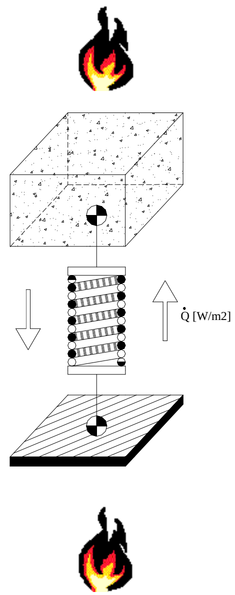
[2] - Ghojel, J, Experimental and analytical technique for estimating interface thermal conductance in composite structural elements under simulated fire conditions, *Experimental Thermal and Fluid Science*, No. 28, 2004, pp. 347-354



Heat flux or thermal flux, sometimes also referred to as heat flux density or heat flow rate intensity is a flow of energy per unit of area per unit of time. In SI units, it is measured in [W/m2]. It has both a direction and a magnitude so it is a vectorial quantity.

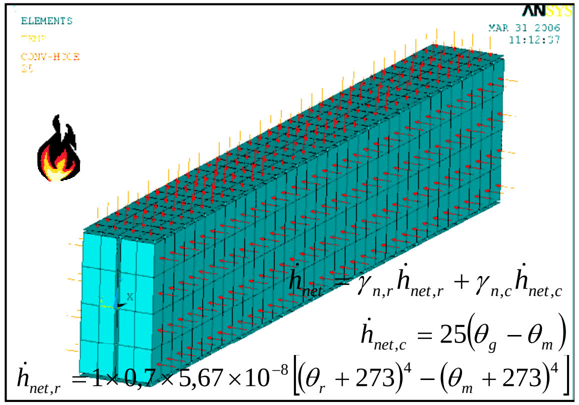
$$\dot{\vec{Q}} = \frac{\vec{Q}}{A} = -K\,\vec{\nabla}(T)$$

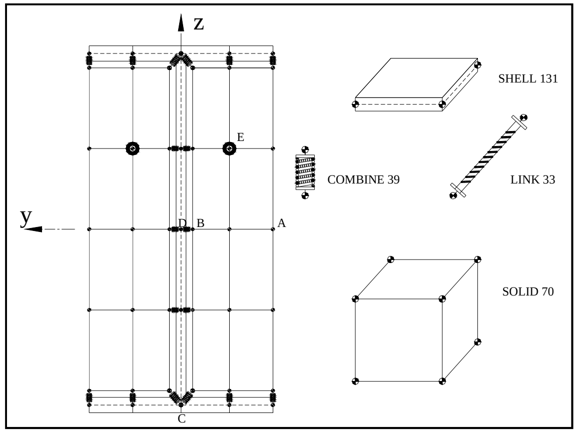
$$\left\|\dot{\vec{Q}}\right\| = COND\,(\Delta T)$$



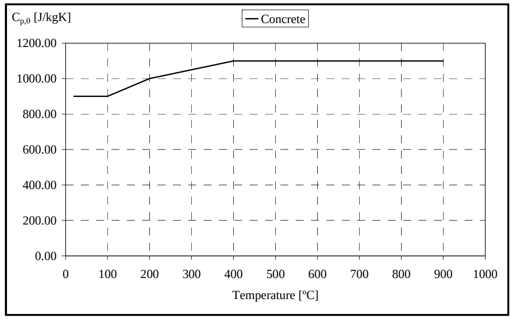
THERMAL MODELLING OF PARTIALLY-ENCASED ELEMENT

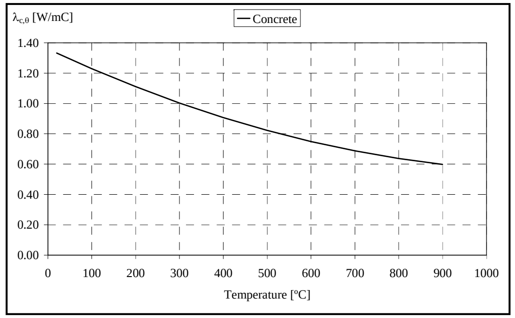
- Steel model:
  - Hot rolled profile: Shell 131, 4 nodes, in-plane and through thickness conduction capacity. In-plane linear shape functions (2x2 integration scheme), through thickness (assuming no temperature variation, 1 integration point).
- Reinforcement:
  - Link 33, 2 nodes, with ability to conduct heat between nodes. Linear shape functions with exact integration scheme.
- Concrete model:
  - Solid70, 8 nodes, 3D conduction capacity. Linear shape functions for each orthogonal direction (2x2x2 integration scheme).
- Bond model (concrete and steel):
  - Combine39, non linear spring, 2 nodes, generalized temperature difference versus heat flux characteristics. No mass or thermal capacitance is considered.

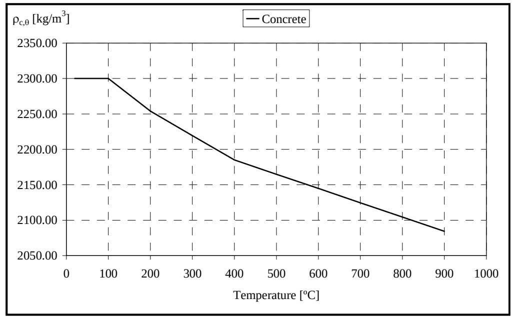


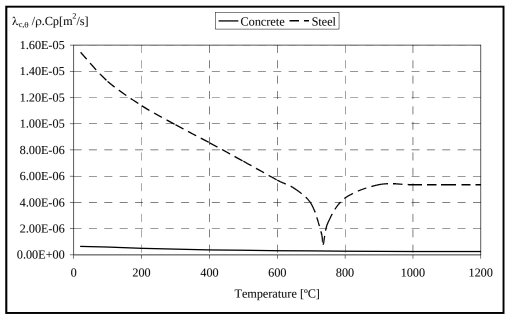


THERMAL MODELLING OF PARTIALLY-ENCASED ELEMENT









- Material Properties for concrete (temperature dependence):
  - Specific Heat for dry concrete of Siliceous and Calcareous aggregates:

$$
\begin{aligned}
C_p[J/kgK] &= 900 \quad ;20[^{\circ}C] \le \theta \le 100[^{\circ}C] \\
&= 900 + (\theta - 100) \quad ;100[^{\circ}C] < \theta \le 200[^{\circ}C] \\
&= 1000 + (\theta - 200)/2 \quad ;200[^{\circ}C] < \theta \le 400[^{\circ}C] \\
&= 1100 \quad ;400[^{\circ}C] < \theta \le 1200[^{\circ}C]
\end{aligned}
$$

  - Thermal Conductivity of Siliceous and Calcareous aggregates:
    - Determined between lower and upper limit values.
    - The value of thermal conductivity may be set by the National annex within the range defined by lower and upper limit. Annex A is compatible with the lower limit.

$$
\lambda_c[W/mK] = 1.36 - 0.136(\theta/100) + 0.0057(\theta/100)^2 \;;20[^{\circ}C] \le \theta \le 1200[^{\circ}C]
$$

  - The variation of density with temperature is influenced by water loss and is defined as follows

$$
\begin{aligned}
\rho[kg/m^3] &= \rho_{20^{\circ}C} \quad ;20[^{\circ}C] \le \theta \le 115[^{\circ}C] \\
&= \rho_{20^{\circ}C}(1 - 0.02(\theta - 115)/85) \quad ;115[^{\circ}C] < \theta \le 200[^{\circ}C] \\
&= \rho_{20^{\circ}C}(0.98 - 0.03(\theta - 200)/200) \quad ;200[^{\circ}C] < \theta \le 400[^{\circ}C] \\
&= \rho_{20^{\circ}C}(0.95 - 0.07(\theta - 400)/800) \quad ;400[^{\circ}C] < \theta \le 1200[^{\circ}C]
\end{aligned}
$$

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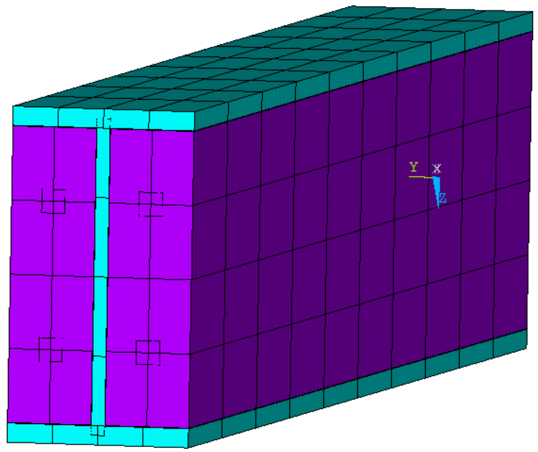
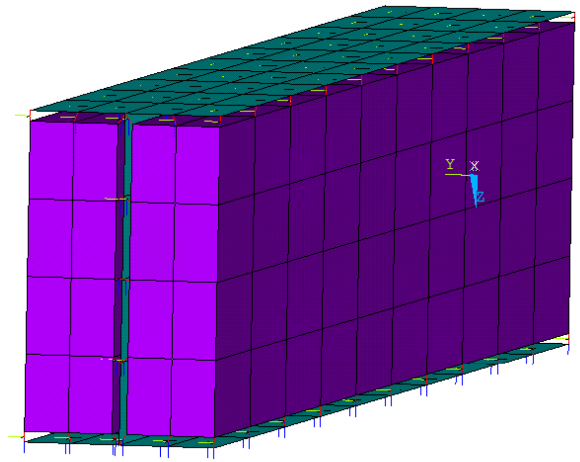
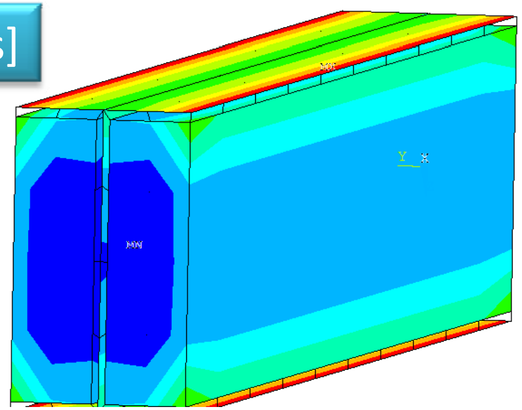
THERMAL MODELLING OF PARTIALLY-ENCASED ELEMENT

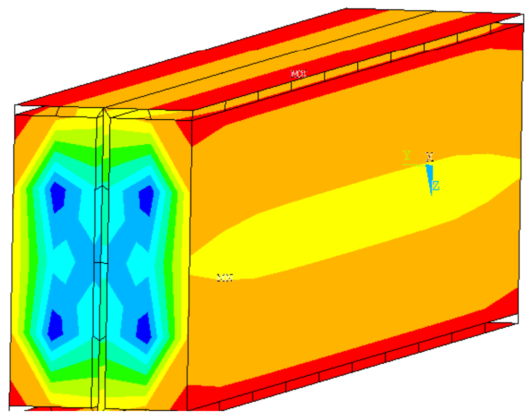
- How to build the finite element model:
  - Geometric modelling to define:
    - Finite shell elements, using keypoints, lines and areas;
    - Finite solid elements, using keypoints, areas and volumes;
    - Finte combine element, using elements, by off set distance.





Time = 60 [s]





Time = 3600 [s]

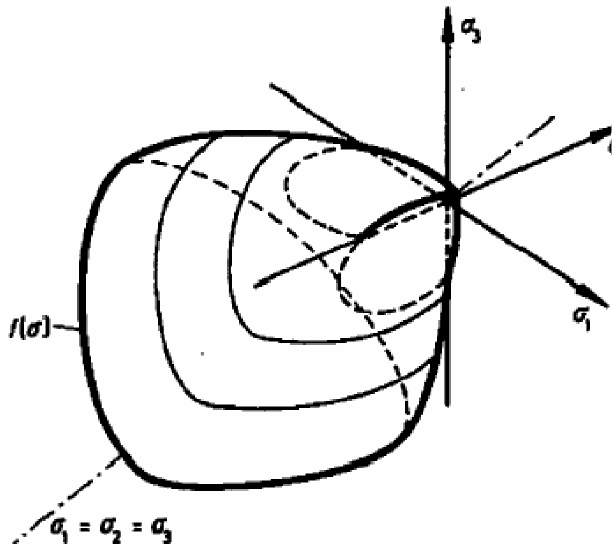
CONSITUTIVE MODEL FOR CONCRETE

- Constitutive modelling of concrete in ANSYS:
  - Failure criteria (surface):
    - Described in terms of the invariants of the stress tensor;
    - Dependent of the hydrostatic component of the stress (σh). (The stress tensor can be separated into two components. One component is a hydrostatic stress that acts to change the volume of the material only. The other is the deviatoric stress that acts to change the shape only).
  - Continuum mechanics provides a mean of modelling at the macroscopic level the material damage that occurs at the microscopic level.
  - Willam and Warnke (1974) developed a widely used model for the triaxial failure surface. The failure surface may be represented in principal stress-space of unconfined plain concrete.
  - Failure surface: $\frac{F}{f_c} - S \geq 0$
    - "F" represents a function of the principal stress state.
    - "S" is the failure surface, written in terms of 5 input parameters.
    - "fc", represents the uniaxial compression stress.

$$\begin{bmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{xy} & \sigma_{yy} & \tau_{yz} \\ \tau_{xz} & \tau_{yz} & \sigma_{zz} \end{bmatrix} = \begin{bmatrix} \sigma_h & o & o \\ o & \sigma_h & o \\ o & o & \sigma_h \end{bmatrix} +$$

$$+ \begin{bmatrix} \sigma_{xx} - \sigma_h & \tau_{xy} & \tau_{xz} \\ \tau_{xy} & \sigma_{yy} - \sigma_h & \tau_{yz} \\ \tau_{xz} & \tau_{yz} & \sigma_{zz} - \sigma_h \end{bmatrix}$$

$$\sigma_h = \frac{1}{3}\left(\sigma_{xx} + \sigma_{yy} + \sigma_{zz}\right)$$



CONSITUTIVE MODEL FOR CONCRETE

- For each domain, independent functions describe F and S:
  - First: Compression Compression Compression: $0 \geq \sigma_1 \geq \sigma_2 \geq \sigma_3$
  - Second: Tension Compression Compression: $\sigma_1 \geq 0 \geq \sigma_2 \geq \sigma_3$
  - Third: Tension Tension Compression: $\sigma_1 \geq \sigma_2 \geq 0 \geq \sigma_3$
  - Fourth: Tension Tension Tension: $\sigma_1 \geq \sigma_2 \geq \sigma_3 \geq 0$
- First domain: $0 \geq \sigma_1 \geq \sigma_2 \geq \sigma_3$
  - The function F assumes the following formula:

$$F = F_1 = \frac{1}{\sqrt{15}}\left[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2\right]^{1/2}$$

  - And the failure surface is defined by:

$$S = S_1 = \frac{2r_2(r_2^2 - r_1^2)\cos\eta + r_2(2r_1 - r_2)[4(r_2^2 - r_1^2)\cos^2\eta + 5r_1^2 - 4r_1 r_2]^{1/2}}{4(r_2^2 - r_1^2)\cos^2\eta + (r_2 - 2r_1)^2}$$

$$\cos\eta = \frac{2\sigma_1 - \sigma_2 - \sigma_3}{\sqrt{2}[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2]^{1/2}}$$

$$r_1 = a_0 + a_1\xi + a_2\xi^2$$

$$r_2 = b_0 + b_1\xi + b_2\xi^2$$

$$\xi = \frac{\sigma_1 + \sigma_2 + \sigma_3}{3f_c} = \frac{\sigma_h}{f_c}$$

- Where the terms of this expression are defined by:
  - The coefficients a0, a1, a2, b0, b1, b2 are determined as a function of the known properties of concrete (Uniaxial tensile strength ft, Uniaxial compression strength fc, biaxial compression strength fcb) .

CONSITUTIVE MODEL FOR CONCRETE

- Second domain: $\sigma_1 \geq 0 \geq \sigma_2 \geq \sigma_3$
  - The function F assumes the following formula:

$$F = F_2 = \frac{1}{\sqrt{15}}\left[\sigma_2^2 + \sigma_3^2 + (\sigma_2 - \sigma_3)^2\right]^{1/2}$$

  - And the failure surface is defined by:

$$S = S_2 = \left(1 - \frac{\sigma_1}{f_t}\right)\frac{2p_2(p_2^2 - p_1^2)\cos\eta + p_2(2p_1 - p_2)[4(p_2^2 - p_1^2)\cos^2\eta + 5p_1^2 - 4p_1p_2]^{1/2}}{4(p_2^2 - p_1^2)\cos^2\eta + (p_2 - 2p_1)^2}$$

    - Where the terms of this expression are defined by:
      - The coefficients a0, a1, a2, b0, b1, b2 are determined as a function of the known properties of concrete.

$$p_1 = a_0 + a_1\chi + a_2\chi^2$$

$$p_2 = b_0 + b_1\chi + b_2\chi^2$$

$$\chi = \frac{\sigma_2 + \sigma_3}{3f_c}$$

  - If the failure criteria is satisfied then cracks at the plane normal to the principal stress σ1 occurs.

CONSITUTIVE MODEL FOR CONCRETE

- Third domain: $\sigma_1 \geq \sigma_2 \geq 0 \geq \sigma_3$
  - The failure criteria is defined by the following functions:

$$F = F_3 = \sigma_i \qquad i = 1,2$$

  - The failure surface is described by:

$$S = S_3 = \frac{f_t}{f_c}\left(1 + \frac{\sigma_3}{S_2(\sigma_i, 0, \sigma_3)}\right) \qquad i = 1,2$$

  - If the failure criteria is satisfied for i=1,2, cracks at normal planes to the principal stresses σ1 and σ2 occur.
  - If the failure criteria is only satisfied for i=1, then cracks will only occur at the plane normal to the principal stress σ1 .
- Fourth domain: $\sigma_1 \geq \sigma_2 \geq \sigma_3 \geq 0$
  - The failure criteria is defined by the following expression:

$$F = F_4 = \sigma_i \qquad i = 1,2,3$$

  - he failure surface S is defined by $S = S_4 = \frac{f_t}{f_c}$
  - f the failure criteria is satisfied for i=1,2 e 3, then cracks occur at the planes normal to the principal stresses σ1 ,σ2 and σ3.
  - Otherwise if criteria is satisfied for i=1,2, then cracks occur at the planes normal to the principal stresses σ1 e σ2. finally if criteria is only satisfied for i=1, then cracks will only appear at the plane normal to the principal stress σ1 .

SOLID 65 (CONCRETE FINITE ELEMENT - ANSYS)

- SOLID 65:
  - The finite element has tri-linear interpolating functions
  - 8 nodes with three degrees of freedom per node.
  - The constitutive model described previously allows cracking in three orthogonal directions for each integration point of the element. The numerical integration schema uses Gauss integration with 2x2x2 integration points.
  - Initially, concrete is assumed as being an isotropic material. As the loading is increased, when a crack occurs at a specific point of integration, the crack is accounted for by the modification of the mechanical properties of the material, which means that it is modeled as a distributed crack or a smeared crack. The presence of a crack at an integration point is represented through the introduction of a weak plane at the direction normal to the crack.
  - Additionally the model allows the inclusion of a shear transfer coefficient (β). This coefficient represents the shear strength reduction factor for the post-cracking loading, which causes a sliding parallel to the crack plane. This shear transfer coefficient can assume values between:
    - 0 – smooth crack with total lost of shear transfer capacity
    - 1 – irregular crack without lost of shear transfer capacity.
    - Best practice:
      - Shear transfer coefficients for an open crack=βt=0.25;
      - Shear transfer coefficients for a closed crack=βc=0.90.

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SOLID 65 (CONCRETE FINITE ELEMENT - ANSYS)

- SOLID 65:
  - Through the inclusion of a stress relaxation factor, it is possible to accelerate the solution convergence process when cracking is imminent. This stress relaxation factor does not introduce any modification in the stress-strain relation at the post-cracking regime. After the convergence to the final cracked state, the stiffness normal to the failure plane is equal to zero.
  - When the material evaluated at an integration point fails in axial, biaxial or tri-axial compression, the material is assumed as crushed at this point.
  - Crushing is defined as the complete deterioration of the structural integrity of the material, and the stiffness contribution of this integration point for the element is ignored. The stiffness normal to the failure plane is equal to zero.

HOW TO RUN FIRE ANALYSIS (THERMAL + MECHANICAL)

- RUN THERMAL ANALYSIS, BASED ON RADIATION AND CONVECTION
  - FILE.RTH should contains time history results.
- SWITCH ELEMENT TYPE: FROM THERMAL TO STRUCTURAL.
  - Correct element options, real constants, materials, etc.
  - Please consider the following:
    - SOLID 70 will be automatically modified to SOLID 185. Please convert SOLID 185 to SOLID 65.
      - Remember for SOLID 65: Keyoption (8) =2 and Keyoption (3)=2
    - SHELL 131 will be modified to Shell 181.
      - This element requires two real constants to substitute shell layer thicknesses.
    - COMBINE 39 will be modifed to COMBINE 39 .
      - Modify the real constant associated with this element, introducing force, relative displacement to model bond behaviour.
    - LINK 33 will be automatically modified to LINK 180. Please convert LINK 180 to LINK 8.
      - Do not modify the real constant associated with this element.
      - Modify the element option related to the type of degree of freedom. Remove TEMP and use UX, UY, UZ.
  - Solution will be performed with STEP LOADS.
    - Build the step load procedure for each time STEP, using:
      - The results of thermal analysis.
      - Introducing mechanical load.

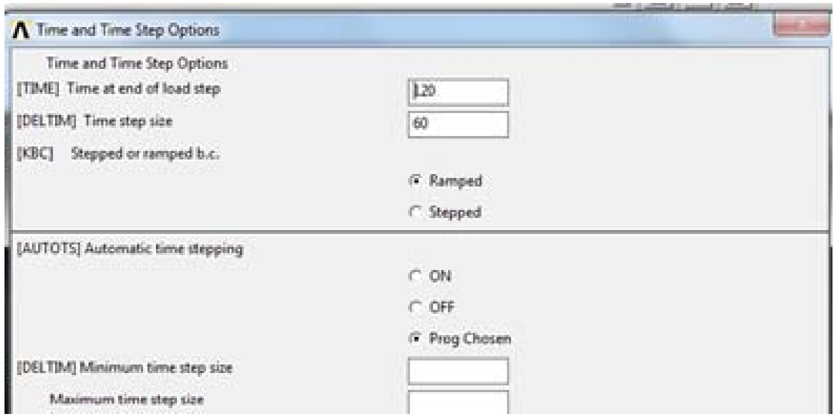
HOW TO RUN FIRE ANALYSIS (THERMAL + MECHANICAL)

- Automatic switch element type and modification procedure.

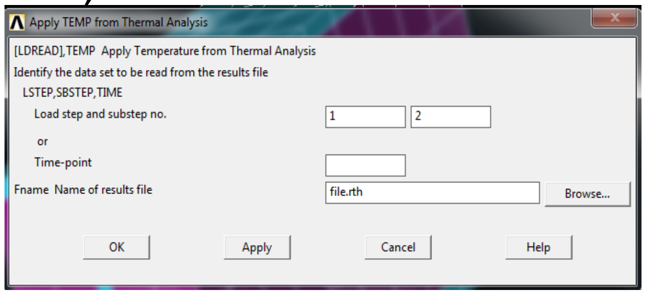
ELEMENT TYPE	THERMAL ANALYSIS	AUTOMATIC SWITCH	MANUALLY CREATE	MECHANICAL ANALYSIS
1	SHELL 131	SHELL 181		SHELL 181
2	SOLID 70	SOLID 185		-
3	COMBIN 39	COMBIN 39		COMBIN 39
4	LINK 33	LINK 180		-
5			SOLID 65	SOLID 65
6			LINK 8	LINK 8

HOW TO RUN FIRE ANALYSIS (THERMAL + MECHANICAL)

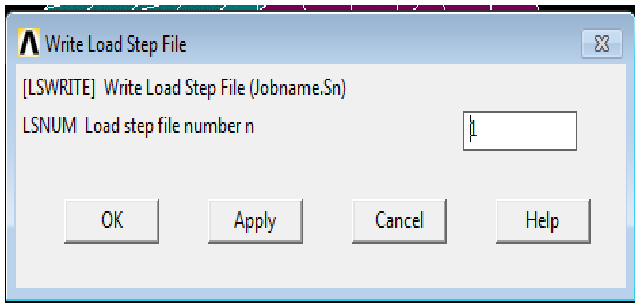
- GUI command from menu:
  - LOAD STEP OPTIONS, TIME FREQUENCY



  - DEFINE LOADS, APPLY , STRUCTURAL , TEMPERATURE, FROM THERMAL ANALYSIS



  - LOAD STEP OPTIONS, WRILE LS FILE , 1, (CRIA FICHEIRO FILE.S01)



HOW TO RUN FIRE ANALYSIS (THERMAL + MECHANICAL)

- Command lines to produce load step files.
  - TIME, 60
  - LDREAD,TEMP,1,1,,,'file','rth',' '
  - LSWRITE,1
  - TIME, 120
  - LDREAD,TEMP,1,2,,,'file','rth',' '
  - LSWRITE,2
  - …
  - …
  - LDREAD,TEMP,1,59,,,'file','rth',' '
  - LSWRITE,59
  - TIME,3600
  - LDREAD,TEMP,1,60,,,'file','rth',' '
  - LSWRITE,60

```
/COM,ANSYS RELEASE 12.0.1  UP20090415     14:33:19    06/10/2010
/NOPR
/TITLE,
_LSNUM=  1
ANTYPE, 0
NLGEOM, 1
…..
DELTIM,  60.0000000    , 1.000000000E-02,  60.0000000    ,
KBC,    0
KUSE,    0
TIME,  3600.0000000
TREF,  0.00000000
ALPHAD,  0.00000000
BETAD,  0.00000000
DMPRAT,  0.00000000
……
D,    56,UX  ,  0.00000000    ,  0.00000000
D,    56,UY  ,  0.00000000    ,  0.00000000
D,    56,UZ  ,  0.00000000    ,  0.00000000
…….
BFE,      1,TEMP,  1,  247.669359
BFE,      1,TEMP,  2,  162.498246
BFE,      1,TEMP,  3,  168.668701
BFE,      1,TEMP,  4,  251.629346

BF,    471,TEMP,  29.6566190
BF,    472,TEMP,  29.6607315
BF,    473,TEMP,  29.9317361
/GOPR
```

File.S60

HOW TO RUN FIRE ANALYSIS (THERMAL + MECHANICAL)

- GUI command from menu:
  - SOLUTION, SOLVE, FROM LS FILES