APPLIED MECHANICS II
MECHANICAL ENGINEERING

INSTITUTO POLITÉCNICO DE BRAGANÇA
Escola Superior de Tecnologia e Gestão

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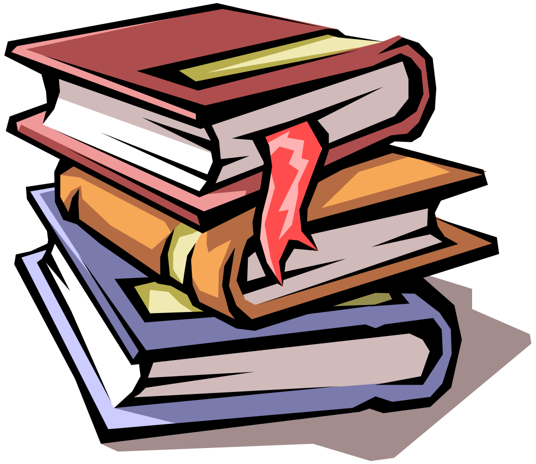
(english version)
05th December 2010

BIBLIOGRAPHY



- Apontamentos fornecidos pelo docente da disciplina
- Beer P. Ferdinand, Johnston Jr. Russel; ‘’Mecânica Vectorial para Engenheiros - Dinâmica”; - 6 edição; McGraw Hill.
- Meriam J.L., Kraige L.G.; “Engeneering Mechanics - Dymanics”, John Wiley & Sons, Inc.

REFERENCES

- Piloto, P.A.G.; Apontamentos da disciplina de Mecânica aplicada II; Estig; 1995.
- Beer P. Ferdinand, Johnston Jr. Russel; ‘’Mecânica Vectorial para Engenheiros - Dinâmica”; - 6 edição; McGraw Hill; electronic edition – instructors manual.

SYSTEM UNITS

Quantity	To change English Units	To Metric Units	Multiply English Units by
Length	Inch [in] Foot [ft] Mile [ml]	Millimeter [mm] Meter [m] Kilometer [km]	25,4 0,3948 1,6093
Area	Square foot [ft^2] Acre [a]	Square meter [m^2] Square meter [m^2]	0,0929 4046,8564929
Volume	Gallon [gal] Cubic foot [ft^3]	Liter [L] or [l] Cubic meter [m^3]	3,7854 0,0283
Pressure	psf [lb/ft^2] psi [lb/in^2]	Pa kPa	47,8803 6,8947
Weight	pound [lb]	kilogram [kg]	0,4536

Bureau International des Poids et Mesures

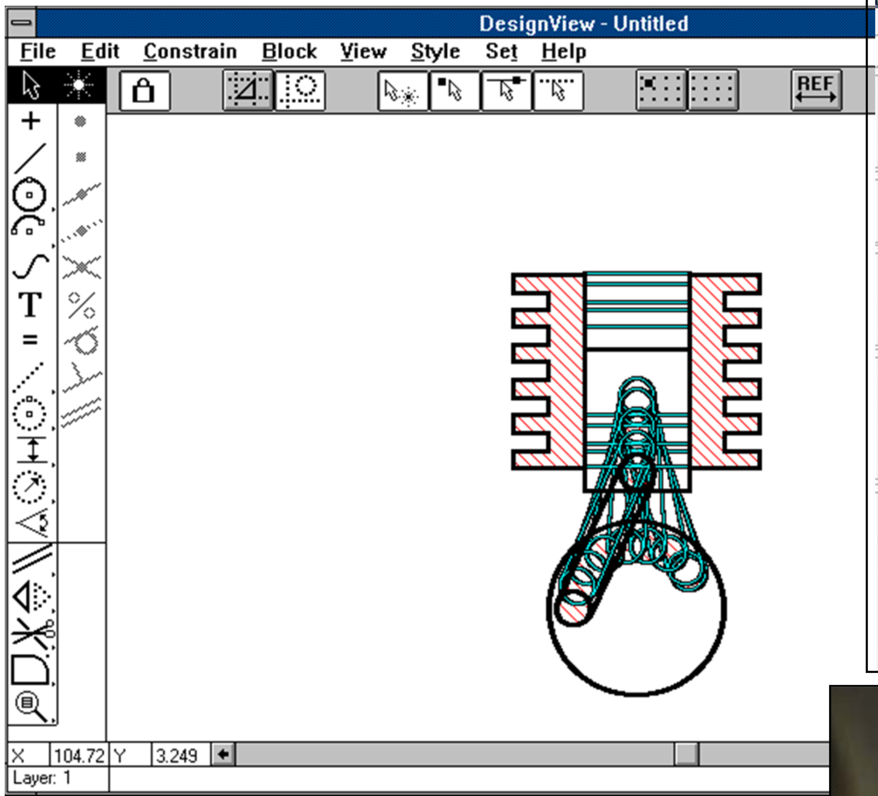
http://www.bipm.fr/enus/welcome.html

MULTIPLES AND SUB-MULTIPLES

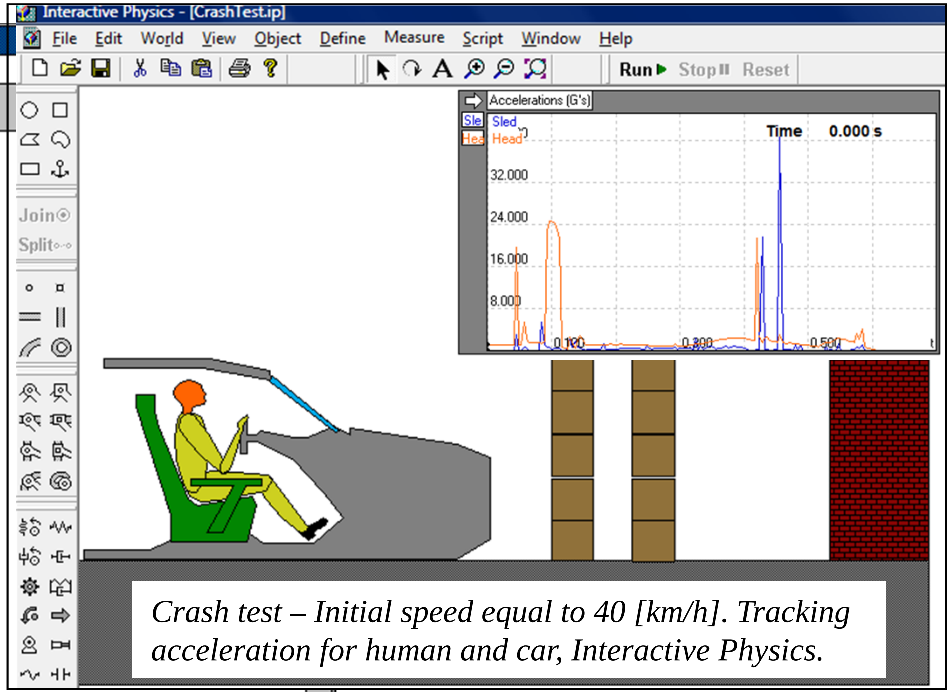
	Power base 10	Symbol	Designation
Multiples	10^{18}	E	exa
	10^{15}	P	peta
	10^{12}	T	tera
	10^{9}	G	giga
	10^{6}	M	mega
	10^{3}	k	kilo
	10^{2}	h	hecto
	10	da	deca
Submultiples	10^{-1}	d	deci
	10^{-2}	c	centi
	10^{-3}	m	mili
	10^{-6}	μ	micro
	10^{-9}	n	nano
	10^{-12}	p	pico
	10^{-15}	f	fento
	10^{-18}	a	ato

MECANISMS – TRAJECTORY

Rigid bodies, assembly together to produce movement.



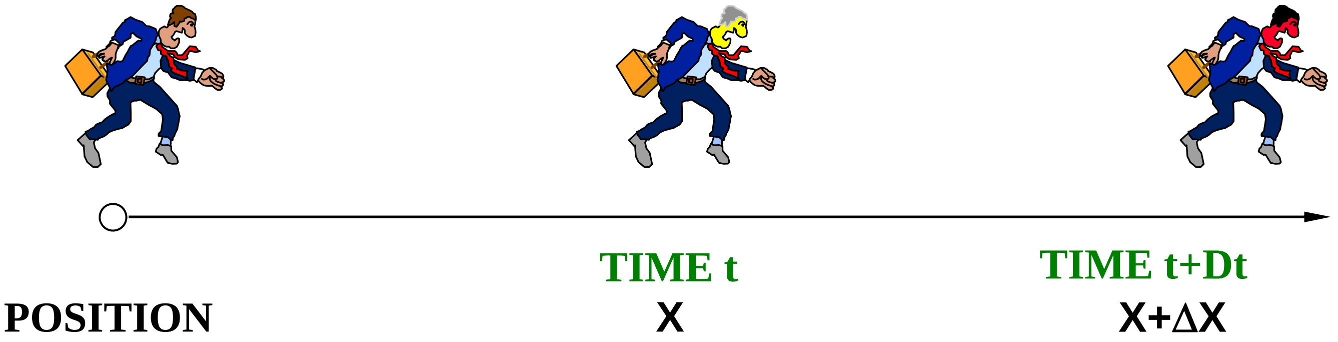
DesignView WAS a trade mark of Computer Vision



Crash test – Initial speed equal to 40 [km/h]. Tracking acceleration for human and car, Interactive Physics.

Accident Analysis & Reconstruction, Exponent Engineering

KINEMATIC
– Revision –



The motion of a particle along a straight line is termed ***rectilinear motion***. To define the position *P* of the particle on that line, we choose a fixed origin *O* and a positive direction. The distance *x* from *O* to *P*, with the appropriate sign, completely defines the position of the particle on the line and is called the ***position coordinate*** of the particle.

The ***velocity*** *v* of the particle is equal to the time derivative of the position coordinate *x*,

$$v = \frac{\Delta x}{\Delta t}[L / T] \qquad v = \lim_{\Delta t -> 0} \frac{\Delta x}{\Delta t}[L / T] = \frac{dx}{dt}$$

ACCELERATION

The ***acceleration a*** is obtained by differentiating ***v*** with respect to ***t***,

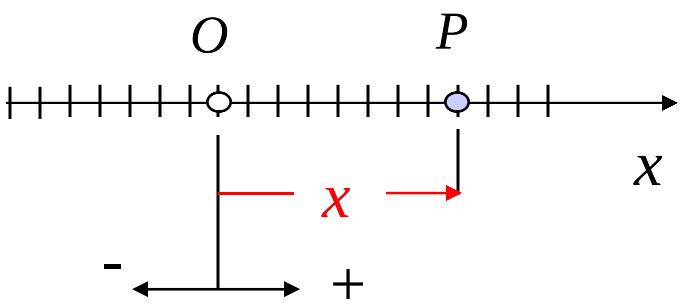
$$a = \lim_{\Delta t -> 0} \frac{\Delta v}{\Delta t} [L / T^2] = \frac{dv}{dt} \qquad a = \frac{dv}{dt} \quad \text{or} \quad a = \frac{d^2 x}{dt^2}$$

we can also express ***a*** as

$$a = \frac{dv}{dt} = \frac{dv}{dx}\frac{dx}{dt} = v\frac{dv}{dx}$$

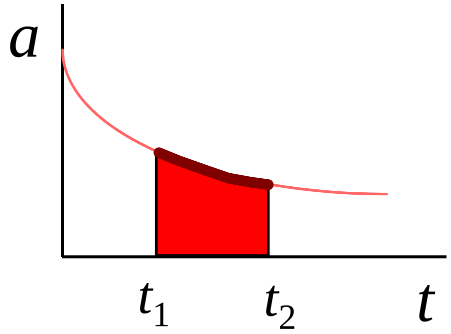
•The velocity v and acceleration a are represented by algebraic numbers which can be positive or negative. A positive value for v indicates that the particle moves in the positive direction, and a negative value that it moves in the negative direction.

•A positive value for a, however, may mean that the particle is truly accelerated (i.e., moves faster) in the positive direction, or that it is decelerated (i.e., moves more slowly) in the negative direction. A negative value for a is subject to a similar interpretation.



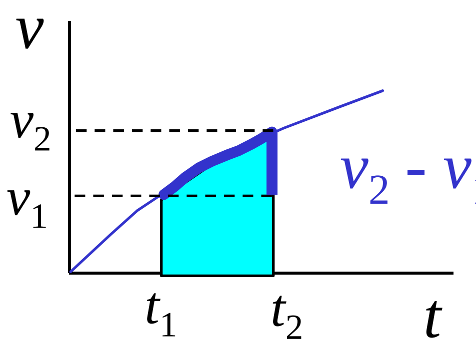
GRAPHICAL SOLUTION

Sometimes it is convenient to use a *graphical solution* for problems involving rectilinear motion of a particle. The graphical solution most commonly involves x - t, v - t , and a - t curves.



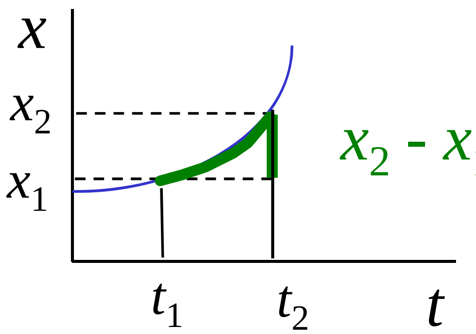
At any given time t,

v = slope of x - t curve
a = slope of v - t curve



$$v_2 - v_1 = \int_{t_1}^{t_2} a \, dt$$

while over any given time interval t_1 to t_2,



$$x_2 - x_1 = \int_{t_1}^{t_2} v \, dt$$

$v_2 - v_1$ = area under a - t curve
$x_2 - x_1$ = area under v - t curve

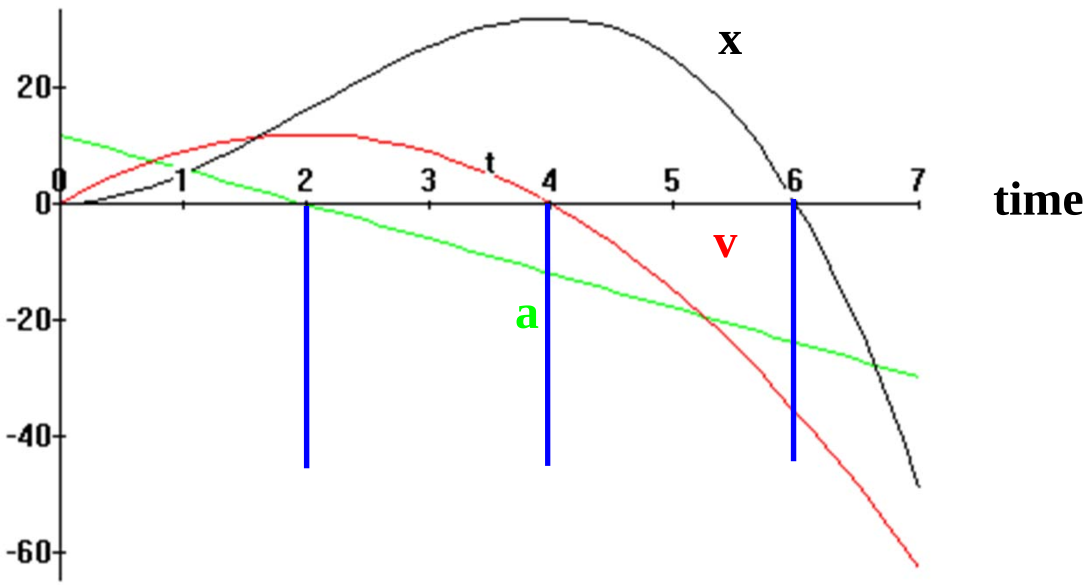
Thematic Exercise 1

Problem: When a point is moving through a straight line, its position is defined by: $x = 6t^2 - t^3$

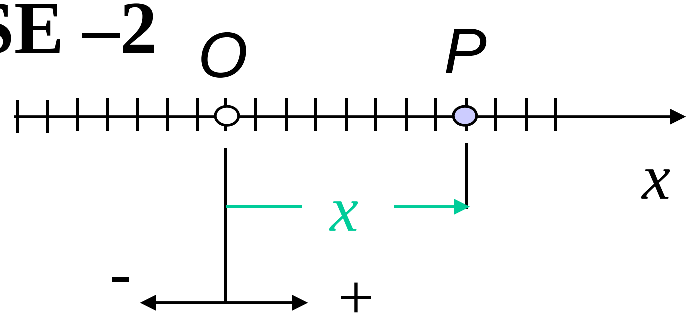
Calculate the instantaneous velocity and acceleration for all time instants.

$$v(t) = \frac{dx}{dt} = 12t - 3t^2$$

$$a(t) = \frac{dv(t)}{dt} = 12 - 6t$$



KINEMATIC EXERCISE –2



Problem: The position of a particle P when moving along a straight line is given by:

$$x = t^3 - 6t^2 - 15t + 40,\ \mathrm{t}[\mathrm{s}], x[m], t \geq 0$$

Calculate:

a) The time for the velocity to vanish.
b) The position and the displacement travel by the point to that instant.
c) The acceleration of that point at that instant.
d) The distance travel by the point from the position at the instant t=4[s] till the instant of t=6[s].

$$v = 3t^2 - 12t - 15\left[{}^{m}\!/\!{}_{s}\right]$$

$$a = 6t - 12\left[{}^{m}\!/\!{}_{s^2}\right]$$

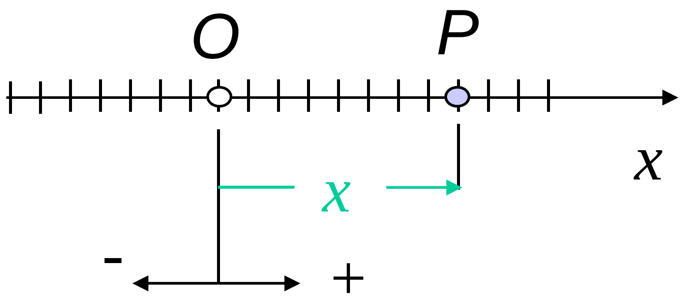
a) $3t^2 - 12t - 15 = 0 \Leftrightarrow t = -1 \wedge t = 5$

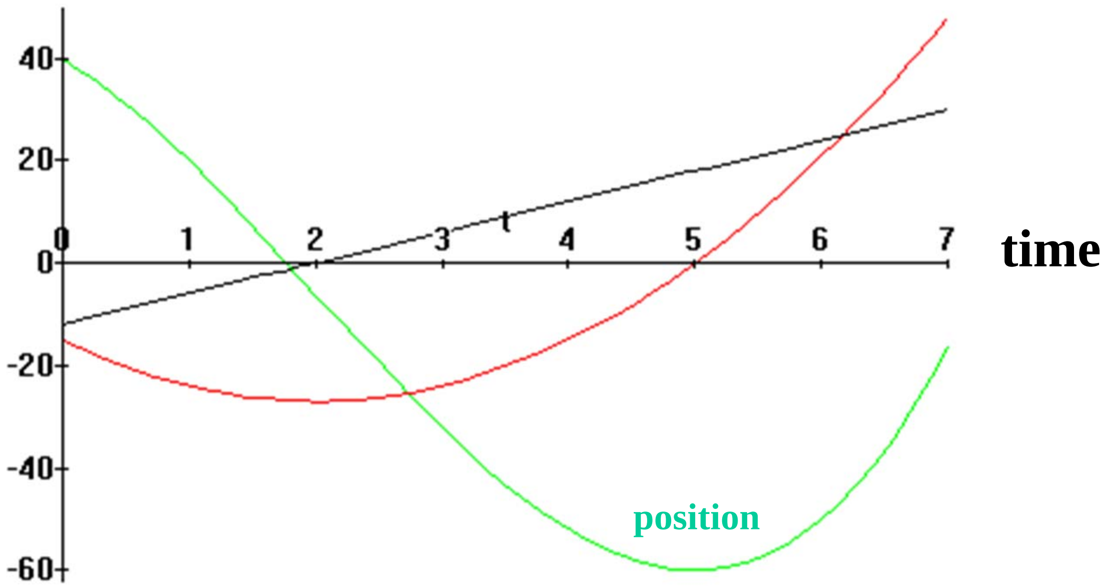
b) $\mathrm{x(t = 5) = \text{-}60(m)}$

Note: the changes to sign velocity should be cheked during the time interval

$\mathrm{x(t = 0) = 40\ (m)}$ $\mathrm{\Delta x(t = 0, t = 5) = 100\ (m)}$

KINEMATIC EXERCISE –2 resolution





c) $a(t = 5) = 18\,(ms^{-2})$

d) $x(t = 4) = -52,\ x(t = 5) = -60,\ x(t = 6) = -50 \Rightarrow total = 18(m)$

TYPES OF RECTILINEAR MOTION

uniform rectilinear motion, in which the velocity *v* of the particle is constant.

$$x = x_o + vt$$

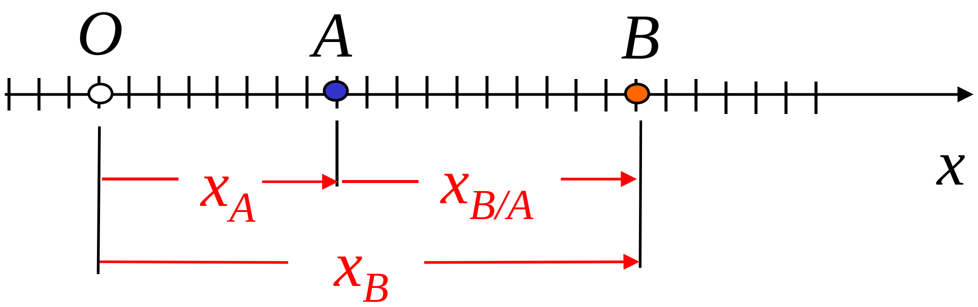
uniformly accelerated rectilinear motion, in which the acceleration a of the particle is constant.

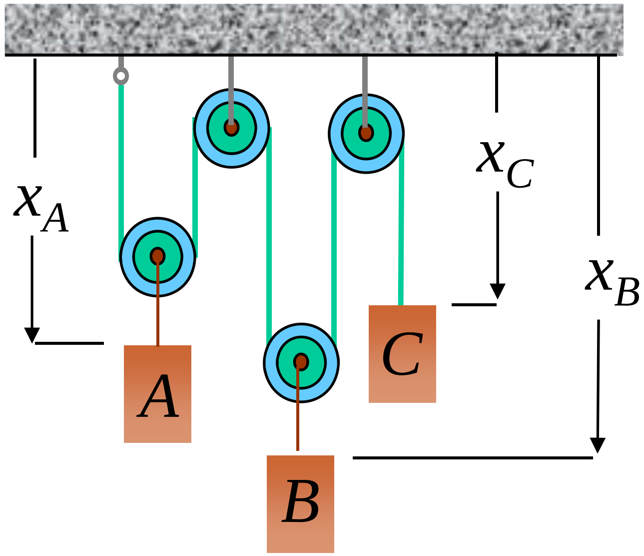
$$v = v_o + at$$

$$x = x_o + v_o t + \frac{1}{2} at^2$$

$$v^2 = v_o^2 + 2a(x - x_o)$$

RELATIVE MOTION





When particles A and B move along the same straight line, the relative motion of B with respect to A can be considered. Denoting by $x_{B/A}$ the relative position coordinate of B with respect to A , we have

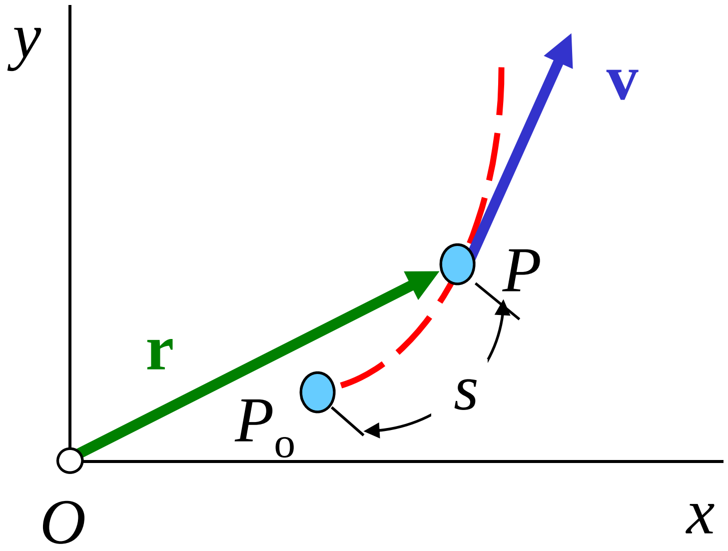
$$x_B = x_A + x_{B/A}$$

Differentiating twice with respect to t, we obtain:

$$v_B = v_A + v_{B/A} \qquad a_B = a_A + a_{B/A}$$

where $v_{B/A}$ and $a_{B/A}$ represent, respectively, the ***relative velocity*** and the ***relative acceleration*** of B with respect to A.

CURVILINEAR MOTION



The ***curvilinear motion of a particle*** involves particle motion along a curved path. The position *P* of the particle at a given time is defined by the ***position vector*** **r** joining the origin ***O*** of the coordinate system with the point ***P***.

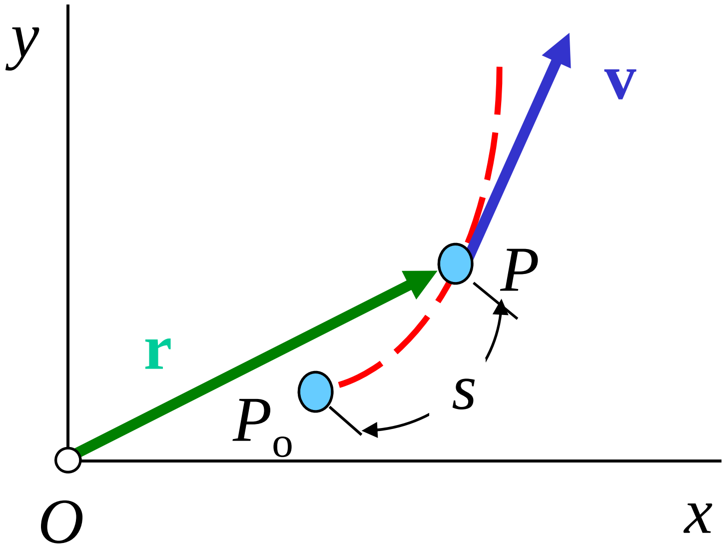
The velocity **v** of the particle is defined by the relation

$$\mathbf{v} = \frac{d\mathbf{r}}{dt}$$

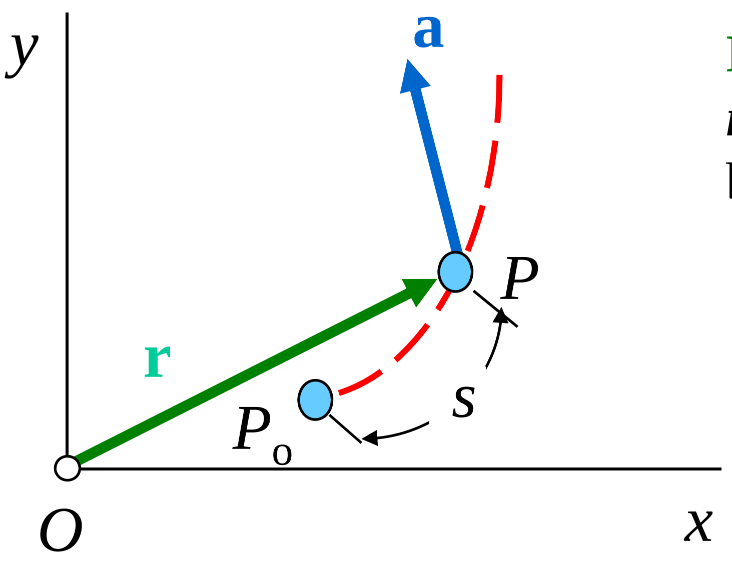
The ***velocity vector is tangent to the path of the particle***, and has a magnitude *v* equal to the time derivative of the length *s* of the arc described by the particle:

$$v = \frac{ds}{dt}$$

CURVILINEAR MOTION – cont.



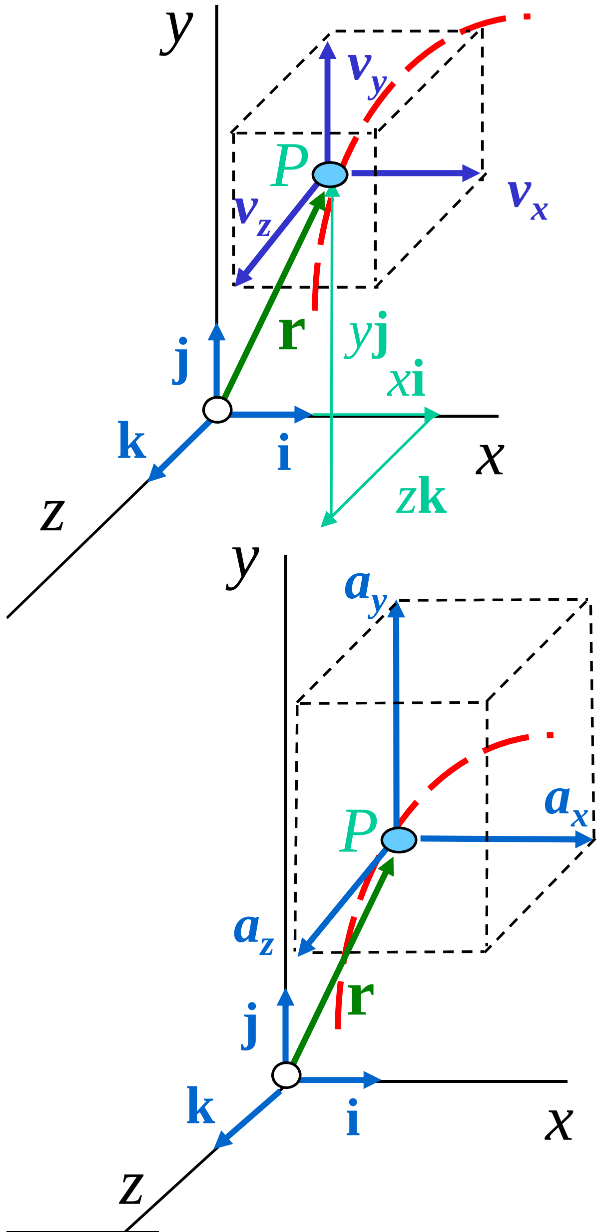
$$\mathbf{v} = \frac{d\mathbf{r}}{dt} \longrightarrow v = \frac{ds}{dt}$$



Note: In general, the acceleration **a** of the particle ***is not tangent to the path of the particle***. It is defined by the relation

$$\mathbf{a} = \frac{d\mathbf{v}}{dt}$$

GENERAL MOVEMENT DISCRIPTION



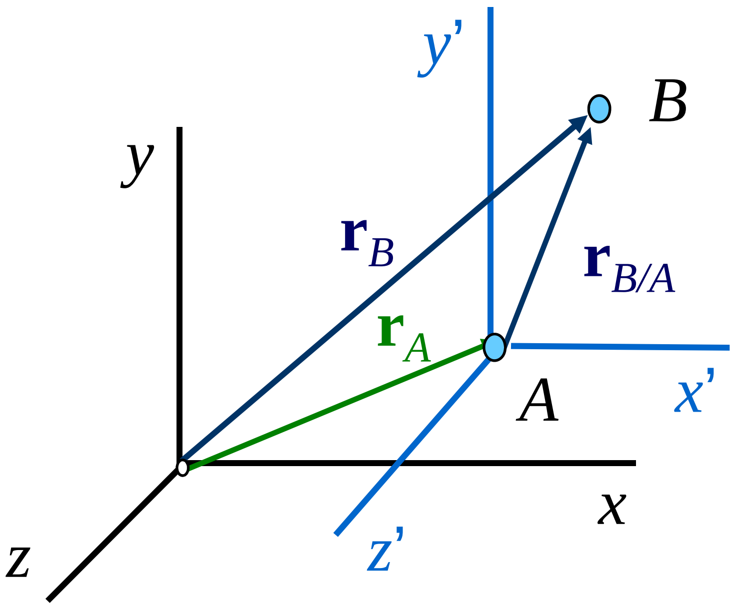
Denoting by *x*, *y*, and *z* the rectangular coordinates of a particle *P*, the rectangular components of velocity and acceleration of *P* are equal, respectively, to the first and second derivatives with respect to *t* of the corresponding coordinates:

$$v_x = \dot{x} \qquad v_y = \dot{y} \qquad v_z = \dot{z}$$

$$a_x = \ddot{x} \qquad a_y = \ddot{y} \qquad a_z = \ddot{z}$$

The use of rectangular components is particularly effective in the study of the motion of projectiles.

RELATIVE MOTION OF TWO PARTICLES



For two particles A and B moving in space, we consider the relative motion of B with respect to A , or more precisely, with respect to a moving frame attached to A and in translation with A. Denoting by $\mathbf{r}_{B/A}$ the relative position vector of B with respect to A , we have

$$\mathbf{r}_B = \mathbf{r}_A + \mathbf{r}_{B/A}$$

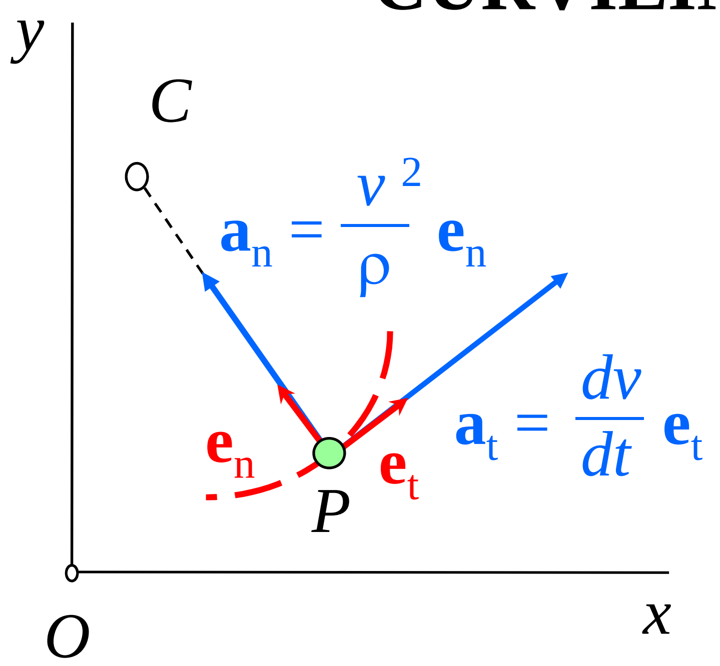
Denoting by $\mathbf{v}_{B/A}$ and $\mathbf{a}_{B/A}$, respectively, the ***relative velocity*** and the relative acceleration of B with respect to A, we also have

$$\mathbf{v}_B = \mathbf{v}_A + \mathbf{v}_{B/A}$$

and

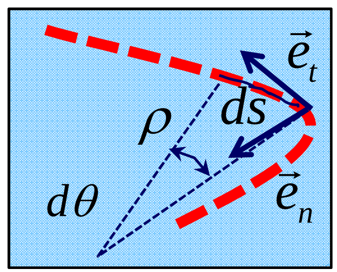
$$\mathbf{a}_B = \mathbf{a}_A + \mathbf{a}_{B/A}$$

CURVILINEAR COORDINATES



It is sometimes convenient to resolve the velocity and acceleration of a particle *P* into components other than the rectangular *x*, *y*, and *z* components. For a particle *P* moving along a path confined to a plane, we attach to *P* the unit vectors $\mathbf{e}_t$ tangent to the path and $\mathbf{e}_n$ normal to the path and directed toward the centre of curvature of the path.

The velocity and acceleration are expressed in terms of tangential and normal components. The velocity of the particle is $\mathbf{v} = v\mathbf{e}_t$



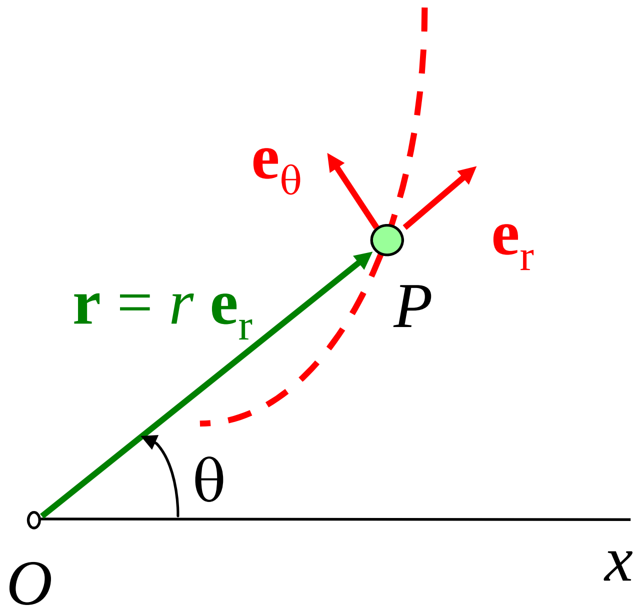
The acceleration is determined by time derivative:

$$\mathbf{a} = \frac{dv}{dt}\mathbf{e}_t + \frac{v^2}{\rho}\mathbf{e}_n$$

$$\frac{d}{dt}\vec{e}_t = \frac{d\vec{e}}{d\theta}\frac{d\theta}{ds}\frac{ds}{dt} = \vec{e}_n \frac{1}{\rho} v$$

Note:
- *v* is the speed of the particle
- ρ is the radius of curvature of its path.
- The velocity vector **v** is directed along the tangent to the path.
- The acceleration vector **a** consists of a component $\mathbf{a}_t$ directed along the tangent to the path and a component $\mathbf{a}_n$ directed toward the center of curvature of the path.

POLAR COORDINATES



When the position of a particle moving in a plane is defined by its polar coordinates r and θ, it is convenient to use radial and transverse components directed, respectively, along the position vector **r** of the particle and in the direction obtained by rotating **r** through 90º counterclockwise.

Unit vectors $\mathbf{e}_r$ and $\mathbf{e}_\theta$ are attached to *P* and are directed in the radial and transverse directions. The velocity and acceleration of the particle in terms of radial and transverse components is:

$$\vec{r} = r \cdot \vec{e}_r$$

$$\vec{v} = \dot{r} \cdot \vec{e}_r + r \frac{d\vec{e}_r}{dt} = \dot{r} \cdot \vec{e}_r + r \frac{d\vec{e}_r}{d\theta} \frac{d\theta}{dt} = \dot{r} \cdot \vec{e}_r + r \cdot \vec{e}_\theta \cdot \dot{\theta}$$

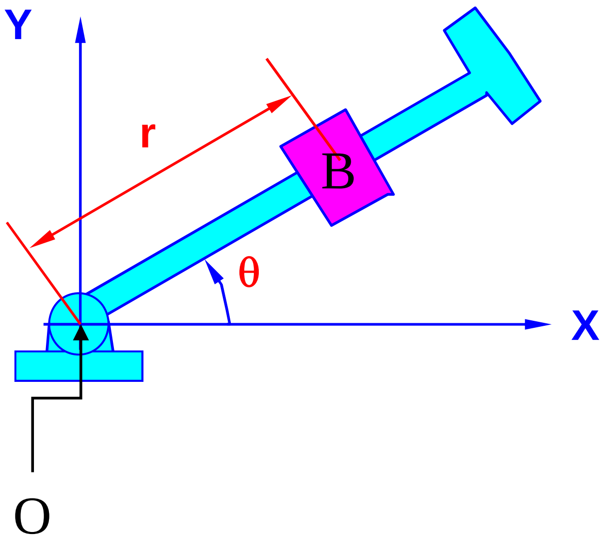
$$\vec{a} = \ddot{r} \cdot \vec{e}_r + \dot{r} \cdot \frac{d\vec{e}_r}{dt} + \dot{r} \cdot \vec{e}_\theta \cdot \dot{\theta} + r \cdot \frac{d\vec{e}_\theta}{dt} \cdot \dot{\theta} + r \cdot \vec{e}_\theta \cdot \frac{d\dot{\theta}}{dt}$$

$$= \ddot{r} \cdot \vec{e}_r + \dot{r} \cdot \dot{\theta} \cdot \vec{e}_\theta + \dot{r} \cdot \dot{\theta} \cdot \vec{e}_\theta + r \cdot \dot{\theta} \cdot \left(-\vec{e}_r\right) \cdot \dot{\theta} + r \cdot \vec{e}_\theta \cdot \ddot{\theta}$$

$$= \left(\ddot{r} - r\dot{\theta}^2\right) \cdot \vec{e}_r + \left(r \cdot \ddot{\theta} + 2 \cdot \dot{r} \cdot \dot{\theta} \cdot\right) \cdot \vec{e}_\theta$$

Note: It is important to note that a_r is ***not*** equal to the time derivative of v_r, and that a_θ is ***not*** equal to the time derivative of v_θ.

RELATIVE MOTION – Thematic Exercise 2



Problem:

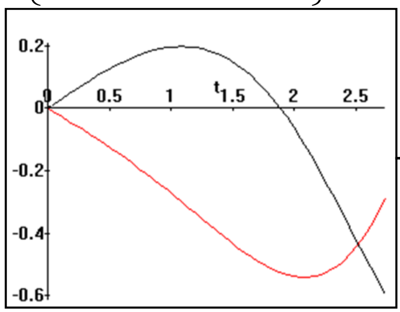
- The rotating 0.9 [m] arm length turns around the point O with the constrained known expression:
  $\theta = 1{,}5 \times 10^{-1} t^2$
- The cursor B travels along the arm being its movement described by the following expression:
- Calculate : $r = 0{,}9 - 0{,}12t^2$

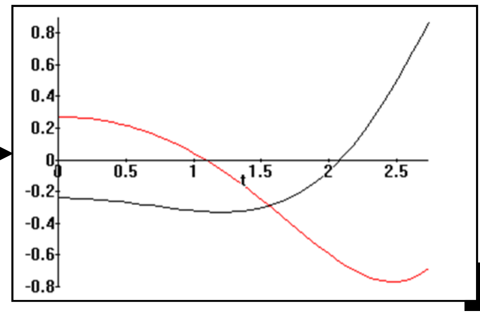
a) the expressions for the instantaneous position, velocity and acceleration of the cursor B.

b) The velocity and accelerations from the cursor B, after rotating the arm 30º.

Resolution in Cartesian coordinates:

$$\vec{r} = (r\cos\theta)\vec{i} + (r\sin\theta)\vec{j} \longrightarrow \vec{v} = \begin{Bmatrix} \dot{r}\cos\theta - r\dot{\theta}\sin\theta \\ \dot{r}\sin\theta + r\dot{\theta}\cos\theta \end{Bmatrix} \longrightarrow \vec{a} = \begin{Bmatrix} \ddot{r}\cos\theta - 2\dot{r}\dot{\theta}\sin\theta - r\dot{\theta}^2\cos\theta - r\ddot{\theta}\sin\theta \\ \ddot{r}\sin\theta + 2\dot{r}\dot{\theta}\cos\theta - r\dot{\theta}^2\sin\theta + r\ddot{\theta}\cos\theta \end{Bmatrix}$$





RELATIVE MOTION – resolution

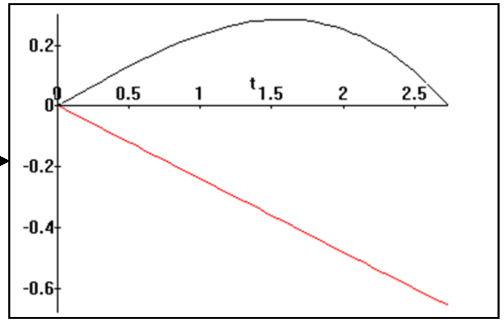
Resolution in Polar coordinates:

$$\vec{r} = r\vec{e}_r$$

$$\vec{v} = \dot{r}\vec{e}_r + r\dot{\theta}\,\vec{e}_\theta$$

$$= \left(-0.24\cdot t\right)\vec{e}_r + \left(0.9-0.12\cdot t^2\right)\left(0.3\cdot \mathrm{t}\right)\vec{e}_\theta$$

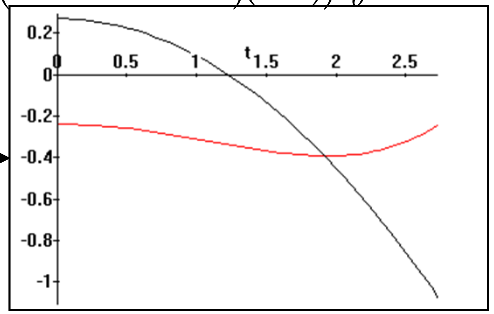
$$= \left(-0.24\cdot t\right)\vec{e}_r + \left(0.27\cdot t-0.036\cdot t^3\right)\vec{e}_\theta$$



$$\vec{a} = \left(\ddot{r}-r\dot{\theta}^2\right)\vec{e}_r + \left(2\dot{r}\dot{\theta}+r\ddot{\theta}\right)\vec{e}_\theta$$

$$= \left(-0.24-\left(0.9-0.12\cdot t^2\right)\left(0.3\cdot t\right)\right)\vec{e}_r + \left(2\left(-0.24\cdot t\right)\left(0.3\cdot t\right)+\left(0.9-0.12\cdot t^2\right)\left(0.3\right)\right)\vec{e}_\theta$$

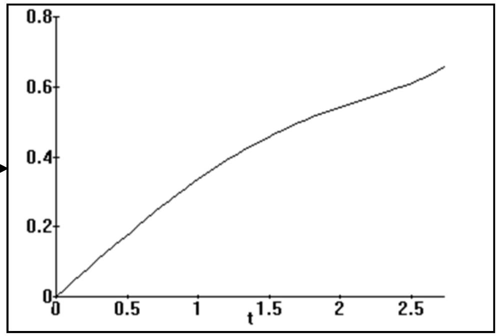
$$= \left(-0.24-0.27\cdot t+0.036\cdot t^3\right)\vec{e}_r + \left(-0.180\cdot t^2+0.27\right)\vec{e}_\theta$$



Scalar velocity and acceleration results:

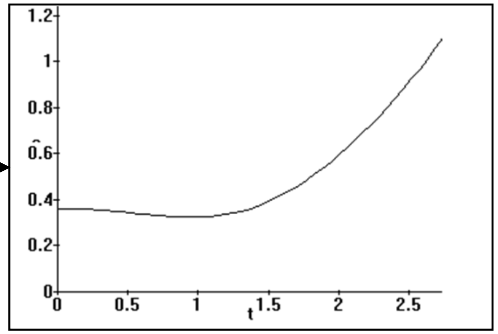
$$v = \sqrt{v_x^2+v_y^2}$$

$$v = \sqrt{v_r^2+v_\theta^2}$$



$$a = \sqrt{a_x^2+a_y^2}$$

$$a = \sqrt{a_r^2+a_\theta^2}$$



NEWTON'S SECOND LAW (linear momentum)

Denoting by *m* the mass of a particle, by Σ **F** the sum, or resultant, of the forces acting on the particle, and by **a** the acceleration of the particle relative to a *newtonian frame of reference*, we write:

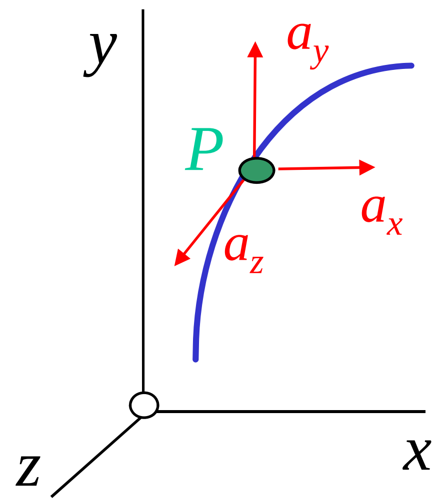
$$\sum \vec{F} = m\vec{a}$$

Introducing the *linear momentum* of a particle, **L** = *m***v**, Newton's second law can also be written as

$$\sum \vec{F} = \dot{\vec{L}}$$

which expresses that: ***the resultant of the forces acting on a particle is equal to the rate of change of the linear momentum of the particle.***

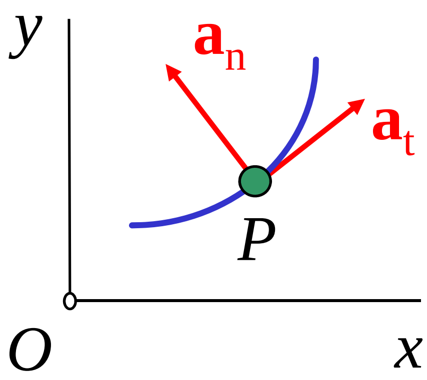
SOLVING A PROBLEM



To solve a problem involving the motion of a particle, $\Sigma \mathbf{F} = m\mathbf{a}$ should be replaced by equations containing scalar quantities.

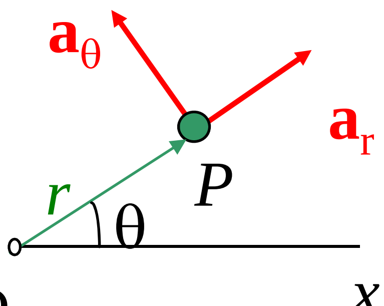
Using *rectangular* or *cartesian* components,

$$\Sigma F_x = ma_x \qquad \Sigma F_y = ma_y \qquad \Sigma F_z = ma_z$$



Using *tangential* and *normal* components,

$$\Sigma F_t = ma_t = m\frac{dv}{dt} \qquad \Sigma F_n = ma_n = m\frac{v^2}{\rho}$$

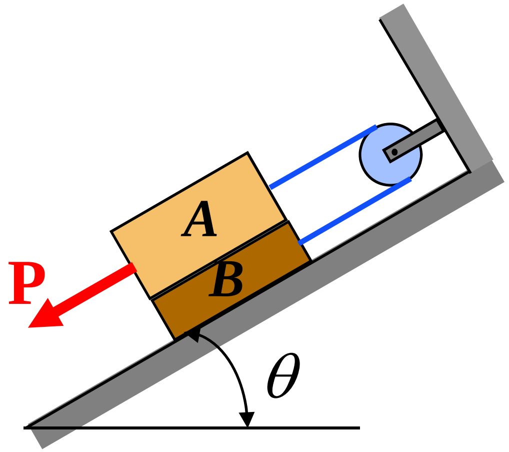


Using *radial* and *transverse* components,

$$\Sigma F_r = ma_r = m(\ddot{r} - r\dot{\theta}^2)$$

$$\Sigma F_\theta = ma_\theta = m(r\ddot{\theta} + 2\dot{r}\dot{\theta})$$

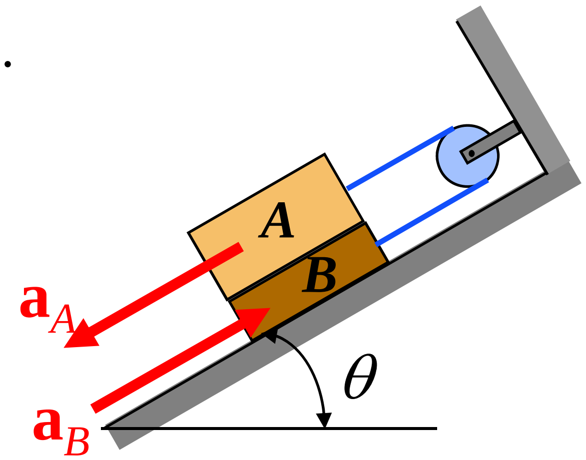
PROBLEM 12.123 – Thematic Exercise 3



Block A has a mass of 30 kg and block B has a mass of 15 kg. The coefficients of friction between all plane surfaces of contact are $\mu_s = 0.15$ and $\mu_k = 0.10$.
Knowing that $\theta = 30^o$ and that the magnitude of the force **P** applied to block A is 250 N, determine:
(a) the acceleration of block A ;
(b) the tension in the cord.

1. ***Kinematics:*** Examine the acceleration of the particles.

Assume motion with block A moving down. If block A moves and accelerates down the slope, block B moves up the slope with the same acceleration.

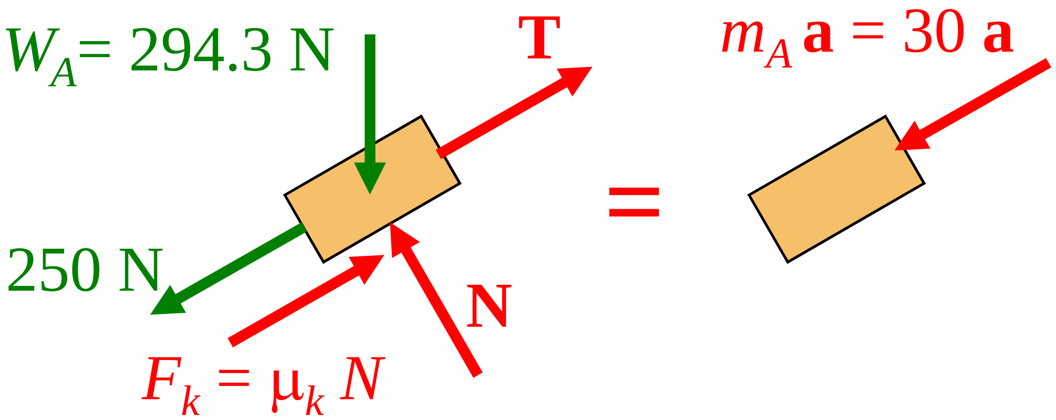


$$a_A = a_B$$

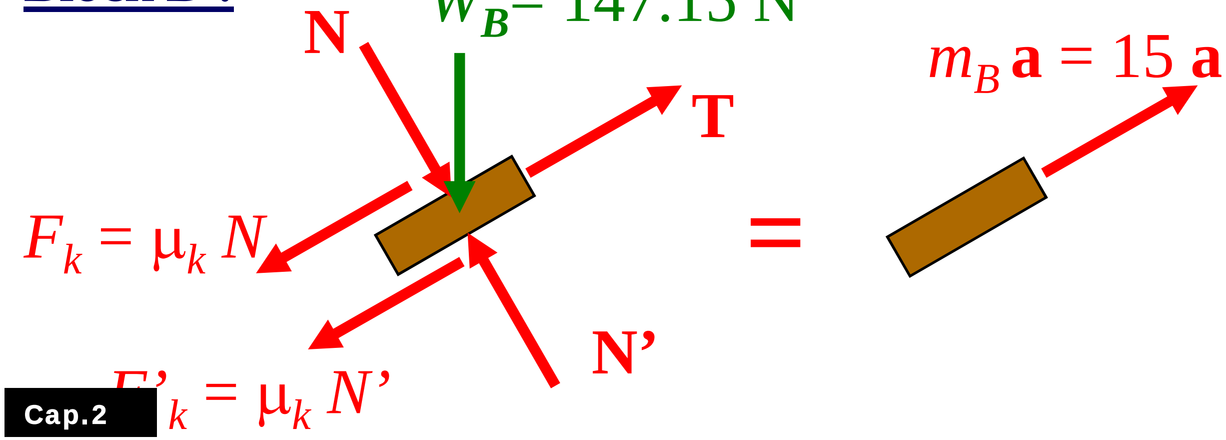
PROBLEM 12.123 (cont.)

2. ***Kinetics:*** Draw a free body diagram showing the applied forces and an equivalent force diagram showing the vector *m***a** or its components.

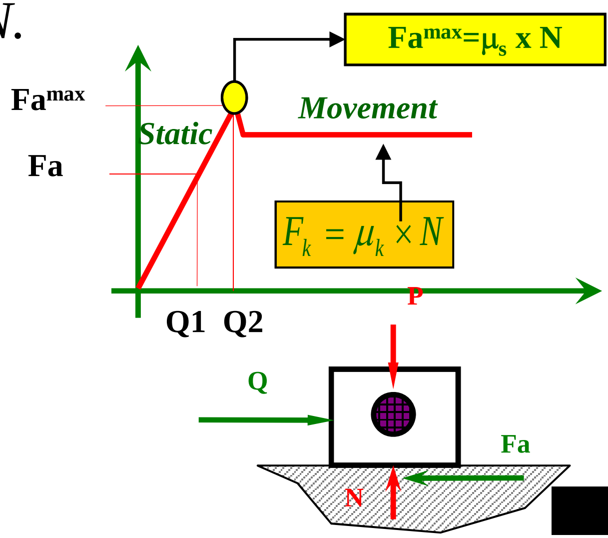
Block *A* :



Block *B* :



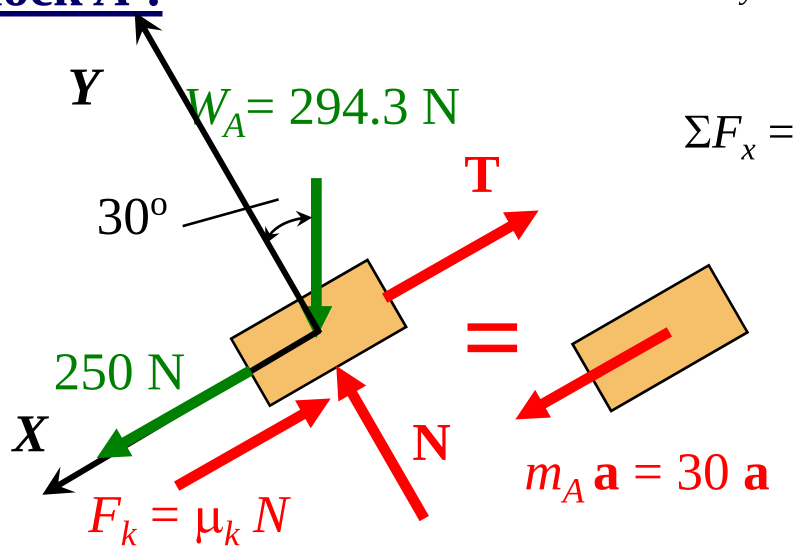
3. ***When a problem involves dry friction:*** It is necessary first to assume a possible motion and then to check the validity of the assumption. The friction force on a moving surface is $F = \mu_k N$. The friction force on a surface when motion is impending is $F = \mu_s N$.



PROBLEM 12.123 (cont.)

4. *Apply Newton's second law:* The relationship between the forces acting on the particle, its mass and acceleration is given by $\Sigma \mathbf{F} = m\,\mathbf{a}$. The vectors **F** and **a** can be expressed in terms of either their rectangular components or their tangential and normal components. Absolute acceleration (measured with respect to a <u>*newtonian frame of reference*</u>) should be used.

Block *A* :



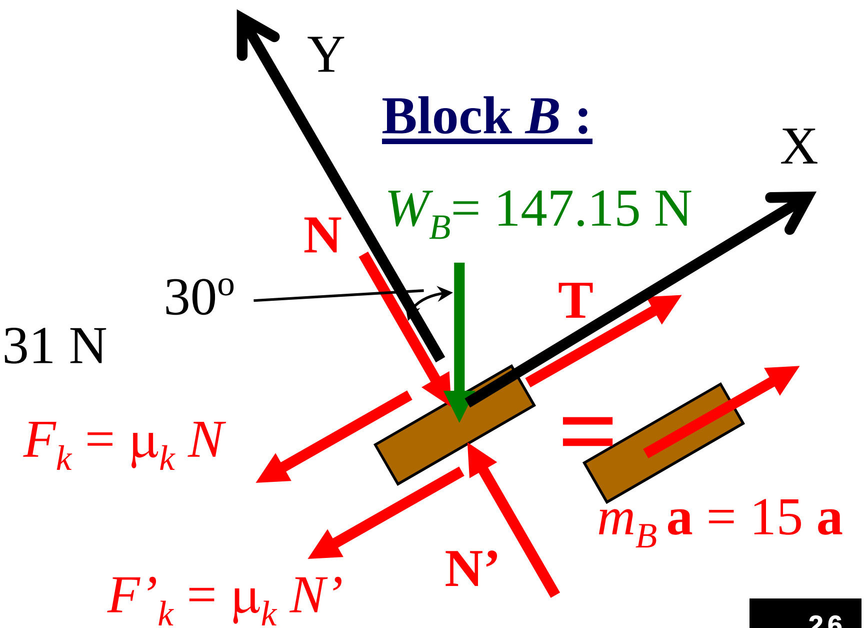
$\Sigma F_y = 0$: N - (294.3) cos 30º = 0 N = 254.87 N

then: $F_k = \mu_k N$ = 0.10 (254.9) = 25.49 N

$\Sigma F_x = ma$: 250 + (294.3) sin 30º - 25.49 - T = 30 a

then: 371.66 - T = 30 a (1)

Block *B* :



$\Sigma F_y = 0$: N' - N - (147.15) cos 30º = 0 : N' = 382.31 N

then: $F'_k = \mu_k N'$ = 0.10 (382.31) = 38.23 N

$\Sigma F_x = ma$: $T - F_k - F'_k$ - (147.15) sin 30º = 15 a

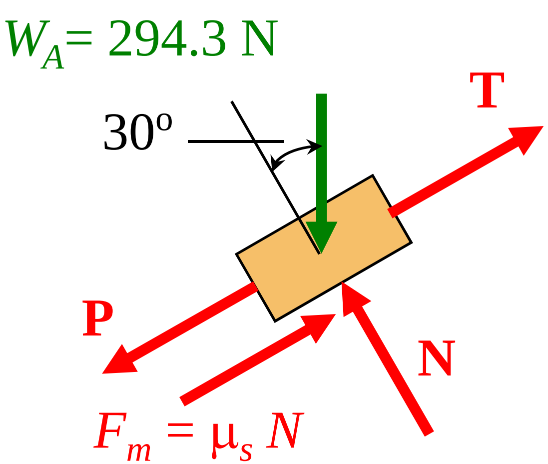
then: T - 137.29 = 15 a (2)

PROBLEM 12.123 - solution

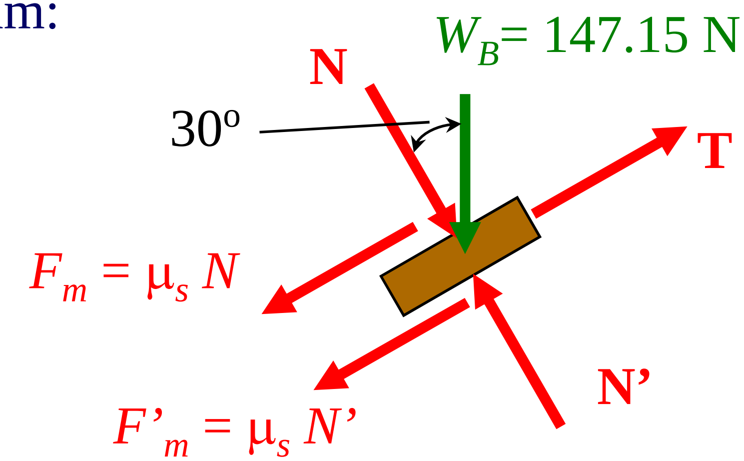
Solving equations (1) and (2) gives: $T = 215$ [N] $a = 5.21$ [m/s^2]

Verify assumption of motion.

<u>Check:</u> We should verify that blocks actually move by determining the value of the force **P** for which motion is impending. Find **P** for impending motion. For impending motion both blocks are in equilibrium:



<u>Block *A:*</u>   **<u>Block *B:*</u>**



From Static equilibrium:

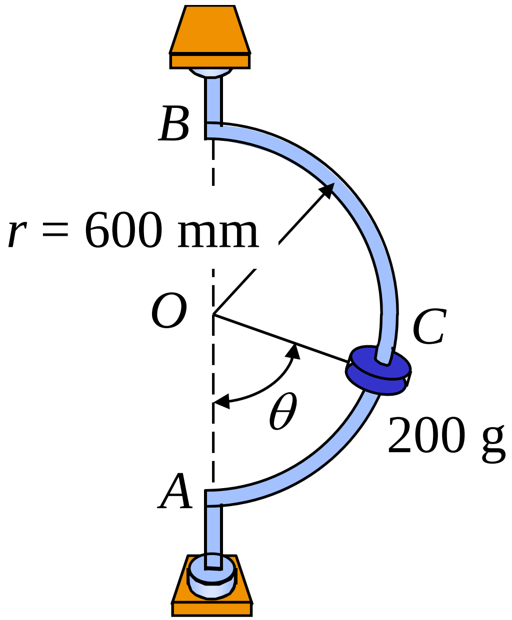
$\Sigma F_y = 0$: $N = 254.87$ N $\rightarrow F_m = \mu_s N = 0.15\,(254.87) = 38.23$ N

$\Sigma F_y = 0$: $N' = 382.31$ N $\rightarrow F'_m = \mu_s N' = 0.15\,(382.31) = 57.35$ N

$\Sigma F_x = 0$: $P + (294.3) \sin 30^\circ - 38.23 - T = 0$ (3)

$\Sigma F_x = 0$: $T - 38.23 - 57.35 - (147.15) \sin 30^\circ = 0$ (4)

Solving equations (3) and (4) gives *P* = 60.2 N. Since the actual value of *P* (250 N) is larger than the value for impending motion (60.2 N), motion takes place as assumed.

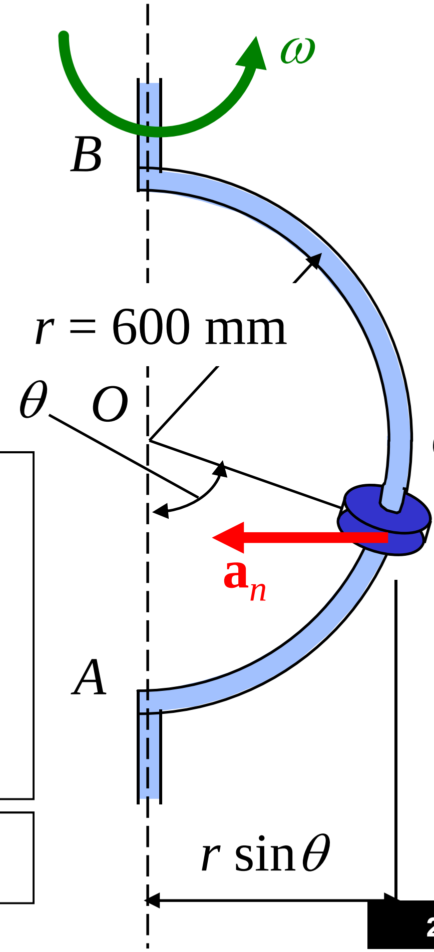


PROBLEM 12.127

A small 200-g collar *C* can slide on a semicircular rod which is made to rotate about the vertical *AB* at the constant rate of 6 [rad/s]. Determine the minimum required value of the coefficient of static friction between the collar and the rod if the collar is not to slide when:

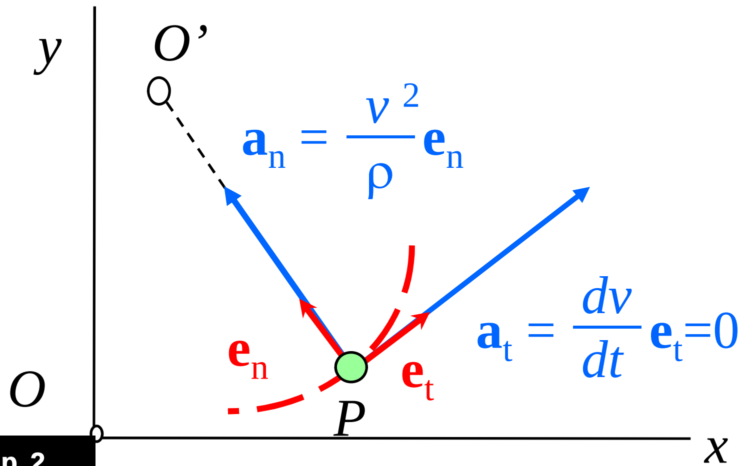
(a) $\theta = 90^o$, (b) $\theta = 75^o$, (c) $\theta = 45^o$.

Indicate in each case the direction of the impending motion.



1. *Kinematics*: Determine the acceleration of the particle.

Using curvilinear coordinates:



$$a_n = (r \sin\theta)\,\omega^2$$

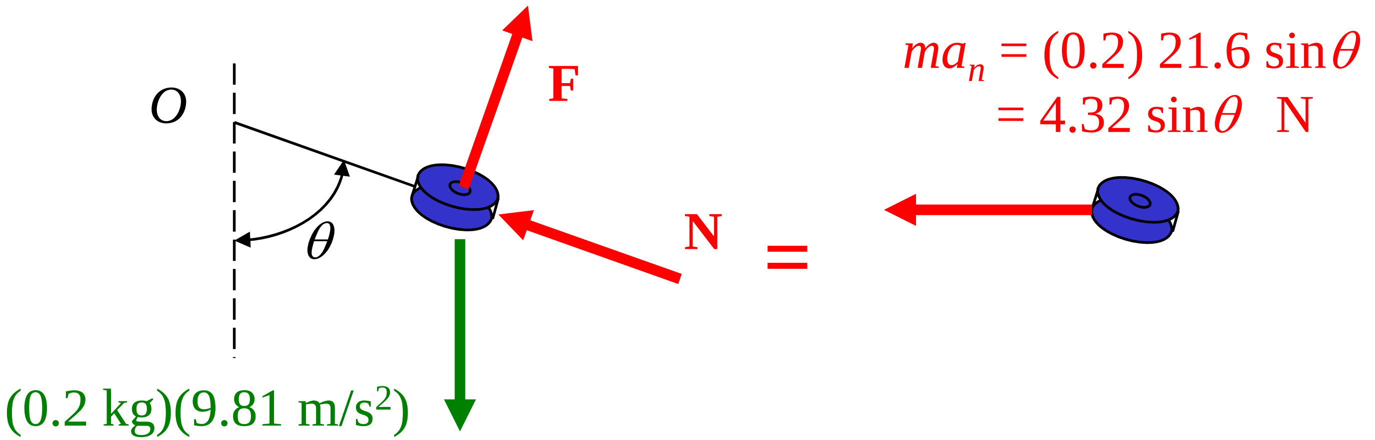
$$a_n = (0.6\text{ m}) \sin\theta\,(\,6\text{ rad/s}\,)^2$$

$$a_n = 21.6 \sin\theta \quad [\text{m/s}^2]$$

$$a_t = 0$$

PROBLEM 12.127 (cont.)

2. *Kinetics:* Draw a free body diagram showing the applied forces and an equivalent force diagram showing the vector *m***a** or its components.



3. *Apply Newton's second law:* The relationship between the forces acting on the particle, its mass and acceleration is given by $\Sigma\, \mathbf{F} = m\, \mathbf{a}$. The vectors **F** and **a** can be expressed in terms of either their rectangular components or their tangential and normal components. Absolute acceleration (measured with respect to a Newtonian frame of reference) should be used.

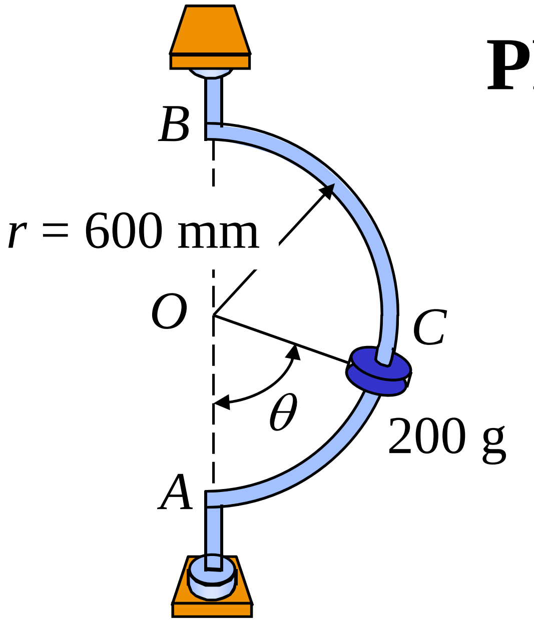
$$\Sigma F_t = ma_t: F - 0.2\,(9.81)\sin\theta = -\,4.32\sin\theta\cos\theta \Leftrightarrow F = 0.2\,(9.81)\sin\theta - 4.32\sin\theta\cos\theta$$

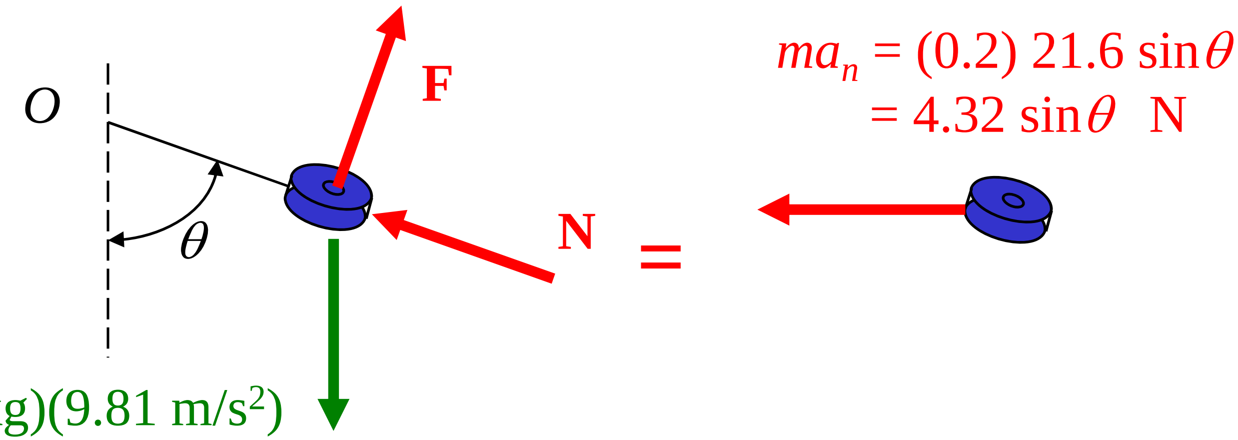
$$\Sigma F_n = ma_n: \quad N - 0.2\,(9.81)\cos\theta = 4.32\sin\theta\sin\theta \Leftrightarrow N = 0.2\,(9.81)\cos\theta + 4.32\sin^2\theta$$

4. *Friction law:* $F = \mu N$

Note: For a given θ, the values of F , N , and μ can be determined!!!

PROBLEM 12.127 (solution)





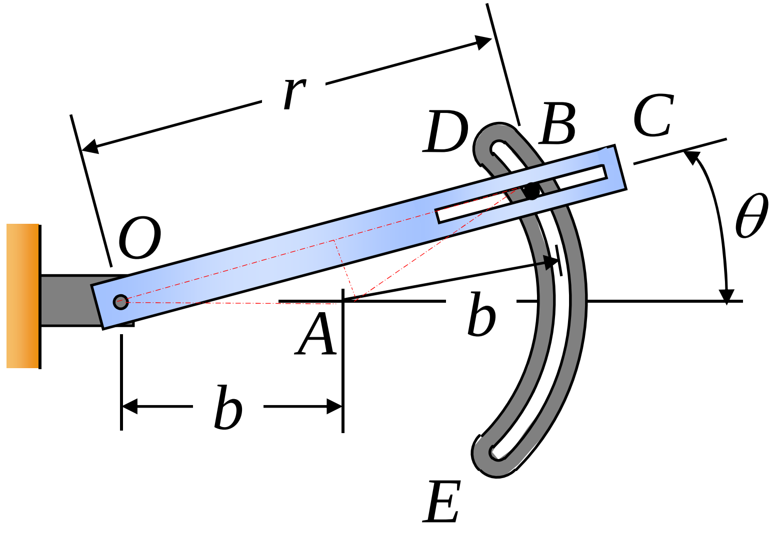
Solution:

(a) $\theta = 90^o$, $F = 1.962$ N, $N = 4.32$ N, $\mu = 0.454$ (down)

(b) $\theta = 75^o$, $F = 0.815$ N, $N = 4.54$ N, $\mu = 0.1796$ (down)

(c) $\theta = 45^o$, $F = -0.773$ N, $N = 3.55$ N, $\mu = 0.218$ (up)

PROBLEM 12.128- Thematic exercise 4



Pin B weighs 4 oz and is free to slide in a horizontal plane along the rotating arm OC and along the circular slot DE of radius b = 20 in. Neglecting friction and assuming that $\dot{\theta} = 15$ rad/s and $\ddot{\theta} = 250$ rad/s^2 for the position $\theta = 20^o$, determine:

(a) the radial and transverse components of the resultant force exerted on pin B;

(b) the forces **P** and **Q** exerted on pin B, respectively, by rod OC and the wall of slot DE.

1. Kinematics: Examine the velocity and acceleration of the particle.

In polar coordinates:

$$\mathbf{v} = \dot{r}\,\mathbf{e}_r + r\,\dot{\theta}\,\mathbf{e}_\theta$$

$$\mathbf{a} = (\ddot{r} - r\,\dot{\theta}^2)\,\mathbf{e}_r + (r\,\ddot{\theta} + 2\,\dot{r}\,\dot{\theta})\,\mathbf{e}_\theta$$

$\theta = 20^o$

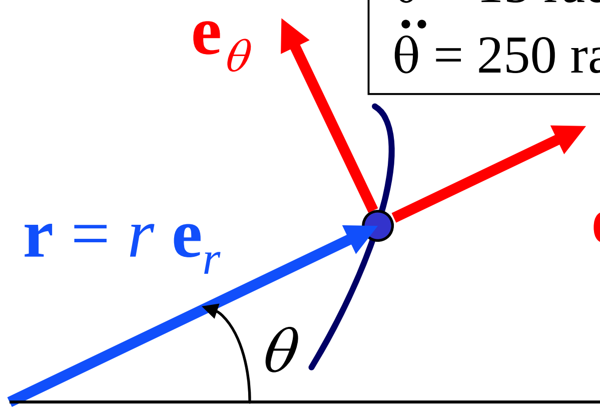
$\dot{\theta} = 15$ rad/s

$\ddot{\theta} = 250$ rad/s^2

$r = 2\,b\cos\theta = 3.13$ [ft]

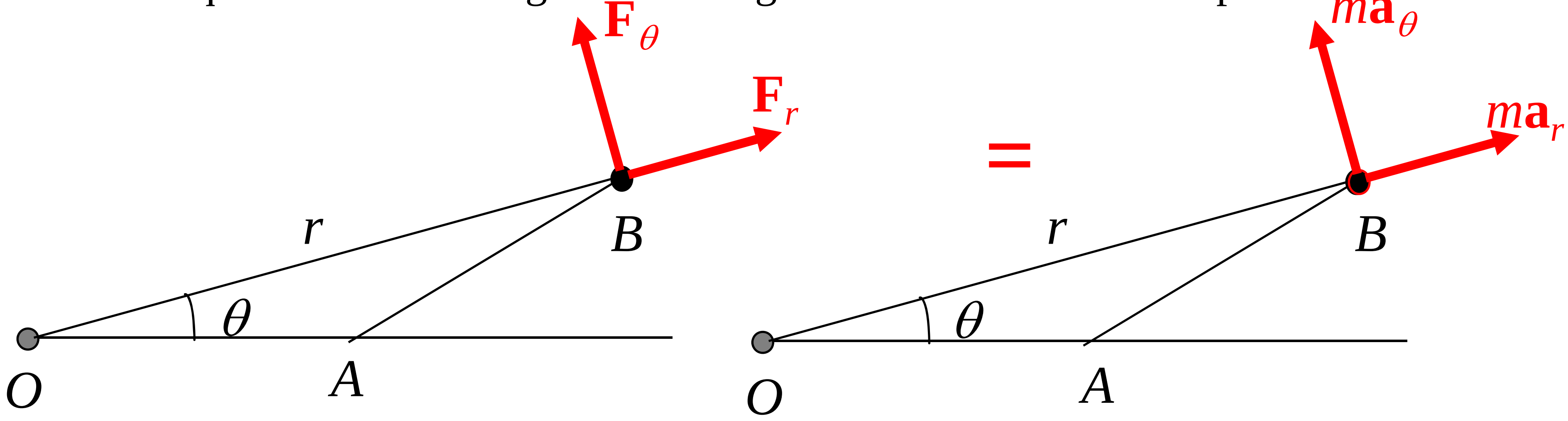
$\dot{r} = -\,2\,b\sin\theta\,\dot{\theta} = -\,17.1$ [ft/s]

$\ddot{r} = -\,2\,b\sin\theta\,\ddot{\theta} - 2\,b\cos\theta\,\dot{\theta}^2 = -\,989.79$ [ft/s^2]



PROBLEM 12.128 (cont.)

2. *Kinetics:* Draw a free body diagram showing the applied forces on pin B and an equivalent force diagram showing the vector $m\mathbf{a}$ or its components.



3. *Apply Newton's second law:* The relationship between the forces acting on the particle, its mass and acceleration is given by $\Sigma \mathbf{F} = m\,\mathbf{a}$. The vectors **F** and **a** can be expressed in terms of either their rectangular components or their radial and transverse components. With radial and transverse components:

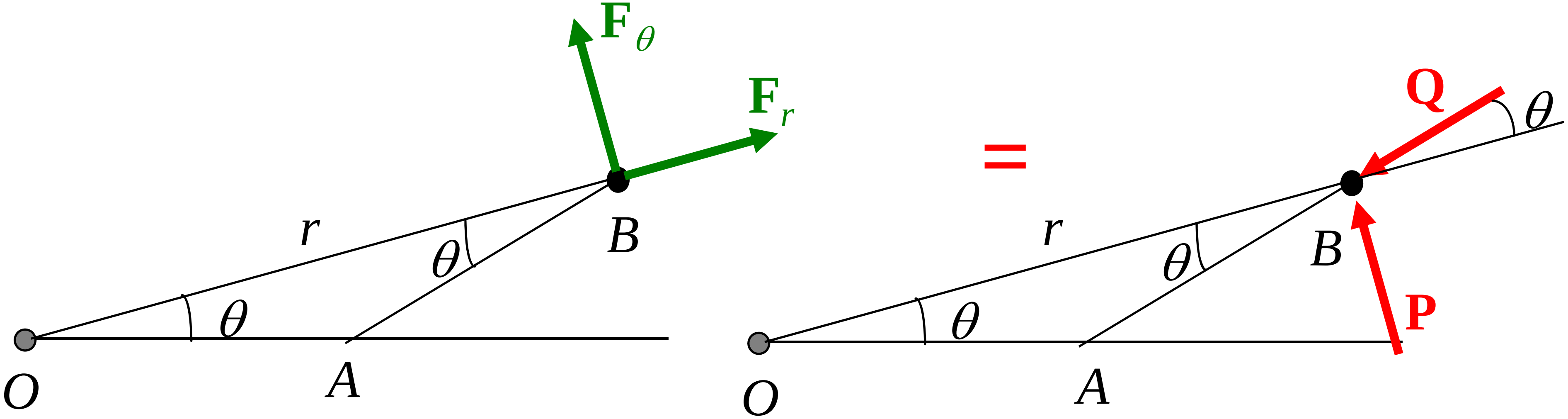
$\Sigma F_r = m\,a_r = m\,(\ddot{r} - r\,\dot{\theta}^2)$ and $\Sigma F_\theta = m\,a_\theta = m\,(r\,\ddot{\theta} + 2\,\dot{r}\,\dot{\theta})$

F_r=(4/16)/32.2* [- 989.79 - (3.13)(15^2)] **and** F_θ = (4/16)/32.2* [(3.13)(250) + 2 (-17.1)(15)]

F_r = -13.16 [lb] and F_θ = 2.10 [lb]

PROBLEM 12.128 (Solution)

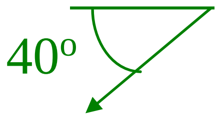
(b) The forces exerted on pin B by both bodies are obtained by vector decomposition



$F_r = -\,Q\cos\theta$

$-13.16 = -\,Q\cos 20^{o}$

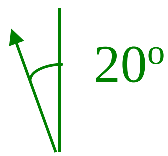
$Q = 14.00$ lb



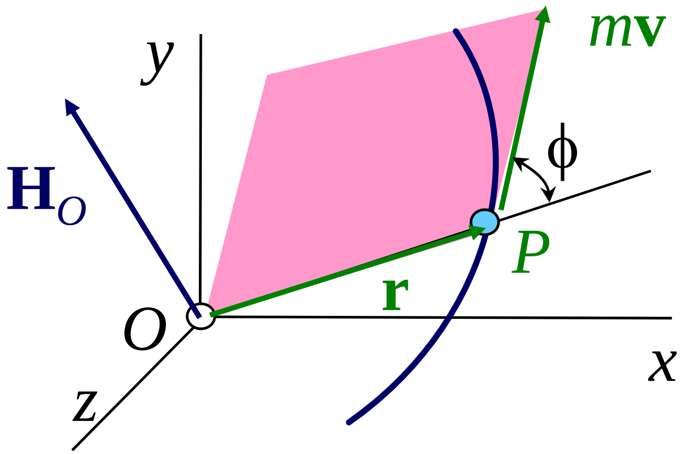
$F_\theta = -\,Q\sin\theta + P$

$2.10 = -\,14.0\sin 20^{o} + P$

P = 6.89 lb



NEWTON'S SECOND LAW (angular momentum)



The *angular momentum* $\mathbf{H}_O$ of a particle about point O is defined as the moment about O of the linear momentum $m\mathbf{v}$ of that particle.

$$\mathbf{H}_O = \mathbf{r} \times m\mathbf{v}$$

We note that $\mathbf{H}_O$ is a vector perpendicular to the plane containing $\mathbf{r}$ and $m\mathbf{v}$ and of magnitude:

$$H_O = rmv \sin \phi$$

Resolving the vectors $\mathbf{r}$ and $m\mathbf{v}$ into rectangular components, we express the angular momentum $\mathbf{H}_O$ in determinant form as:

$$\mathbf{H}_O = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x & y & z \\ mv_x & mv_y & mv_z \end{vmatrix}$$

NEWTON'S SECOND LAW (cont.)

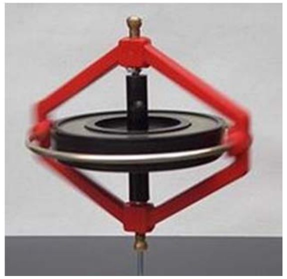
In the case of a particle moving in the *xy* plane, we have $z = v_z = 0$. The angular momentum is perpendicular to the *xy* plane and is completely defined by its magnitude

$$H_O = H_z = m(xv_y - yv_x)$$

Computing the rate of change $\dot{\mathbf{H}}_O$ of the angular momentum $\mathbf{H}_O$, and applying Newton's second law, we write

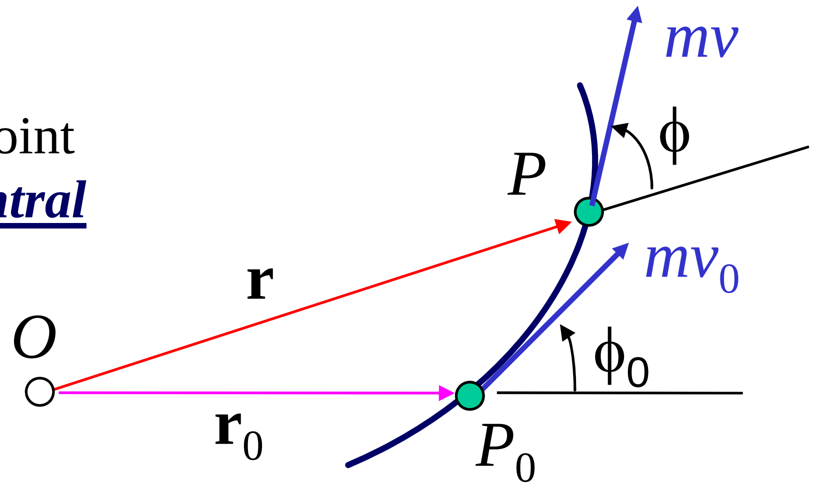
$$\dot{H}_O = \dot{\vec{r}} \times m\vec{v} + \vec{r} \times m\dot{\vec{v}} = \vec{v} \times m\vec{v} + \vec{r} \times m\vec{a} \Leftrightarrow \sum \mathrm{M_O} = \dot{\mathrm{H}}_\mathrm{O}$$

which states that : ***the sum of the moments about O of the forces acting on a particle is equal to the rate of change of the angular momentum of the particle about O*.**



NEWTON'S SECOND LAW – special cases

When the only force acting on a particle P is a force $\mathbf{F}$ directed toward or away from a fixed point O, the particle is said to be moving ***under a central force***. Since $\Sigma\, \mathbf{M}_O = 0$ at any given instant, it follows that $\dot{\mathbf{H}}_O = 0$ for all values of t, and



$$\mathbf{H}_O = \text{constant}$$

We conclude that ***the angular momentum of a particle moving under a central force is constant, both in magnitude and direction***, and that the particle moves in a plane perpendicular to $\mathbf{H}_O$.

Recalling that $H_O = rmv \sin \phi$, for the motion of any particle under a central force, we have, for points P_O and P:

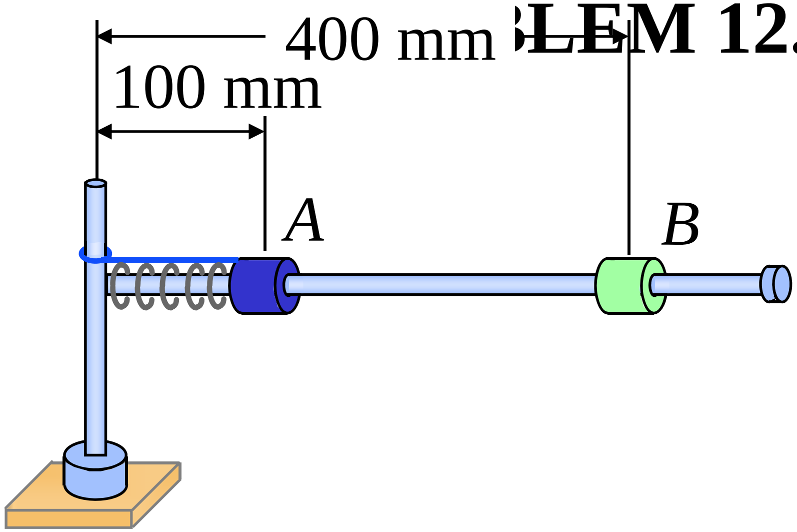
$$rmv \sin \phi = r_o m v_o \sin \phi_o$$

Using polar coordinates and recalling that $v_\theta = r\dot{\theta}$ and $H_O = mr^2\dot{\theta}$, we have

$$r^2\dot{\theta} = h$$

where h is a constant representing the angular momentum per unit mass H_o/m, of the particle.

BLEM 12.131 -Thematic Exercise 5

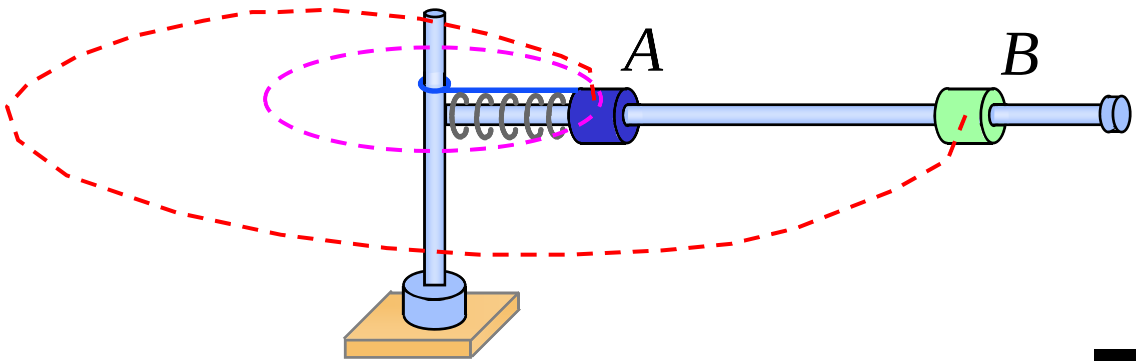


A 250-g collar can slide on a horizontal rod which is free to rotate about a vertical shaft. The collar is initially held at *A* by a cord attached to the shaft and compresses a spring of constant 6 [N/m], which is undeformed when the collar is located 500 [mm] from the shaft. As the rod rotates at the rate $\dot{\theta}_o$ = 16 [rad/s], the cord is cut and the collar moves out along the rod. Neglecting friction and the mass of the rod, <u>determine for the position *B* of the collar</u>:

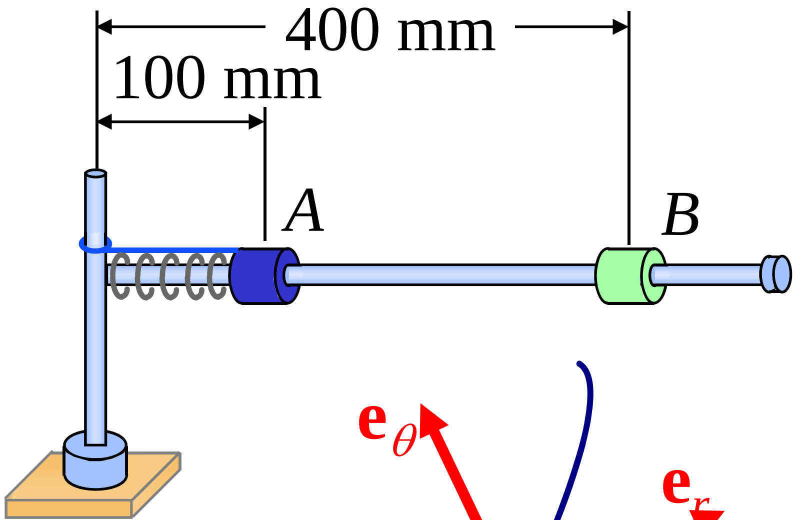
(a) The transverse component of the velocity of the collar;

(b) The radial and transverse components of its acceleration;

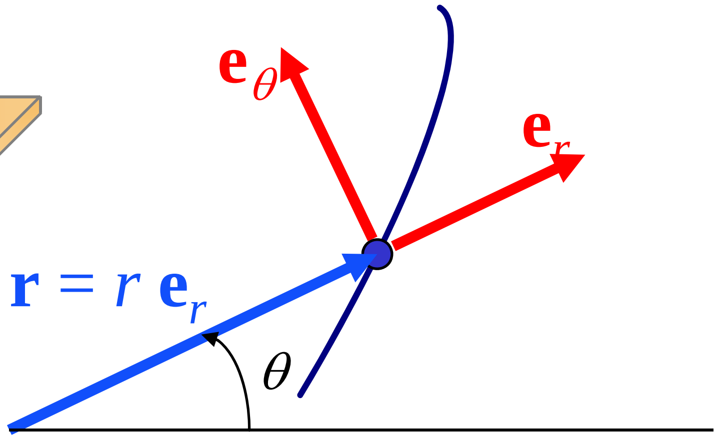
(c) The acceleration of the collar relative to the rod.

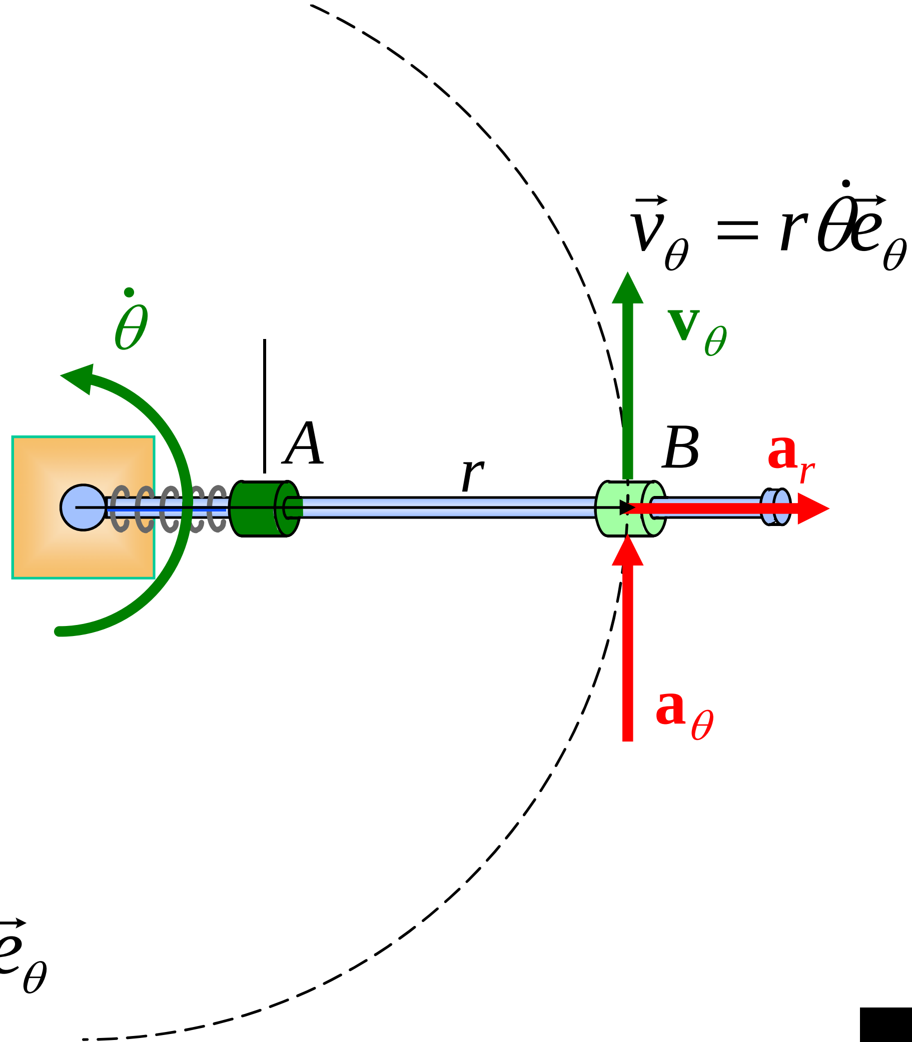


PROBLEM 12.131 (solution)



1. ***Kinematics:*** Examine the velocity and acceleration of the particle.
In polar coordinates:





$$\vec{v} = \dot{r}\vec{e}_r + r\dot{\theta}\vec{e}_\theta$$

$$\vec{a} = \left(\ddot{r} - r\dot{\theta}^2\right)\vec{e}_r + \left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right)\vec{e}_\theta$$

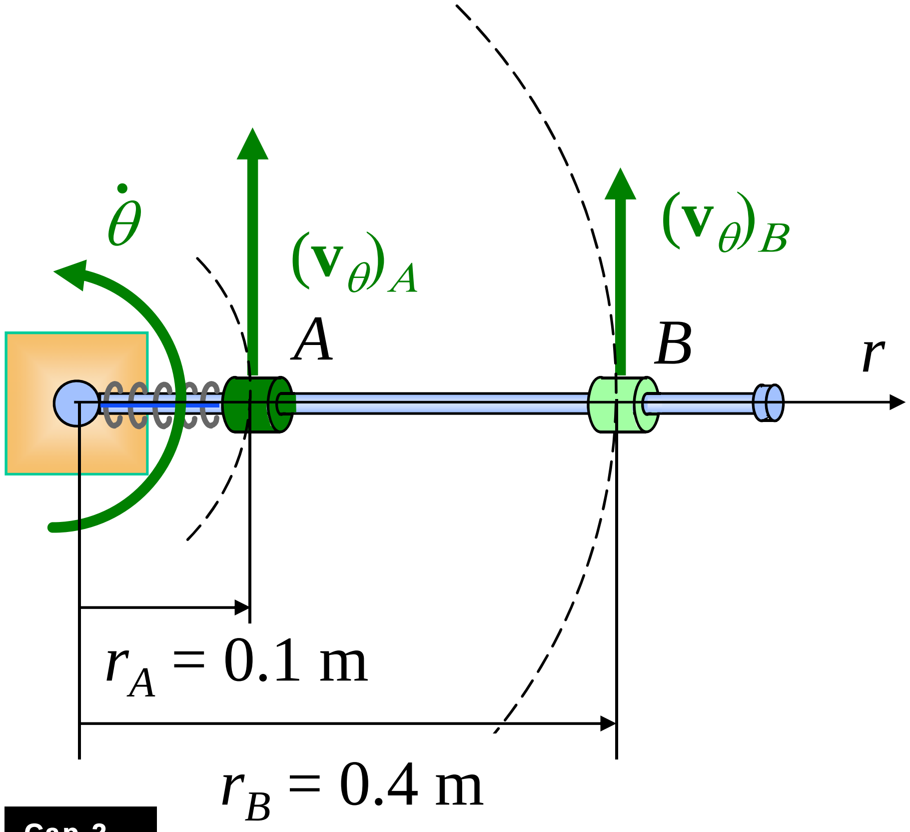
PROBLEM 12.131 (cont.)

2. ***Angular momentum of a particle:*** Determine the particle velocity at B using conservation of angular momentum. In polar coordinates, the angular momentum H_O of a particle about O is given by:

$$H_O = m\, r\, v_\theta$$

The rate of change of the angular momentum is equal to the sum of the moments about O of the forces acting on the particle.

$$\Sigma\, \mathbf{M}_O = \dot{\mathbf{H}}_O$$



If the sum of the moments is zero, the angular momentum is conserved and the velocities at A and B are related by:

$$m\,(\,r\, v_\theta)_A = m\,(\,r\, v_\theta)_B$$

Since

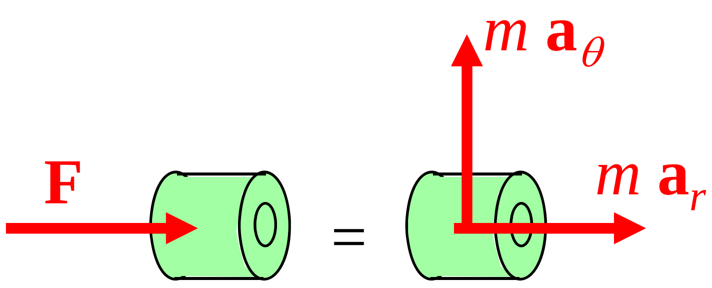
$$(v_\theta)_A = r_A \dot{\theta}$$

$$(v_\theta)_B = \frac{(r_A)^2}{r_B}\dot{\theta} \Leftrightarrow (v_\theta)_B = \frac{(0.1)^2}{0.4}(16)$$

$$\Leftrightarrow (v_\theta)_B = 0.4[m/s]$$

PROBLEM 12.131 (cont.)

3. ***Kinetics:*** Draw a free body diagram showing the applied forces and an equivalent force diagram showing the vector *m***a** or its components.



Only radial force **F** (exerted by the spring) is applied to the collar.

For $r = 0.4$ m:

$F = k\,x = (6 \text{ N/m})(0.5 \text{ m} - 0.4 \text{ m})$

$F = 0.6$ [N]

$\Sigma F_r = ma_r \quad \Leftrightarrow \quad 0.6 \text{ N} = (0.25 \text{ kg})\, a_r \quad \Leftrightarrow \quad \mathbf{a}_r = 2.4 \text{ m/s}^2$

$\Sigma F_\theta = ma_\theta \quad \Leftrightarrow \quad 0 = (0.25 \text{ kg})\, a_\theta \quad \Leftrightarrow \quad \mathbf{a}_\theta = 0$

Kinematics. (c) The acceleration of the collar relative to the rod.

$$v_\theta = r\dot{\theta} \Leftrightarrow \dot{\theta} = \frac{v_\theta}{r} = \frac{0.4}{0.4} = 1[rad/s]$$

$$a_r = \ddot{r} - r\dot{\theta}^2 \Leftrightarrow 2.4[m/s^2] = \ddot{r} - (0.4[m])(1[rad/s^2]) \Leftrightarrow \ddot{r} = 2.8[m/s^2]$$

Conclusion: The relative acceleration is equal to the collar radial acceleration

KINETICS OF PARTICLES: ENERGY AND MOMENTUM METHODS

The linear momentum of a particle is defined as the product $m\mathbf{v}$ of the mass m of the particle and its velocity $\mathbf{v}$. From Newton's second law, $\mathbf{F} = m\mathbf{a}$, we derive the relation

$$m\mathbf{v}_1 + \int_{t_1}^{t_2} \mathbf{F}\, dt = m\mathbf{v}_2$$

where $m\mathbf{v}_1$ and $m\mathbf{v}_2$ represent the momentum of the particle at a time t_1 and a time t_2 , respectively, and where the integral defines the *linear impulse of the force* $\mathbf{F}$ during the corresponding time interval. Therefore,

$$m\mathbf{v}_1 + \mathbf{Imp}_{1\rightarrow 2} = m\mathbf{v}_2$$

which expresses the principle of impulse and momentum for a particle.

KINETICS OF PARTICLES: ENERGY AND MOMENTUM METHODS

When the particle considered is subjected to several forces, the sum of the impulses of these forces should be used;

$$m\mathbf{v}_1 + \Sigma\mathbf{Imp}_{1\to 2} = m\mathbf{v}_2$$

Since ***vector quantities*** are involved, it is necessary to consider their *x* and *y* components separately.

The method of impulse and momentum is effective in the study of impulsive motion of a particle, when very large forces, called impulsive forces, are applied for a very short interval of time Δt, since this method involves impulses $\mathbf{F}\Delta t$ of the forces, rather than the forces themselves. Neglecting the impulse of any nonimpulsive force, we write:

$$m\mathbf{v}_1 + \Sigma\mathbf{F}\Delta t = m\mathbf{v}_2$$

In the case of the impulsive motion of several particles, we write

$$\Sigma m\mathbf{v}_1 + \Sigma\mathbf{F}\Delta t = \Sigma m\mathbf{v}_2$$

where the second term involves only impulsive, external forces.

KINETICS OF PARTICLES:
ENERGY AND MOMENTUM METHODS

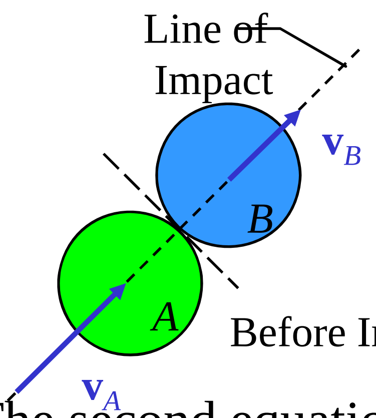
In the particular case ***when the sum of the impulses of the external forces is zero***, the equation above reduces to:

$$\Sigma m\mathbf{v}_1 = \Sigma m\mathbf{v}_2$$

that is, ***the total momentum of the particles is conserved***.

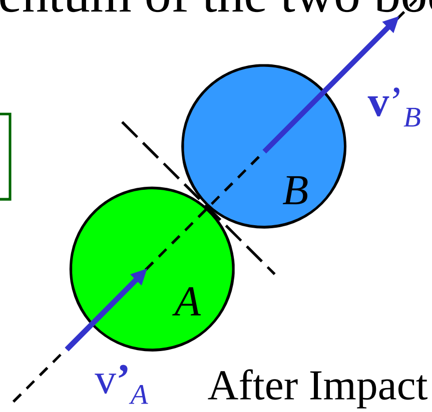
➢In the case of ***direct central impact***, two colliding bodies A and B move along the *line of impact* with velocities $\mathbf{v}_A$ and $\mathbf{v}_B$, respectively. Two equations can be used to determine their velocities $\mathbf{v}'_A$ and $\mathbf{v}'_B$ after the impact.

1- The first represents the conservation of the total momentum of the two bodies,



Before Impact

$$m_A v_A + m_B v_B = m_A v'_A + m_B v'_B$$



After Impact

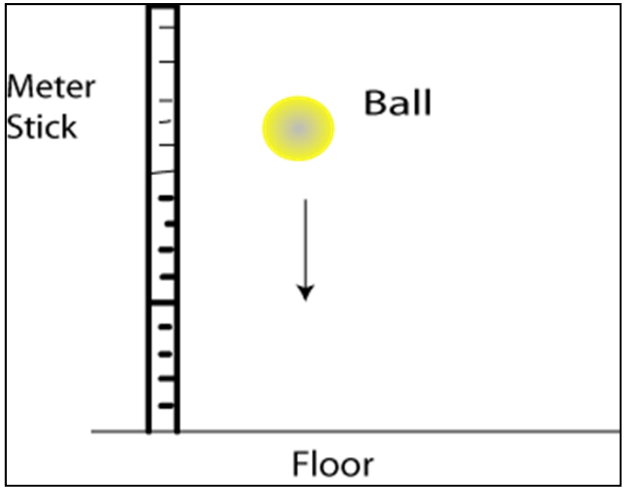
2- The second equation relates the *relative velocities* of the two bodies before and after impact,

$$v'_B - v'_A = e\,(v_A - v_B)$$

COEFFICIENTS OF RESTITUTION

- The coefficient of restitution is the ratio of speeds of a falling object, from when it hits a given surface to when it leaves the surface.
- Procedure:
  - This experiment was carried out in Midwood High School, on the second floor, on an concrete surface.
  - Take the ball and hold it at a set height above the surface. (height of 92 cm for all trials.)
  - Drop the ball and record how high it bounces.
  - Repeat for 5 trials.
  - Repeat with different balls: Practice golf ball, Wilson tennis ball, rubber band ball - many rubber bands put together in ball form, Red plastic ball, Generic unpainted billiard ball, Rubber blue ball, Painted wood ball, Steel ball bearing, Glass marble.
  - Coefficient of Restitution = speed up / speed down=SQRT(h(ave)/H).

object	H (cm)	h_1 (cm)	h_2 (cm)	h_3 (cm)	h_4 (cm)	h_5 (cm)	h_{ave} (cm)	c.o.r.
range golf ball	92	67	66	68	68	70	67.8	0.858
tennis ball	92	47	46	45	48	47	46.6	0.712
billiard ball	92	60	55	61	59	62	59.4	0.804
hand ball	92	51	51	52	53	53	52.0	0.752
wooden ball	92	31	38	36	32	30	33.4	0.603
steel ball bearing	92	32	33	34	32	33	32.8	0.597
glass marble	92	37	40	43	39	40	39.8	0.658
ball of rubber bands	92	62	63	64	62	64	63.0	0.828
hollow, hard plastic ball	92	47	44	43	42	42	43.6	0.688



COEFFICIENT OF RESTITUTION

The constant e is known as the ***coefficient of restitution***; its value lies between 0 and 1 and depends on the material involved. When e = 0, the impact is termed *perfectly plastic*; when e = 1 , the impact is termed *perfectly elastic*.

➢In the case of ***oblique central impact***, the velocities of the two colliding bodies before and after impact are resolved into "*n*" components along the line of impact and "*t*" components along the common tangent to the surfaces in contact.

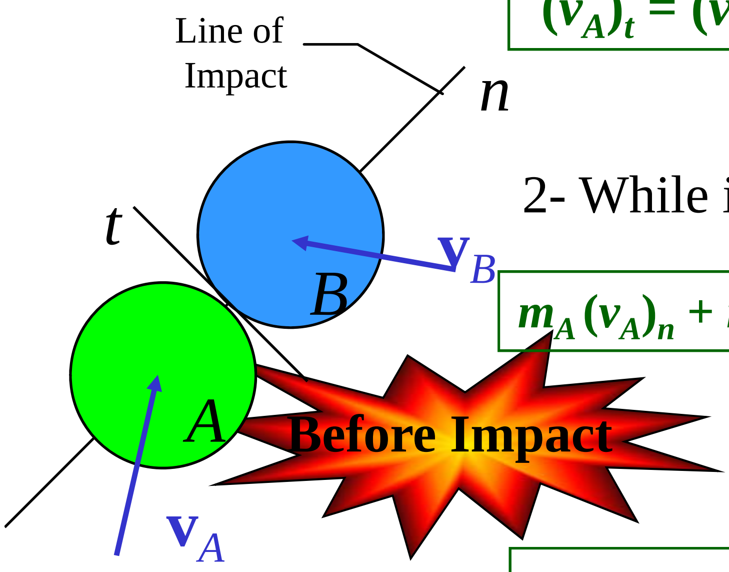
1- In the *t* direction,

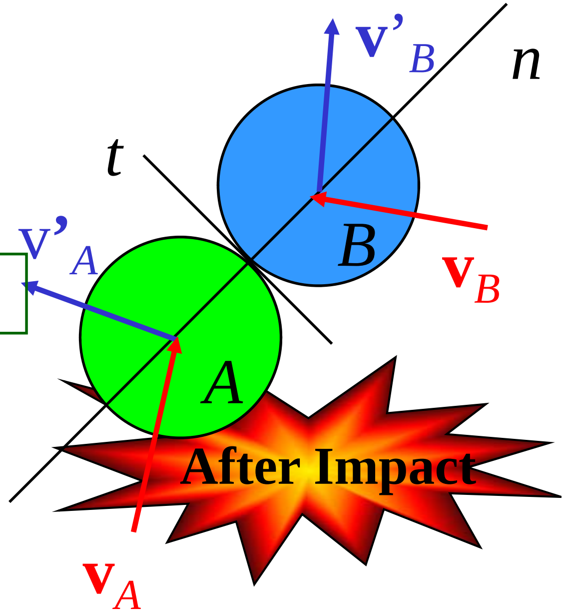
$$(v_A)_t = (v'_A)_t \qquad (v_B)_t = (v'_B)_t$$

2- While in the *n* direction:

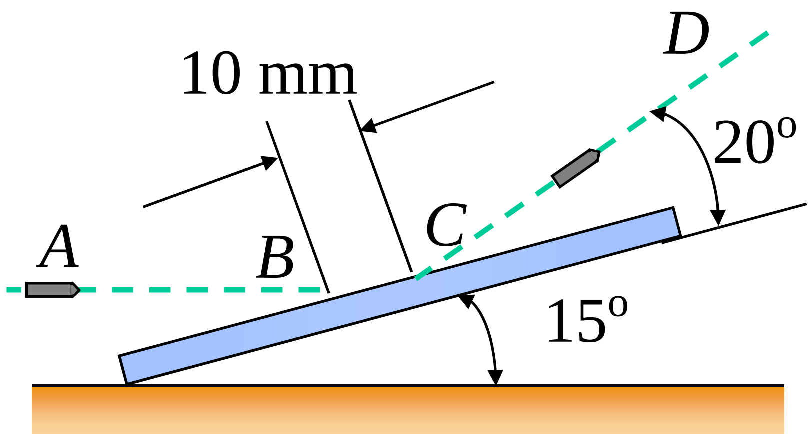
$$m_A (v_A)_n + m_B (v_B)_n = m_A (v'_A)_n + m_B (v'_B)_n$$

$$(v'_B)_n - (v'_A)_n = e\,[(v_A)_n - (v_B)_n]$$





PROBLEM 13.195 – Thematic Exercise 6



A 25-g steel-jacket bullet is fired horizontally with a velocity of 600 m/s and ricochets off a steel plate along the path *CD* with a velocity of 400 m/s. Knowing that the bullet leaves a 10-mm scratch on the plate and assuming that its average speed is 500 m/s while it is in contact with the plate, determine the magnitude and direction of the average impulsive force exerted by the bullet on the plate.

1. ***Draw a momentum impulse diagram:*** The diagram shows the particle, its momentum at t_1 and at t_2, and the impulses of the forces exerted on the particle during the time interval t_1 to t_2.

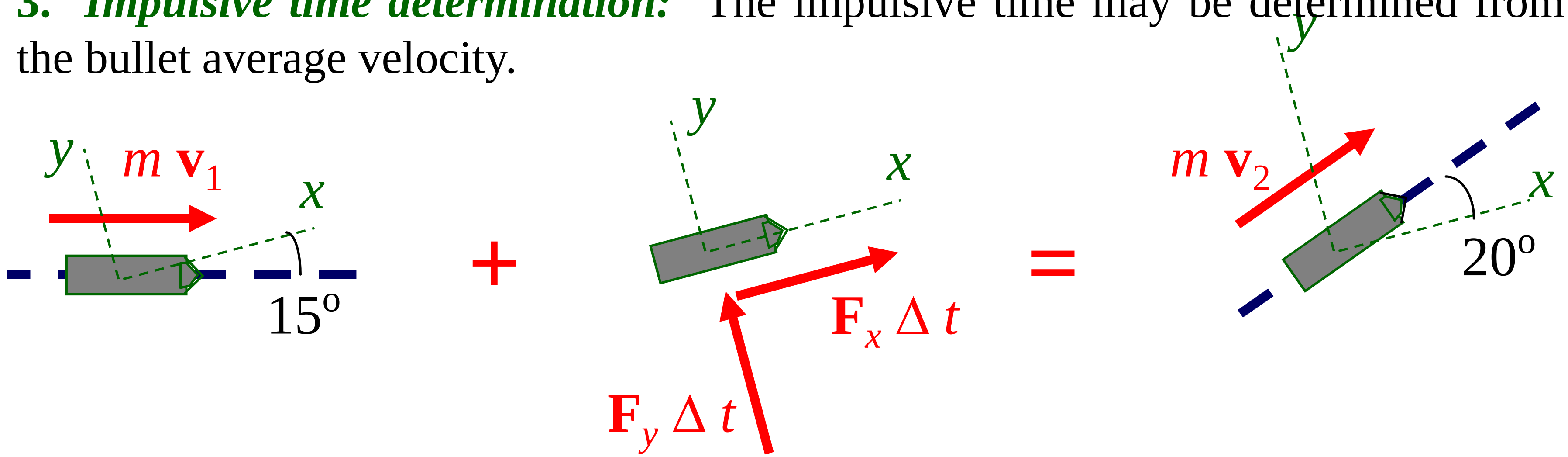
PROBLEM 13.195 – solution 1/2

2. ***Apply the principle of impulse and momentum:*** The final momentum $m\mathbf{v}_2$ of the particle is obtained by adding its initial momentum $m\mathbf{v}_1$ and the impulse of the forces **F** acting on the particle during the time interval considered.

$$m\mathbf{v}_1 + \Sigma\, \mathbf{F}\, \Delta t = m\mathbf{v}_2$$

$\Sigma\, \mathbf{F}$ is sum of the impulsive forces (the forces that are large enough to produce a definite change in momentum).

3. ***Impulsive time determination:*** The impulsive time may be determined from the bullet average velocity.

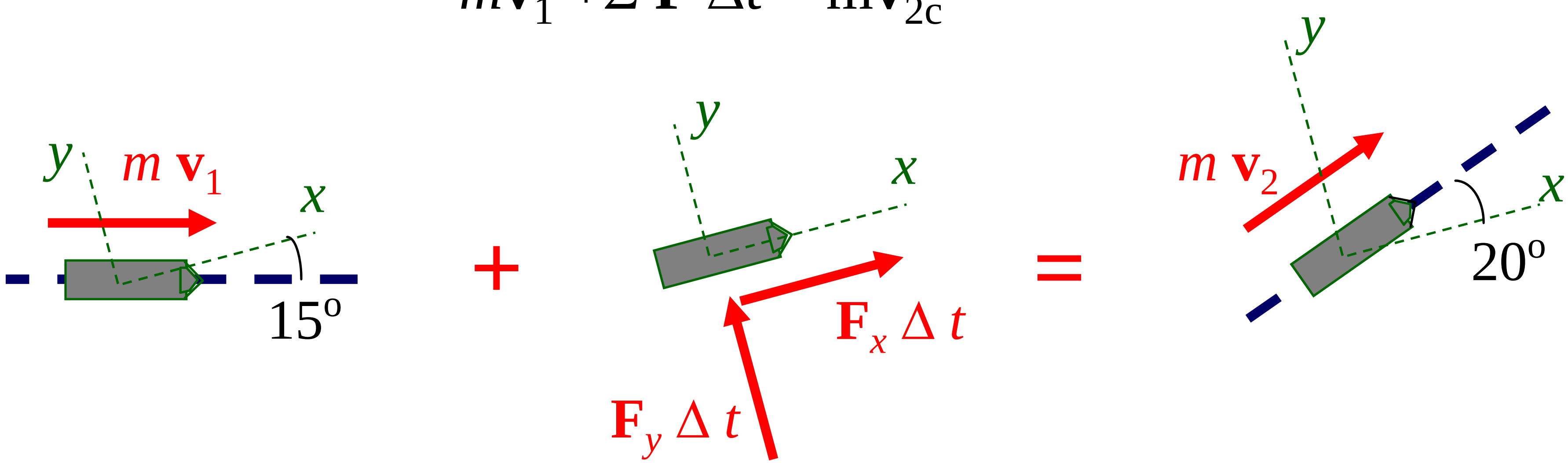


Since the bullet leaves a 10-mm scratch and its average speed is 500 m/s, the time of contact Δt is: $\Delta t = (0.010 \text{ m}) / (500 \text{ m/s}) = 2\text{x}10^{-5}$ s

PROBLEM 13.195 – solution 2/2

4. ***Apply the principle of impulse and momentum.***

$$m\mathbf{v}_1 + \Sigma\,\mathbf{F}\,\Delta t = m\mathbf{v}_{2c}$$



X: $(0.025\ \text{kg})(600\ \text{m/s})\cos 15^o + F_x(2\text{x}10^{-5}\text{s}) = (0.025\ \text{kg})(400\ \text{m/s})\cos 20^o$

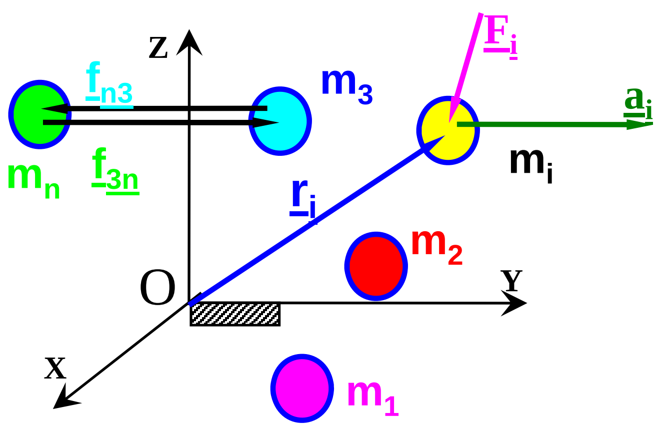
$$F_x = -\ 254.6\ \text{kN}$$

Y: $-(0.025\ \text{kg})(600\ \text{m/s})\sin 15^o + F_y(2\text{x}10^{-5}\text{s}) = (0.025\ \text{kg})(400\ \text{m/s})\ \sin 20^o$

$$F_y = 365.1\ \text{kN}$$

KINETICS OF PARTICLES
NEWTON'S SECOND LAW - system of particles

Denoting by *m* the mass of a particle, by Σ **F** the sum, or resultant, of the forces acting on the particle, and by **a** the acceleration of the particle relative to a ***newtonian frame of reference***, we can write:



$$\Sigma\, \mathbf{F} = \Sigma\, m\mathbf{a}$$

m_i – generic mass material point;
$\underline{a}_i$ – generic acceleration material point;
$\underline{f}_{ij}$ – exerted force by point *j* into point *i*;
$\underline{F}_i$ – External forces resultant over point *i*;
$\underline{r}_i$ – point *i* vector position.

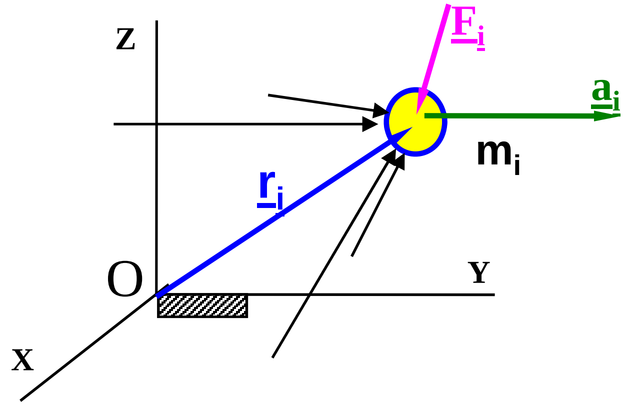
Internal resultant forces exerted over point *i* is: $\sum_{j=1}^{n} \vec{fij}$ Admitted: $\underline{f}_{ii}=\underline{0}$

Newton's second law: $$\vec{F}i + \sum_{j=1}^{n} \vec{f}_{ij} = m_i.\vec{a}_i$$

NEWTON’S SECOND LAW – system of particles

All forces acting on the ***particle*** *i*:

$$\vec{F}i + \sum_{j=1}^{n} \vec{f}_{ij} = m_i.\vec{a}_i$$

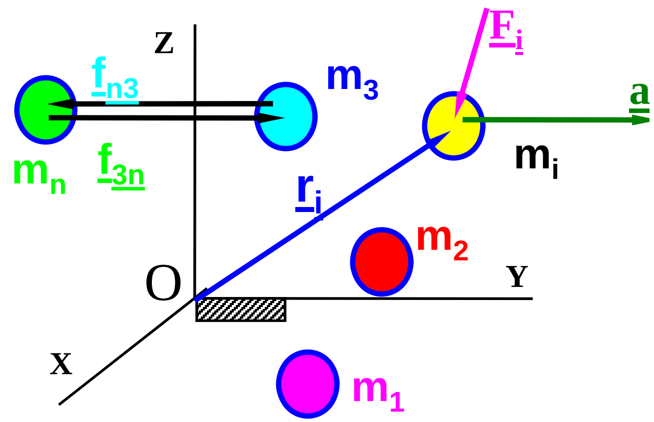


Moments of all forces acting on the ***particle*** *i*:

$$\vec{r}_i \times \vec{F}_i + \sum_{j=1}^{n} \vec{r}_i \times \vec{f}_{ij} = \vec{r}_i \times m_i.\vec{a}_i$$

Moments of all internal forces in the ***system of particles***:

$$\vec{r}_i \times \vec{f}_{ij} + \vec{r}_j \times \vec{f}_{ji} = \left(\vec{r}_j - \vec{r}_i\right) \times \vec{f}_{ji}$$



Conclusion: By the 3rd Newton’s law: $f_{ij}=-f_{ji}$, so the product will vanish, because vectors are parallel.

$$\sum_{i=1}^{n} (\mathbf{r}_i \times \mathrm{F}_i) = \sum_{i=1}^{n} (\mathbf{r}_i \times m_i \mathbf{a}_i)$$

NEWTON'S SECOND LAW – system of particles

Taking into account all the ***system of particles***:

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\vec{f}_{ij}=\vec{0} \qquad \sum_{i=1}^{n}\sum_{j=1}^{n}\vec{r}_i\times\vec{f}_{ij}=\vec{0}$$

Doing the summation of the previous equations ***for all system particles***:

$$\sum_{i=1}^{n}\vec{F}_i=\sum_{i=1}^{n}m_i\vec{a}_i \qquad \sum_{i=1}^{n}\vec{r}_i\times\vec{F}_i=\sum_{i=1}^{n}\vec{r}_i\times m_i\vec{a}_i$$

The ***linear momentum*** **L** and the ***angular momentum*** $\mathbf{H}_o$ about **FIXED** point *O* are defined as:

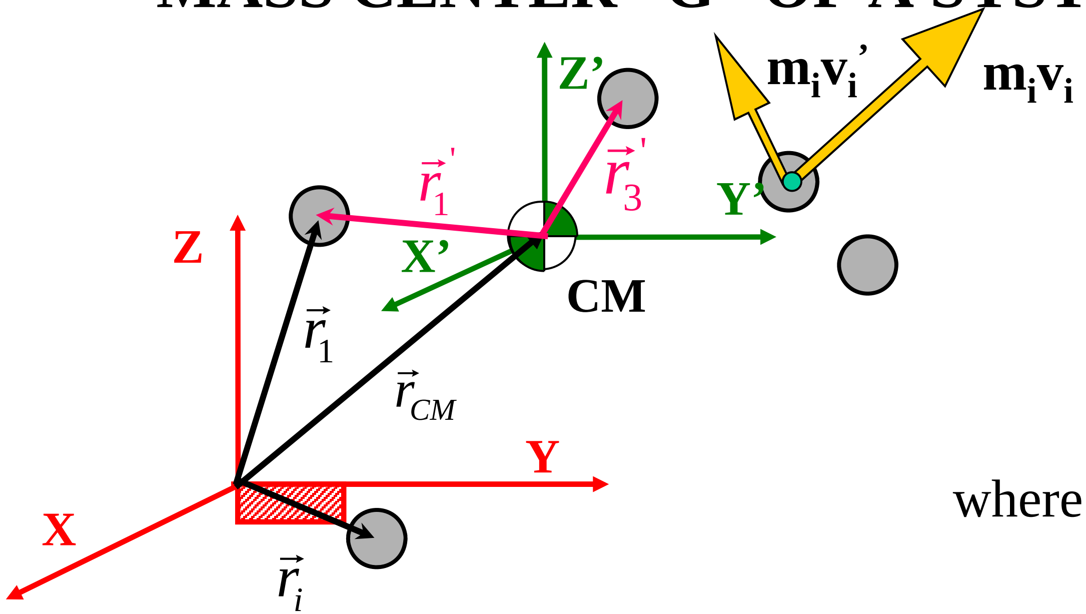
$$\vec{L}=\sum_{i=1}^{n}\left(m_i\vec{v}_i\right) \qquad \vec{H}_0=\sum_{i=1}^{n}\left(\vec{r}_i\times m_i\vec{v}_i\right)$$

differentiating, it can be shown that

$$\boxed{\dot{\vec{L}}=\sum\vec{F}} \qquad \boxed{\dot{\vec{H}}_o=\sum\vec{M}_o}$$

This expresses that ***the resultant and the moment resultant about O of the external forces are, respectively, equal to the rates of change of the linear momentum and of the angular momentum about O of the system of particles.***

ANGULAR AND DYNAMIC MOMENT ON MASS CENTER "G" OF A SYSTEM OF PARTICLES



$$M.\vec{r}_{cm} = \sum_{i=1}^{n} m_i \vec{r}_i$$

$$\sum \vec{F} = M.\vec{a}_{cm}$$

where M represents the total mass: $\Sigma\, m_i$

Being the centroidal coordinate system parallel to the Newtonian system ***(in translation with respect to the newtonian frame Oxyz)***, it is possible to write the value of the angular momentum of the system *about its mass centre G:*

$$\vec{H}_{CM}\Big|_{System-S'} = \sum_{i=1}^{n} \vec{r}\,'_i\Big|_{S'} \times m_i \vec{v}_i'\Big|_{S'}$$

Differentiating the last equation, the ***dynamic moment*** will be calculated according to:

$$\dot{\vec{H}}_{CM}\Big|_{System-S'} = K_{CM} = \sum_{i=1}^{n} \vec{r}_i'\Big|_{S'} \times m_i \vec{a}_i'\Big|_{S'} + \sum_{i=1}^{n} \dot{\vec{r}}_i'\Big|_{S'} \times m_i \vec{v}_i'\Big|_{S'} = \sum_{i=1}^{n} \vec{r}_i'\Big|_{S'} \times m_i \vec{a}_i'\Big|_{S'}$$

EQUALTITY OF THE ANGULAR MOMENTUM ON “MASS CENTER” for different coo. systems

The following demonstration will be used to show that the ***angular momentum relative to the centroidal position is equal when calculated relative to the Newtonian reference or relative to a parallel moving system.***

Being a_i’ the acceleration on the ***moving system S’***.

$$\vec{a}_i = \vec{a}_{CM} + \vec{a}_i^{\,'}$$

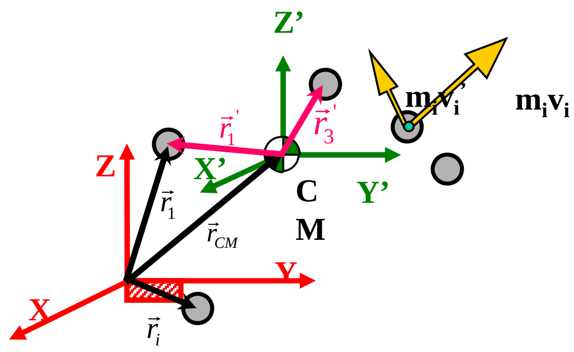
Substituting the acceleration expression into the angular momentum expression:

$$\dot{\vec{H}}_{CM}\Big|_{sistema'} = K_{CM} = \sum_{i=1}^{n} \vec{r}_i\,'\Big|_{S'} \times m_i\left(\vec{a}_i\Big|_{fixo} - \vec{a}_{CM}\Big|_{fixo}\right)$$

$$K_{CM} = \sum_{i=1}^{n} \vec{r}_i\,'\Big|_{S'} \times m_i\vec{a}_i - \sum_{i=1}^{n} m_i\vec{r}_i\,'\Big|_{S'} \times \vec{a}_{CM}$$

$$= \sum_{i=1}^{n} \vec{r}_i\,'\Big|_{S'} \times \left[F_i + \sum_{j=1}^{n} f_{ij}\right]$$

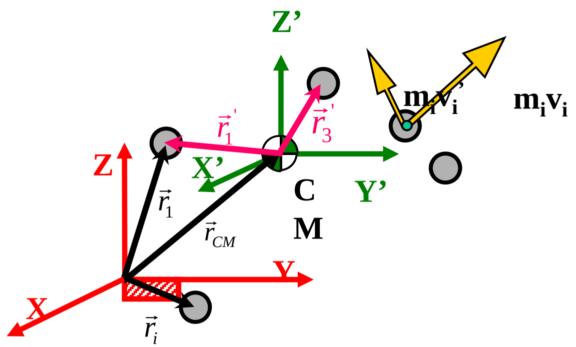
$$= \sum_{i=1}^{n} M_{CM}$$



Equal zero!!!, because:

$$\vec{c}_m\Big|_{S'} \times \sum_{i=1}^{n} m_i = \sum_{i=1}^{n} m_i \times \vec{r}_i\,'$$

DEMONSTRATION



$$\vec{H}_{CM} = \vec{H}'_{CM}$$

Using the concept of relative velocity:

$$\vec{v}_i = \vec{v}_{CM} + \vec{v}_i' \;\longrightarrow\; \vec{H}'_{CM} = \sum_{i=1}^{n} \vec{r}_i' \times m\vec{v}_i'$$

Calculating the angular momentum in mass centre:

$$\vec{H}_{CM} = \left(\sum_{i=1}^{n} m_i r_i'\right) \times \vec{v}_{CM} + \left(\sum_{i=1}^{n} r_i' \times m_i \vec{v}_i'\right) \;\longrightarrow\; \vec{H}_{CM} = \vec{H}'_{CM}$$

Equal zero!!!

$$\vec{c}_m\big|_{S'} \times \sum_{i=1}^{n} m_i = \sum_{i=1}^{n} m_i \times \vec{r}_i'$$

Important Note: This property is valid for centroidal coordinate systems, and in general is not valid for other coordinate systems.

SPECIAL CASES

Case 1: Inexistence of external forces:

$\dot{\vec{L}} = \vec{0}$ Derivative of the linear momentum

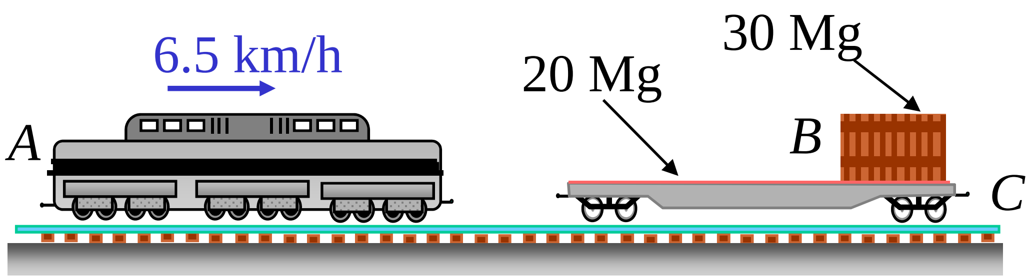
$\dot{\vec{K}}_o = \vec{0}$ Derivative of the angular momentum

Conclusion: Linear and angular momentum conservation.

Case 2: Existence of a unique external central force:

Conclusion: Angular momentum conservation.

PROBLEM 14.106 - Thematic Exercise 7



An 80-Mg railroad engine *A* coasting at 6.5 km/h strikes a 20-Mg flatcar *C* carrying a 30-Mg load *B* which can slide along the floor of the car (μ_k =0.25). Knowing that the flatcar was at rest with its brakes released and that it automatically coupled with the engine upon impact, determine the velocity of the car C:

(a) immediately after impact;

(b) after the load has slid to a stop position relative to the car.

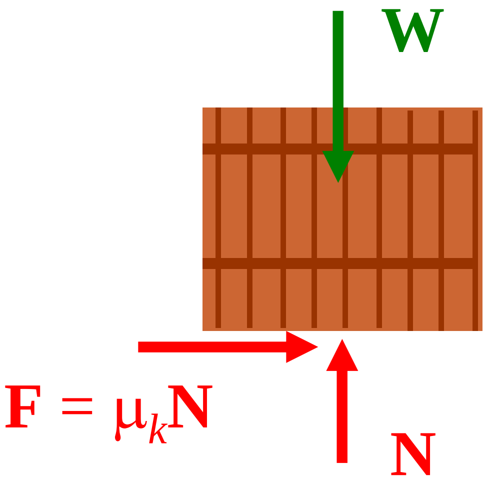
Attention!!!

Conservation of linear momentum of a system of particles is used to determine the final velocity of the system of particles, immediately after coupling and after the load slides to a stop position.

PROBLEM 14.106 (solution)

(a) Velocity immediately after impact

Conservation of linear momentum of a system of particles is used to determine the final velocity of the system of particles.



First consider the load B.
We have $F = \mu_k N = 0.20N$.
Since coupling occurs in $\Delta t \approx 0$: $F\,\Delta t \approx 0$

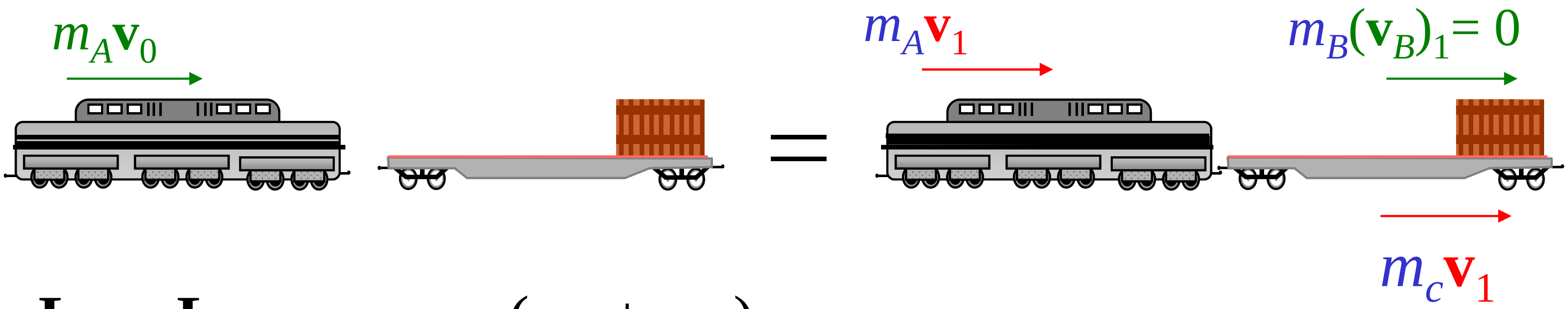
$$m_B (v_B)_O + F\Delta t = m_B (v_B)_1$$

$$0 + 0 = m_B (v_B)_1$$

$$(v_B)_1 = 0$$

PROBLEM 14.106 (solution)

We apply the principle of conservation of linear momentum to the entire system.



$$\mathbf{L}_O = \mathbf{L}_1: \quad m_A v_O = (m_A + m_C)\, v_1$$

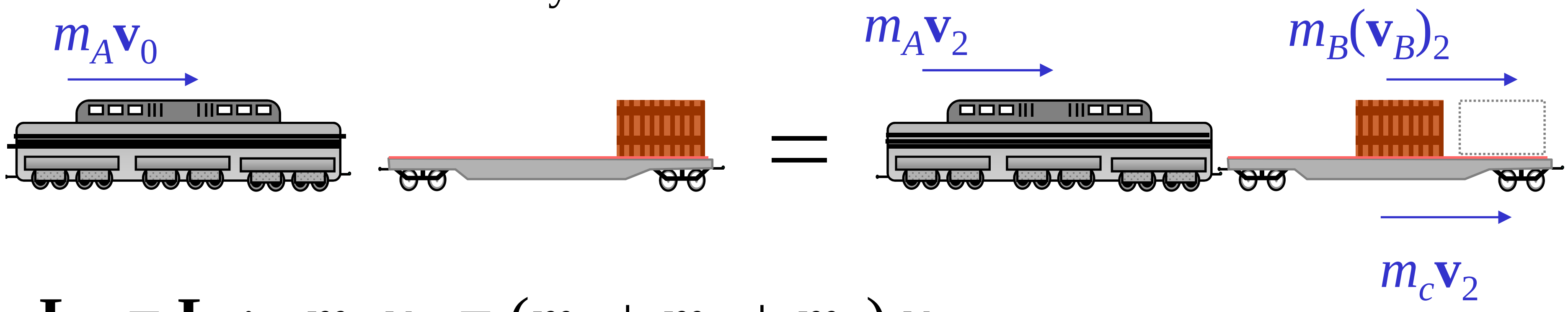
$$v_1 = \frac{m_A}{m_A + m_C} v_O = \frac{80}{80 + 20} \ (6.5 \text{ km/h})$$

$$v_1 = 5.2 \text{ km/h}$$

PROBLEM 14.106 (solution)

(b) <u>Velocity after load B has stopped moving in the car</u>

The engine, car, and load have the same velocity $\mathbf{v}_2$. Using conservation of linear momentum for the entire system:

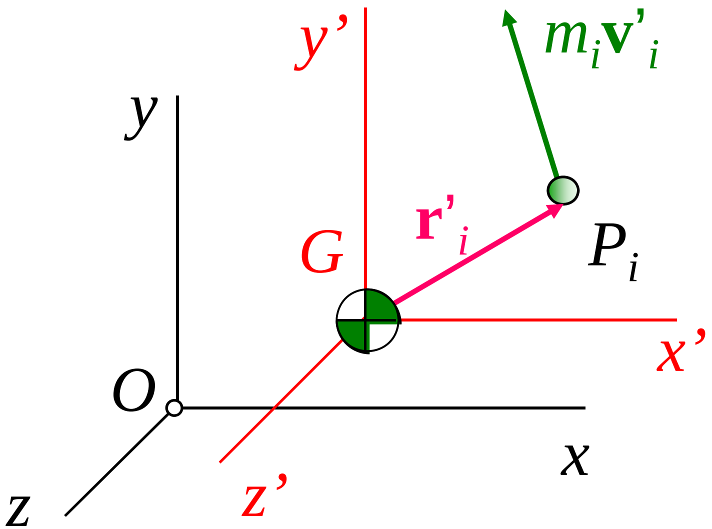


$$\mathbf{L}_O = \mathbf{L}_2: \quad m_A v_O = (m_A + m_C + m_B)\, v_2$$

$$v_2 = \frac{m_A}{m_A + m_C + m_B}\, v_O = \frac{80}{80 + 20 + 30}(6.5 \text{ km/h})$$

$$v_2 = 4 \text{ km/h}$$

WORK AND ENERGY PRINCIPLE



The kinetic energy T of a system of particles is defined as the sum of the kinetic energies of all the particles.

$$T = \frac{1}{2}\sum_{i=1}^{n} m_i v_i^2$$

Using the centroidal reference frame *Gx'y'z'* we note that the kinetic energy of the system can also be obtained by adding the kinetic energy $1/2\, m\bar{v}^2$ associated with the motion of the mass center *G* and the kinetic energy of the system in its motion relative to the frame *Gx'y'z'*:

$$T = \frac{1}{2} m\bar{v}^2 + \frac{1}{2}\sum_{i=1}^{n} m_i {v'}_i^2$$

The ***principle of work and energy*** can be applied to a system of particles as well as to individual particles.

$$T_1 + U_{12} = T_2$$

WORK AND ENERGY PRINCIPLE

$$\boxed{T_1 + U_{12} = T_2}$$

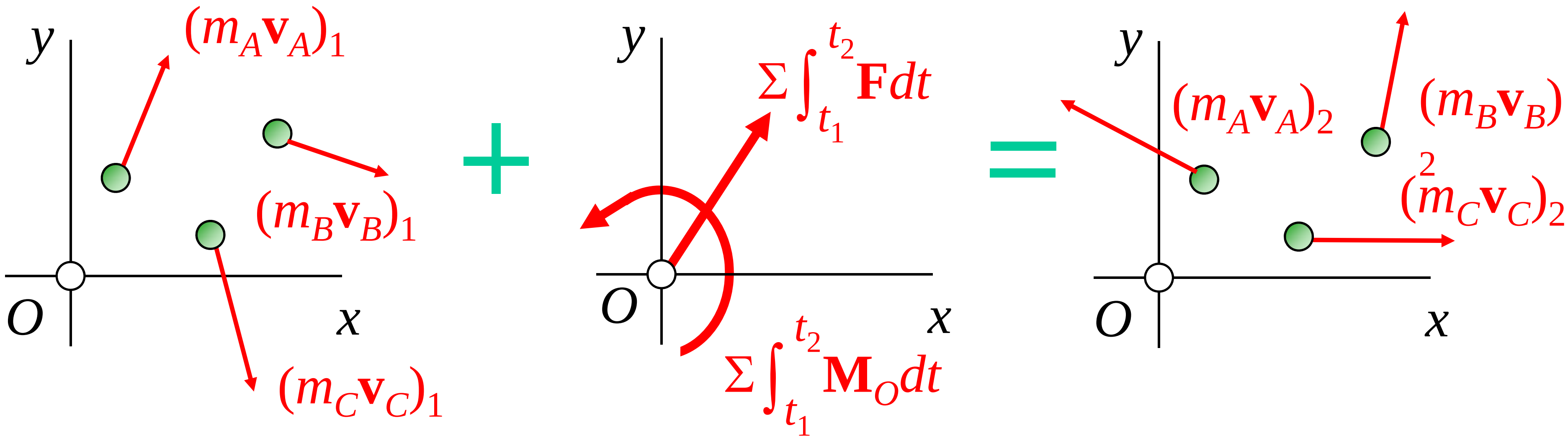
T_1 - kinetic energy of the system points (instant 1)
T_2 - kinetic energy of the system points (instant 2)
U_{12} - Work done by external forces and internal forces **, acting on the particles of the system

However, $\vec{f}_{ij} = -\vec{f}_{ji}$, the work of those internal forces may be different from zero, if the i and j point displacements are not the same.

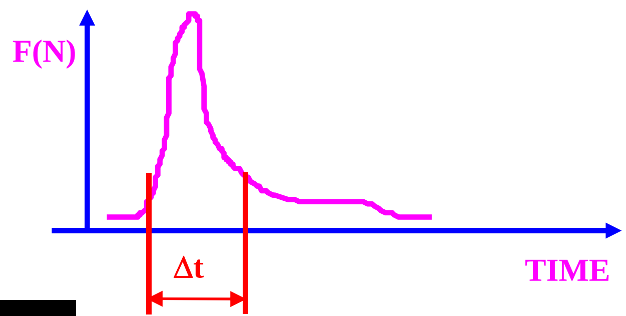
If all the forces acting on the particles of the system are <u>*conservative*</u>, the <u>*principle of conservation of energy*</u> can be applied to the system of particles

$$\boxed{T_1 + V_1 = T_2 + V_2}$$

PRINCIPLE OF IMPULSE AND MOMENTUM FOR A SYSTEM OF PARTICLES

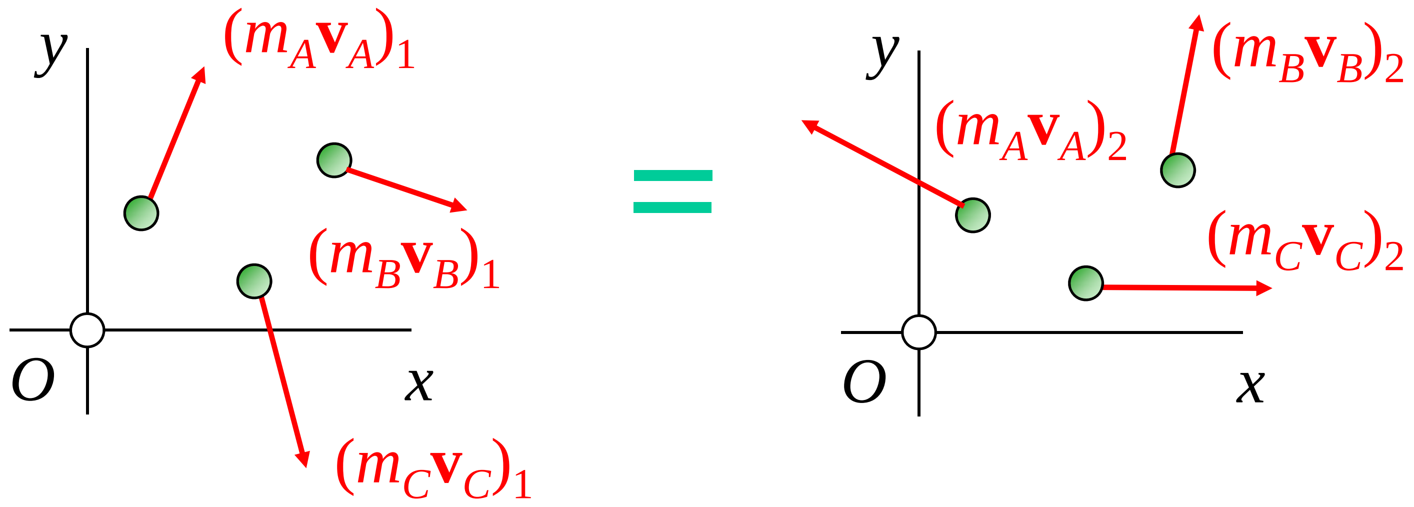


The ***principle of impulse and momentum*** for a system of particles can be expressed graphically as shown above. The momenta of the particles at time t_1 and the impulses of the external forces from t_1 to t_2 form a system of vectors equipollent to the system of the momenta of the particles at time t_2 .



Impulse definition: High amplitude force acting on a small period of time.

PRINCIPLE OF IMPULSE AND MOMENTUM FOR A SYSTEM OF PARTICLES (cont.)



If no external forces act on the system of particles, the systems of momenta shown above are equipollent and we expect the conservation of momenta (linear and angular):

$$\mathbf{L}_1 = \mathbf{L}_2 \quad \text{and} \quad (\mathbf{H}_O)_1 = (\mathbf{H}_O)_2$$

Many problems involving the motion of systems of particles can be solved by applying simultaneously the <u>principle of impulse and momentum</u> and the <u>principle of conservation of energy</u> **or** by expressing that the <u>linear momentum, angular momentum, and energy of the system are conserved</u>.

PRINCIPLE OF IMPULSE AND MOMENTUM FOR A SYSTEM OF PARTICLES (cont.)

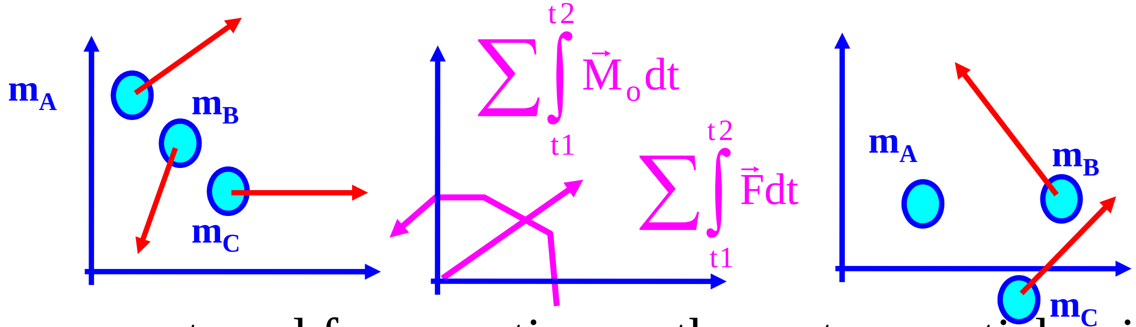
From the dynamic equilibrium equations:

$$\sum \vec{F} = \dot{\vec{L}} \qquad\qquad \sum \vec{M}_o = \dot{\vec{K}}_o$$

Time integration   Time integration

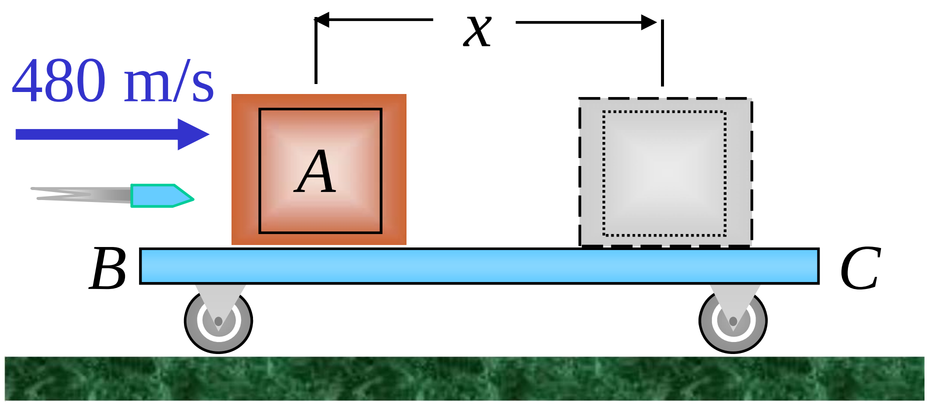
$$\sum \int_{t1}^{t2} \vec{F} dt = \vec{L}_2 - \vec{L}_1 \qquad\qquad \sum \int_{t1}^{t2} M_0 dt = \vec{H}_{o_2} - \vec{H}_{o_1}$$

Those quantities are the so called: linear impulse and angular impulse!!!



Note: If there are no external forces acting on the system particles, it is expected to have *linear and angular momentum conservation*.

PROBLEM 14.105

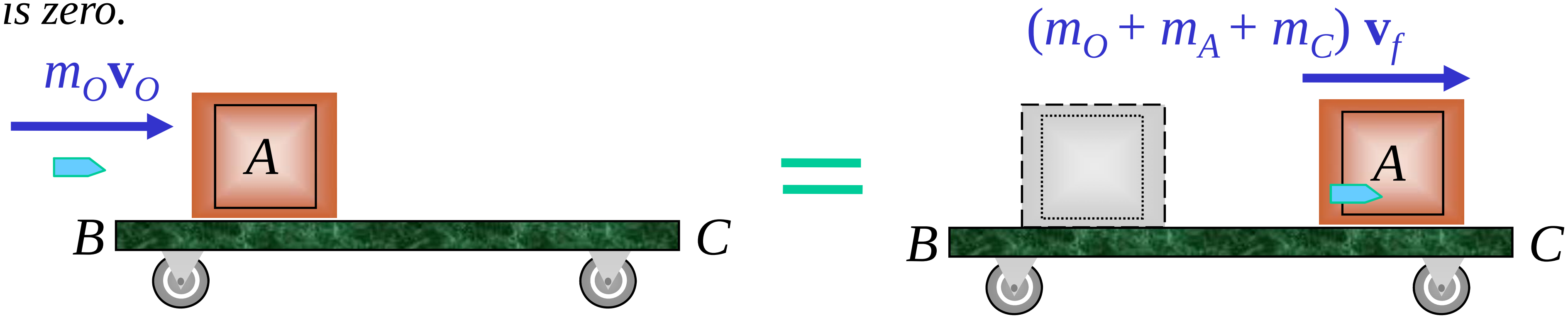


A 30-g bullet is fired with a velocity of 480 m/s into block *A*, which has a mass of 5 kg. The coefficient of kinetic friction between block *A* and cart *BC* is 0.5. Knowing that the cart has a mass of 4 kg and can roll freely, determine:

(a) The final velocity of the cart and block;

(b) The final position of the block on the cart.

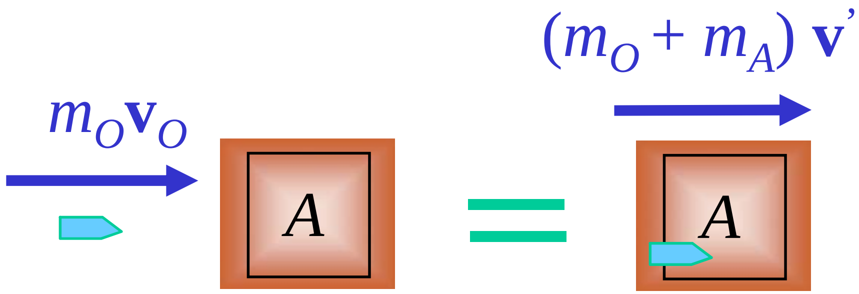
1. *Conservation of linear momentum of a system of particles is used to determine the final velocity of the system of particles.* Conservation of linear momentum occurs *when the resultant of the external forces acting on the particles of the system is zero.*



$$m_O \mathbf{v}_O = (m_O + m_A + m_C)\,\mathbf{v}_f \Leftrightarrow 0.03(480) = (0.03 + 5 + 4)\,\mathbf{v}_f$$

$$\mathbf{v}_f = 1.595 \text{ m/s}$$

PROBLEM 14.105 - SOLUTION



2. ***Conservation of linear momentum during impact is used to determine the kinetic energy immediately after impact.*** The kinetic energy T immediately after the collision is computed from $T = \frac{1}{2}\Sigma\, m_i v_i^2$.

Conservation of linear mementum:

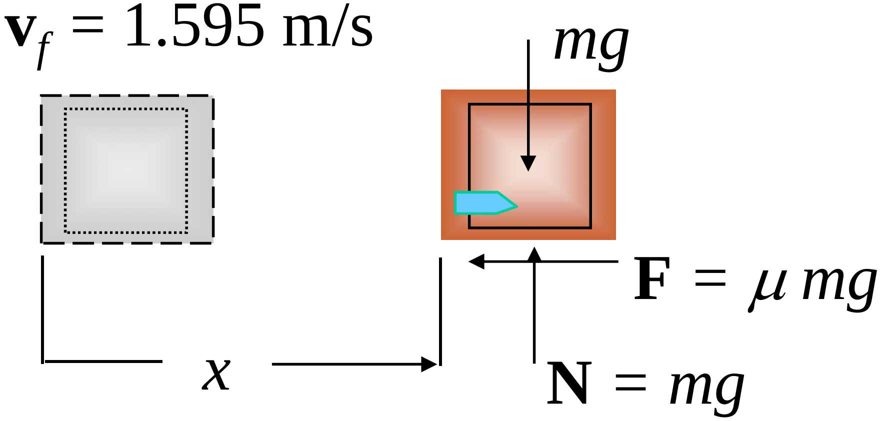
$$m_O \mathbf{v}_O = (m_O + m_A)\,\mathbf{v}'$$

$$0.03(480) = (0.03 + 5)\,\mathbf{v}' \;\Leftrightarrow\; \mathbf{v}' = 2.86 \text{ m/s}$$

Kinetic energy after impact = T' :

$$T' = \frac{1}{2}(m_O + m_A)(v')^2 = 0.5(5.03)(2.86)^2 = 20.61 \text{ N-m}$$

PROBLEM 14.105 - SOLUTION



3. *The work-energy principle is applied to determine how far the block slides.*

The final kinetic energy of the system T_f is determined knowing the final velocity of the system of particles (from step 1). The work is done by the friction force.

Final kinetic energy= T_f: $T = 20.61$ N-m

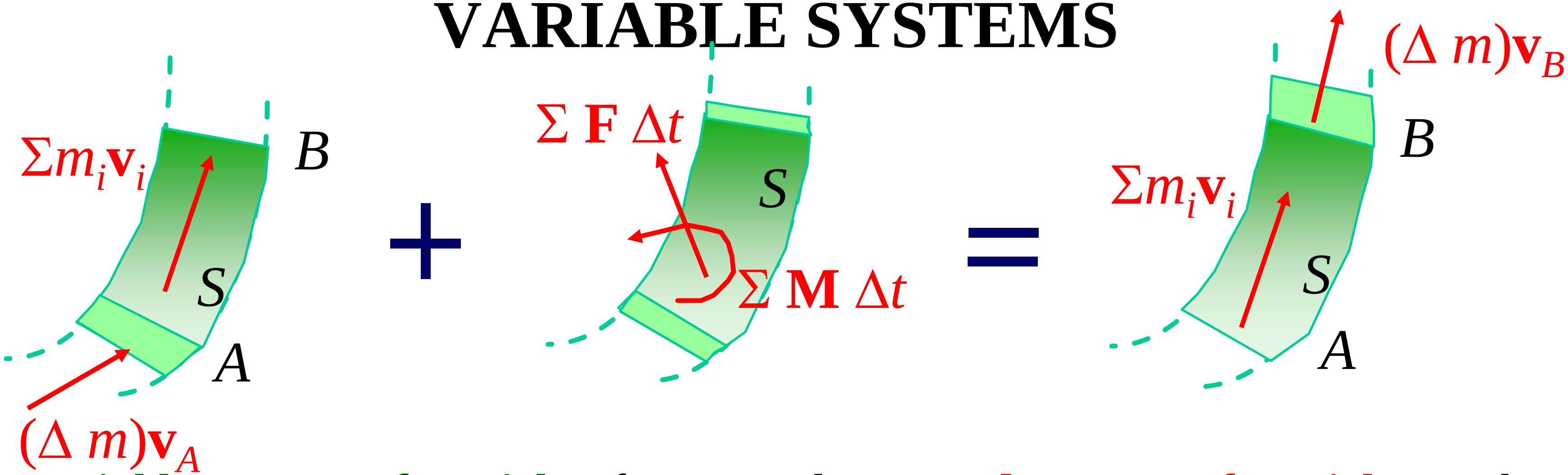
$$T_f = \frac{1}{2}(m_O + m_A + m_C)(v_f)^2 = 0.5(9.03)(1.595)^2 = 11.48 \text{ N-m}$$

The only force to do work is the friction force **F**.

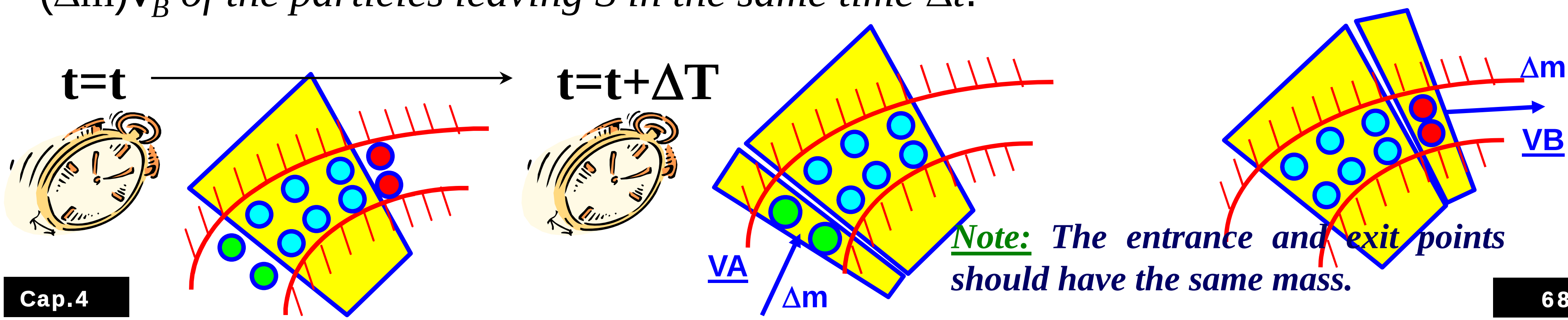
$$T' + U_{1\rightarrow 2} = T_f: \quad 20.61 - \mu(mg)(x) = 11.48 \Leftrightarrow 20.61 - 0.5(5.03)(9.81)(x) = 11.48$$

$$\rightarrow x = 0.370 \text{ m}$$

VARIABLE SYSTEMS



For ***variable systems of particles***, first consider a ***steady stream of particles***, such as a stream of water diverted by a fixed vane or the flow of air through a jet engine. The principle of impulse and momentum is applied to a system S of particles during a time interval Δt, including particles which enter the system at A during that time interval and those (of the same mass Δm) which leave the system at B. *The system formed by the momentum* $(\Delta m)\mathbf{v}_A$ *of the particles entering S in the time* Δt *and the impulses of the forces exerted on S during that time is equipollent to the momentum* $(\Delta m)\mathbf{v}_B$ *of the particles leaving S in the same time* Δt.



***Note:* The entrance and exit points should have the same mass.**

VARIABLE SYSTEMS – stationary systems

Equating the *x* components, *y* components, and moments about a fixed point of the vectors involved, we could obtain as many as three equations, which could be solved for the desired unknowns. From this result, we can derive the expression:

$$\vec{L}_1 + \sum \vec{F}\Delta t = \vec{L}_2 \Leftrightarrow \Delta m.\vec{V}_A + \sum \vec{F}\Delta t = \Delta m.\vec{V}_B$$

$$\Leftrightarrow \sum \vec{F} = \frac{\Delta m}{\Delta t}\left(\vec{V}_B - \vec{V}_A\right)$$

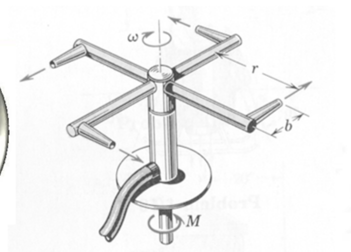
In the limit, when Δt moves toward zero:

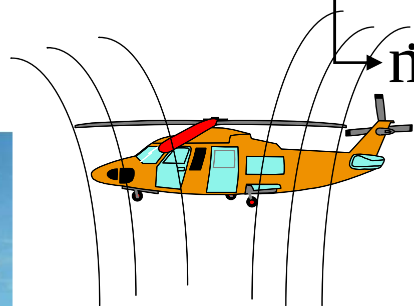
$$\boxed{\sum \vec{F} = \dot{m}\left(\vec{V}_B - \vec{V}_A\right)}$$

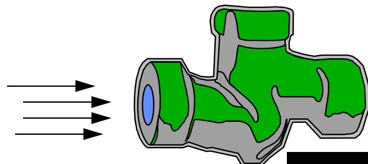
Mass flow rate of the stream

$$\dot{m} = \dot{Q} \times \rho$$

Applications: Flux in a turbine, flow into a pipe, ventilator, flow in a helicopter.

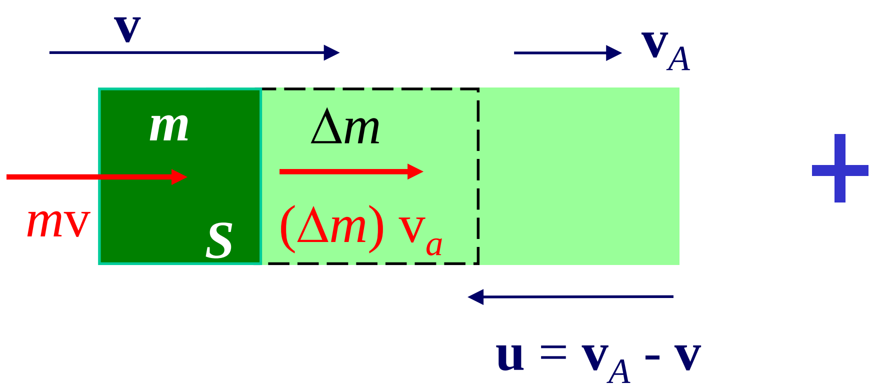


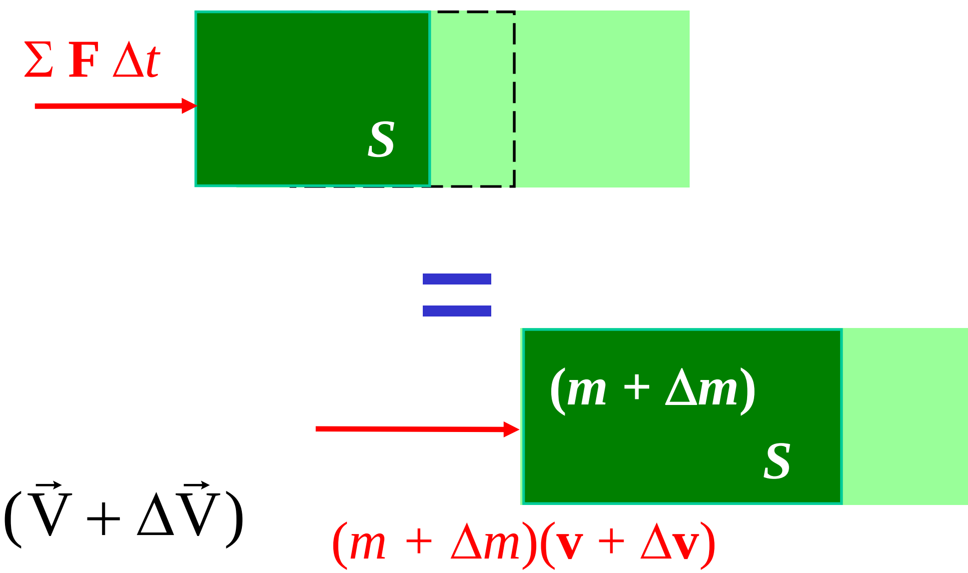




VARIABLE SYSTEMS – non stationary systems

Consider a system of particles gaining mass by continually absorbing particles or losing mass by continually expelling particles (as in the case of a rocket). Applying the principle of impulse and momentum to the system during a time interval Δt, we take care to include particles gained or lost during the time interval. The action on a system S of the particles being absorbed by S is equivalent to a thrust.



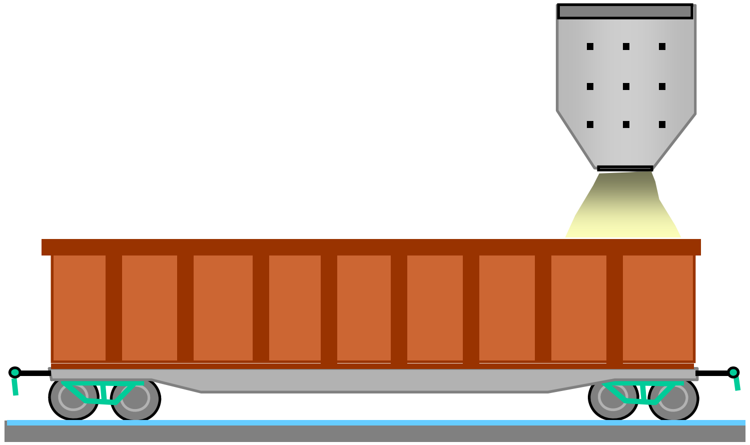


$$m\vec{V} + \Delta m\vec{V}_A + \sum \vec{F}\Delta t = (m + \Delta m)(\vec{V} + \Delta\vec{V})$$

$$\boxed{\sum \vec{F} + \dot{m}\vec{u} = m\frac{d\vec{V}}{dt}}$$

Note: $\underline{u} = \underline{V_A} - \underline{V}$

PROBLEM 14-115 – Thematic exercise 8



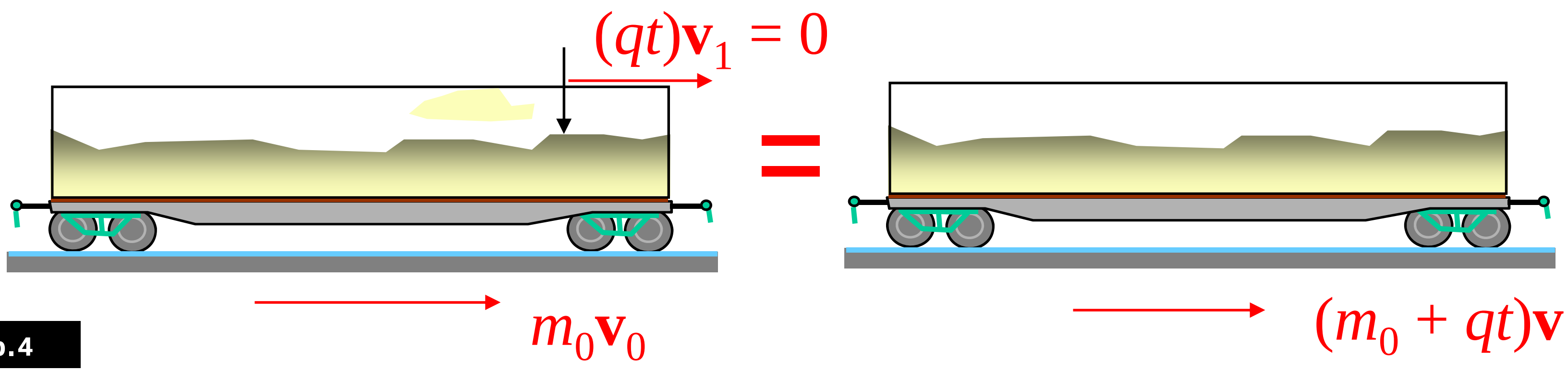
A railroad car of length L and a mass m_0 when empty is moving freely on a horizontal track while being loaded with sand from a stationary chute at a rate $dm/dt = q$. Knowing that the car was approaching the chute at a speed v_0 ,

determine:

(a) The mass of the car and its load after the car has cleared the chute;

(b) The speed of the car at that time.

To solve problems involving a variable system of particles, the principle of impulse and momentum is used.

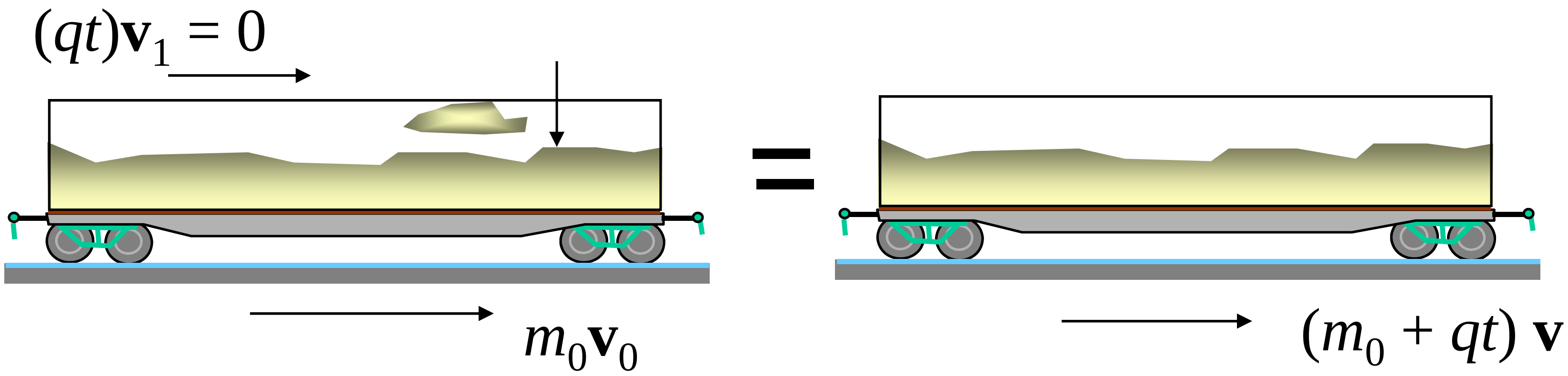


PROBLEM 14-115 - solving

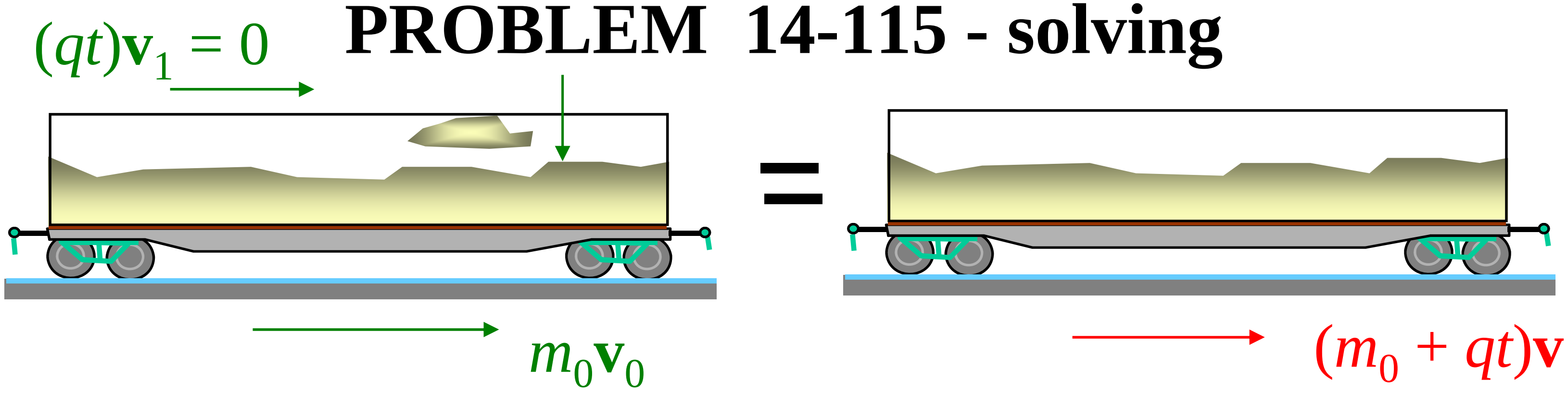
We consider the system consisting of the mass m_0 of the car and its contents at $t = 0$ and of the additional mass qt which falls into the car in the time interval t.

Conservation of linear momentum in the horizontal direction

$$m_0 v_0 = (m_0 + qt)\, v \qquad v = \frac{m_0 v_0}{(m_0 + qt)}$$



PROBLEM 14-115 - solving



$$v = \frac{m_0 v_0}{m_0 + qt}$$

Letting $v = \dfrac{dx}{dt} = \dfrac{m_0 v_0}{m_0 + qt}$

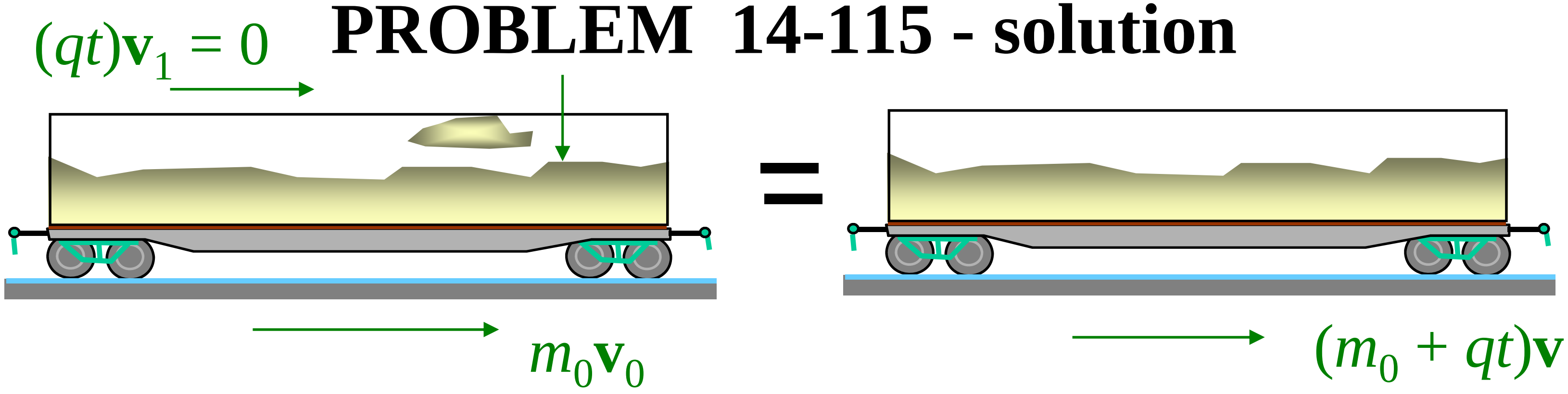
$$dx = \frac{m_0 v_0 \, dt}{m_0 + qt} \qquad x = m_0 v_0 \int_0^t \frac{dt}{m_0 + qt}$$

$$x = \frac{m_0 v_0}{q}\left[\ln(m_0 + qt)\right]_0^t = \frac{m_0 v_0}{q} \ln \frac{m_0 + qt}{m_0}$$

Using the exponential form: $m_0 + qt = m_0 \, e^{qx/m_0 v_0}$

where $m_0 + qt$ represents the mass at time t and after the car has moved through x.

PROBLEM 14-115 - solution



(a) making $x = L$, we obtain the final mass:

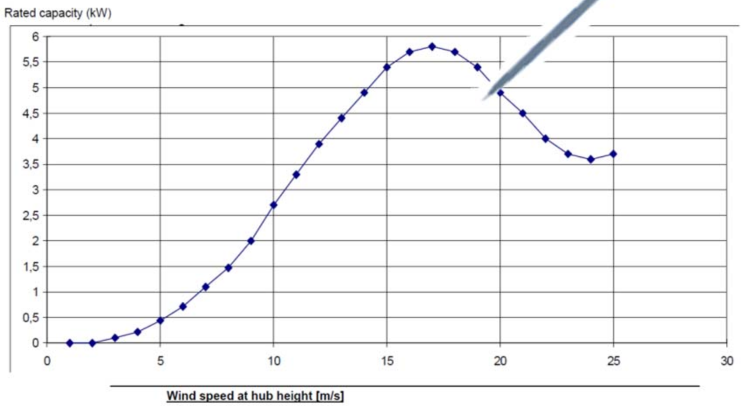
$$m_f = m_0 + qt_f = m_0\, e^{qL/m_0 v_0}$$

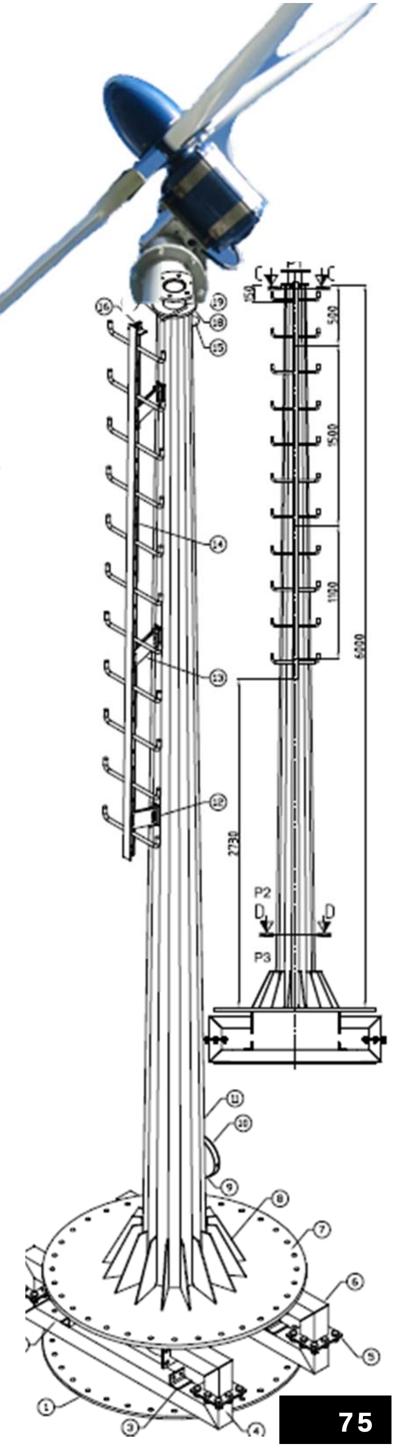
(b) making $t = t_f$ in the velocity equation we obtain the final velocity:

$$v = \frac{m_0 v_0}{m_0 + qt_f} = \frac{m_0}{m_f}\, v_0 = v_0\, e^{-qL/m_0 v_0}$$

Practical exercise

- Roof mounted turbines (Montana FORTIS model).
  - Determine the forces produced by the wind on the top of the main tower for the wind generator.
  - The wind generator main characteristics are:
    - Rated Power: 5800 [W];
    - Rotor Diameter: 5 [m];
    - Swept area: 19,63 [m2];
    - Rated wind speed: 17 [m/s]
    - Cut wind speed: 2.5 [m/s]





- Notes about roof installation:
  - Turbine needs a laminar air flow to work properly, so the existence of other buildings in the surrounding are not too good.
  - The roof needs to be flat and strong enough to cope with the weight.
  - The roof needs to be stiff enough to counter vibrations that might enter into the building.

Practical exercise

- Define control volume for air (system of particles)
  - Apply the principle of impulse and momentum.

$$\sum \vec{F} = \dot{m}\left(\vec{V}_B - \vec{V}_A\right)$$

  - Assume Swept area=19,63 [m2];
  - Assume volumetric flow rate = 329,12 [m3/s]

$$\dot{Q} = V \times A_{swept}$$

  - Assume mass flow rate = 425,5 [kg/s]

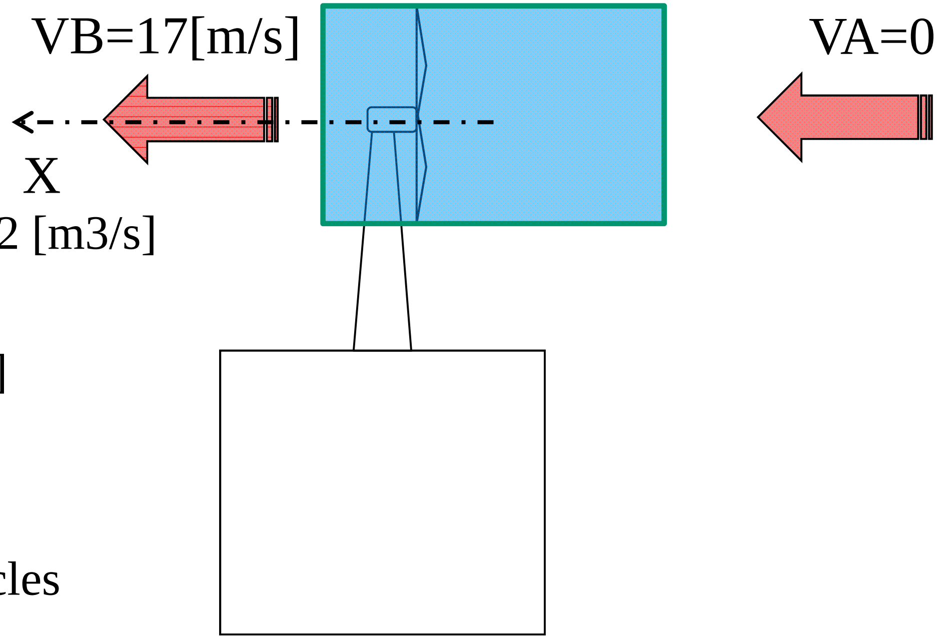
$$\dot{m} = \dot{Q} \times \rho_{air}$$

  - Forces that act on the system of particles

$$Fx = 7234{,}4\,[N]$$

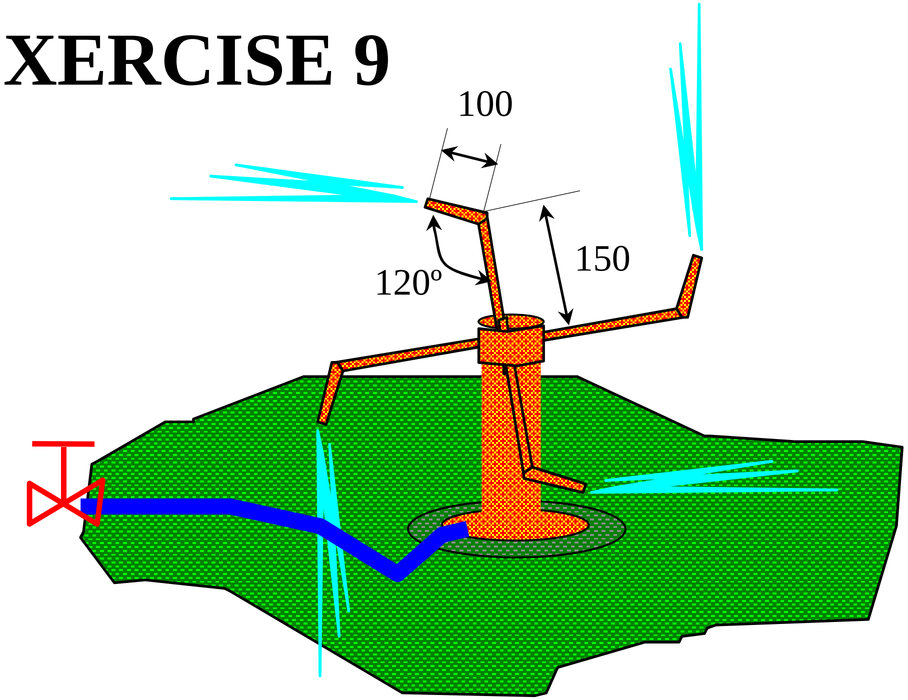
  - Force that act on the column structure:

$$Fx = -7234{,}4\,[N]$$

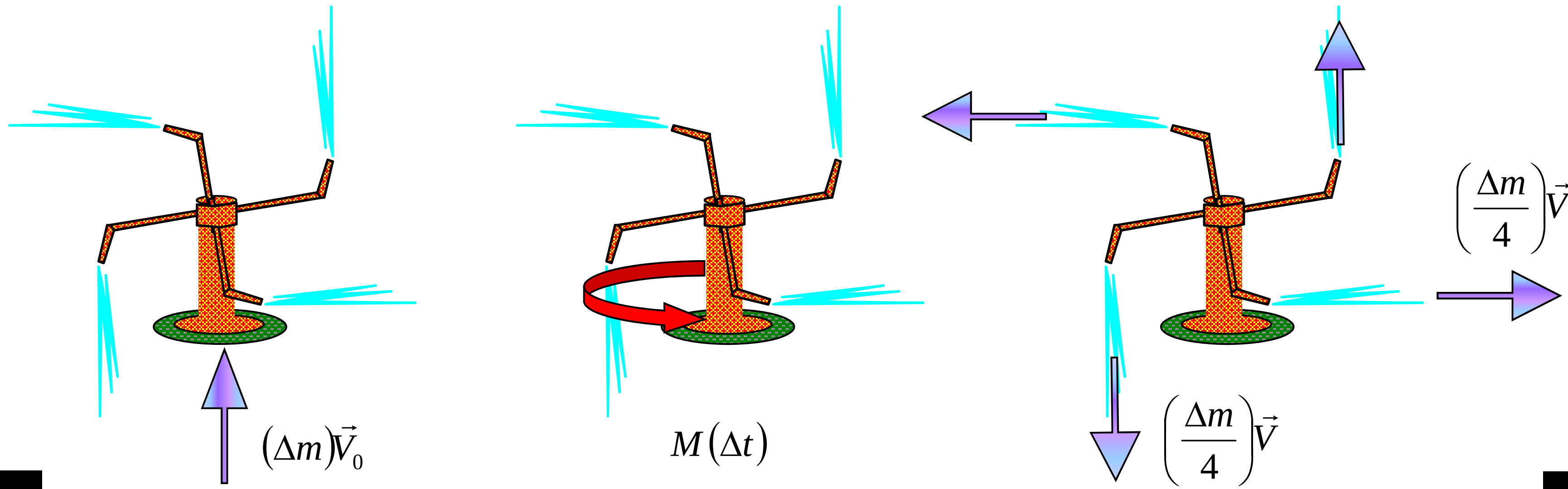


THEMATIC EXERCISE 9

Each of the four rotating arms of sprinkler consists of two straight portions of pipe forming 120 º angle. Each arm discharges water at the rate of 20 [l/min] with relative velocity of 18 [m/s]. Friction is equivalent to a couple of M=0.375 [N.m]. Determine the angular velocity at which sprinkler rotates.



Applying the principle of impulse and momentum to the water sprinkler.



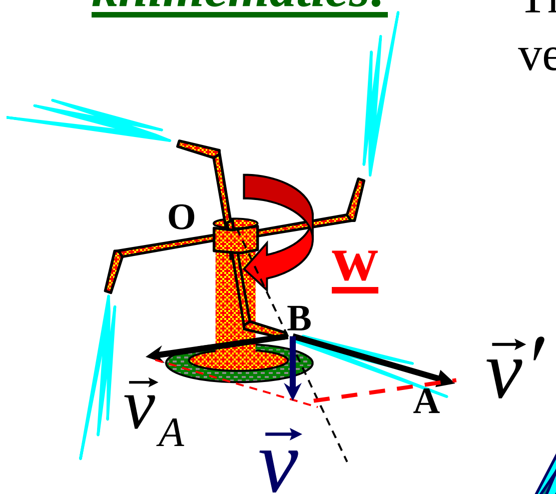
THEMATIC EXERCISE 9 - resolution

Equating moments about axis of rotations:

$$\sum \int_{t1}^{t2} M_0 dt = \vec{H}_{o_2} - \vec{H}_{o_1} \quad \Leftrightarrow \quad \begin{cases} 0 + M\Delta t = 4 \text{ moment of } (\Delta m / 4) v \\ \vec{H}_{01} \quad \text{Vanish, because } \mathbf{r} // \mathbf{v} \end{cases}$$

knimematics:

The velocity “v” of the water leaving the arm is the resultant of the velocity “**v´**” relative to the arm and the velocity “$\mathbf{v_A}$” of the nozzle.

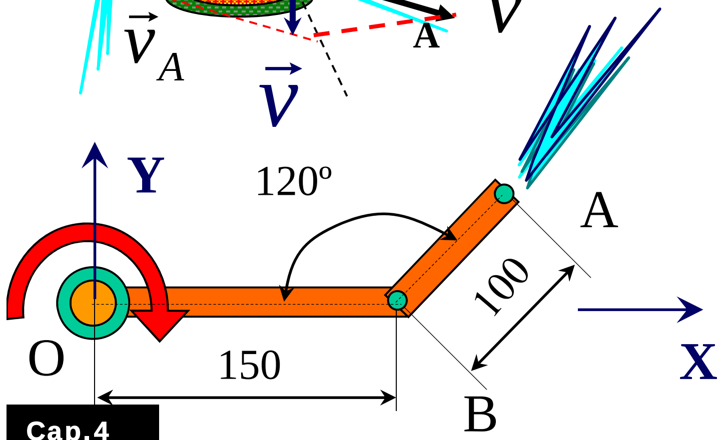


$$\boxed{\vec{v} = \vec{v}' + \vec{v}_A}$$

$$\vec{v}' = 18 \vec{\lambda}_{BA} [m/s]$$

$$\vec{v}_A = \vec{v}_O + \vec{w} \times \vec{OA} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} + \begin{Bmatrix} 0 \\ 0 \\ -w \end{Bmatrix} \times \begin{Bmatrix} 150 + 100 Cos(60^o) \\ 100 Sin(60^o) \\ 0 \end{Bmatrix}$$

$$\vec{v}_A = \begin{Bmatrix} w 100 Sin(60^o) \\ -w[150 + 100 Cos(60^o)] \\ 0 \end{Bmatrix} [mm/s]$$



THEMATIC EXERCISE 9 - solution

$$\begin{Bmatrix} 0 \\ 0 \\ M\Delta t \end{Bmatrix} = \vec{H}_{o_2} - \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \Leftrightarrow \begin{Bmatrix} 0 \\ 0 \\ M\Delta t \end{Bmatrix} = \vec{O}A \times 4\frac{\Delta m}{4}\begin{Bmatrix} 9+0.0866w \\ 15.6-0.2w \\ 0 \end{Bmatrix} \Leftrightarrow$$

$$\begin{Bmatrix} 0 \\ 0 \\ M\Delta t \end{Bmatrix} = \begin{Bmatrix} 0.15+0.1Cos(60^o) \\ 0.1Sin(60^o) \\ 0 \end{Bmatrix} \times 4\frac{\Delta m}{4}\begin{Bmatrix} 9+0.0866w \\ 15.6-0.2w \\ 0 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ \Delta m[0.2(15.6-0.2w)-0.0866(9+0.0866w)] \end{Bmatrix}$$

$$M\Delta t = \Delta m[2.34 - 0.0475w]$$

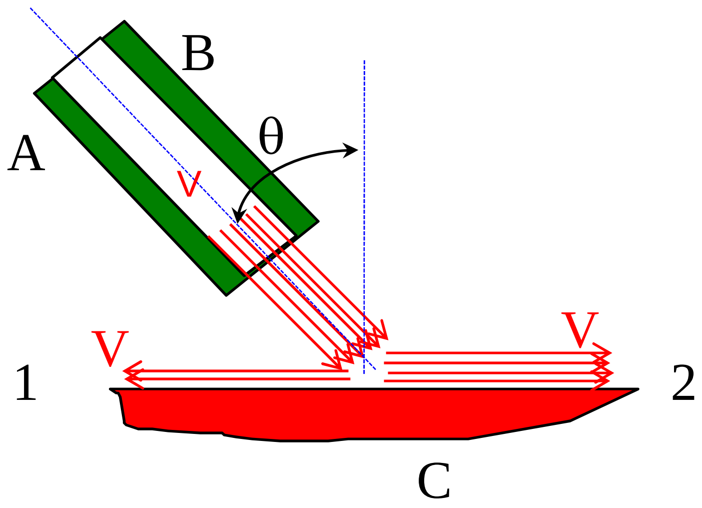
From the hydraulic equivalence:

$$\frac{\Delta m}{\Delta t} = \dot{m} = \rho\dot{Q} \Leftrightarrow \dot{m} = 1.4 \times 80[l/\min].\frac{1\min}{60s} = 4/3[kg/s]$$

Result:

$$w \cong 42[rad/s] = 400[rpm]$$

SAMPLE PROBLEM - 2



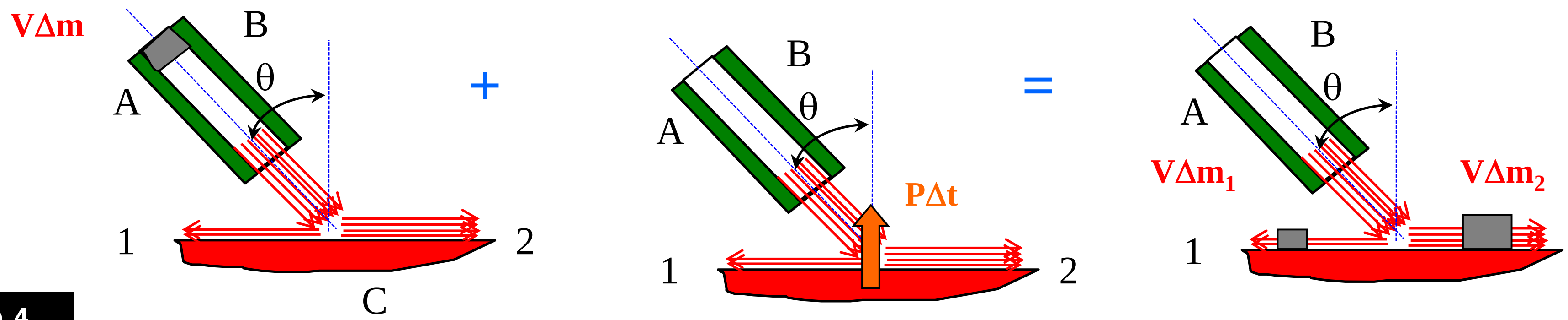
The water coming through two parallel distinct plates A and B flows continuously with a constant velocity of 30 [m/s]. The flow will be divided into two horizontal separate zones, due to the plane plate. Knowing volumetric flow rate for each of the resultant fluxes, Q1=100 [l/min] and Q2=500 [l/min], determine:

a) The theta angle;
b) The total force exerted by the flux over the plane plate.

Conservation of mass :

$$\dot{m}_{IN} = \dot{m}_{OUT\_1} + \dot{m}_{OUT\_2}$$

Impulse and Momentum impulse :



SAMPLE PROBLEM – 2 – solution

Principle of impulse and momentum

$$\sum \vec{F} = \dot{\vec{L}} \quad \Leftrightarrow \quad \sum \vec{F} = \dot{m}_{OUT}\vec{v}_{OUT} - \dot{m}_{IN}\vec{v}_{IN}$$

$$\begin{Bmatrix} 0 \\ F_y \\ 0 \end{Bmatrix} = 8{,}33\times10^{-3}\rho\begin{Bmatrix} v_{2x} \\ 0 \\ 0 \end{Bmatrix} + 1{,}66\times10^{-3}\rho\begin{Bmatrix} -v_{1x} \\ 0 \\ 0 \end{Bmatrix} - 0.01\rho\begin{Bmatrix} 30.\sin\Theta \\ -30.\cos\Theta \\ 0 \end{Bmatrix}$$

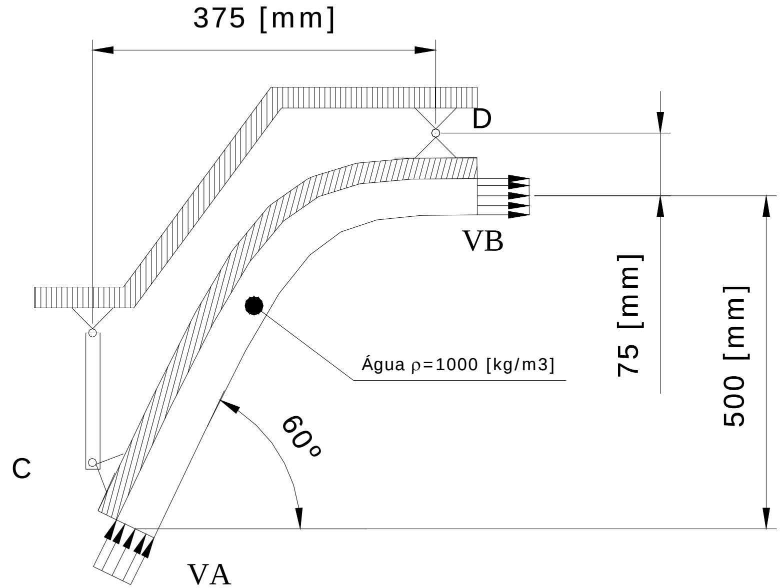
Aditional data: ρ^{water}=1000 (kg/m^3), v_{2x}=30(m/s), v_{1x}=30(m/s)

Solution: Fy = 224(N), θ ≅ 41.8°

Note: The force exerted by the stream into the plate is a force of equal amplitude but from up to down.

TEST EXERCISE

The water coming through a duct is inject at point A at 25 [m/s], with a volumetric flow rate of 1.2 [m3/min]. For the same output velocity B, determine the resultant forces exerted by the flux over the support.



Solution: Applying the principle of impulse and momentum

$$\vec{L}_{IN} + \sum \vec{F}\Delta t = \vec{L}_{OUT} \Leftrightarrow m\begin{Bmatrix} 25\cos(60^{o}) \\ 25\sin(60^{o}) \\ 0 \end{Bmatrix} + \sum \vec{F}\Delta t = m\begin{Bmatrix} 25 \\ 0 \\ 0 \end{Bmatrix} \Leftrightarrow \frac{m}{\Delta t}\left[\begin{Bmatrix} 25\cos(60^{o}) \\ 25\sin(60^{o}) \\ 0 \end{Bmatrix} - \begin{Bmatrix} 25 \\ 0 \\ 0 \end{Bmatrix}\right] = \sum \vec{F}$$

$$\sum \vec{F} = \begin{Bmatrix} 250{,}00 \\ -433{,}01 \\ 0 \end{Bmatrix}$$

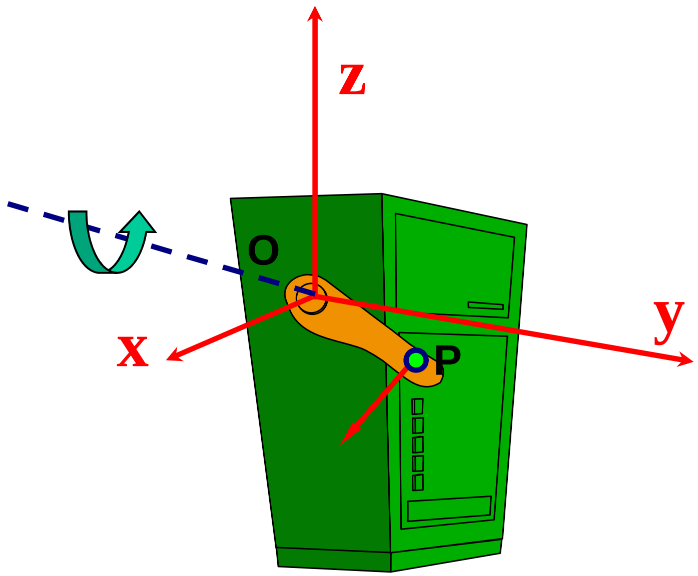
Applying the third Newton’s Law, the exerted resultant force over support is the oppsite vector.

$$\sum \vec{F} = \begin{Bmatrix} -250{,}00 \\ +433{,}01 \\ 0 \end{Bmatrix}$$

RIGID BODIES – inertial matrices

Mass Matrix definition of a rigid body in a point, in reference to a specific coordinate system – Mathematical operator which reports the inertial three dimensional state of a body trough their moments and products of inertia .

$$\begin{bmatrix} I_{xx} & -P_{xy} & -P_{xz} \\ & I_{yy} & -P_{yz} \\ & & I_{zz} \end{bmatrix}$$



- The figure represents a rotational body, being “O” the rotational instantaneous centre point.
- Each point “P” from the yellow arm has a linear and angular momentum equal to:

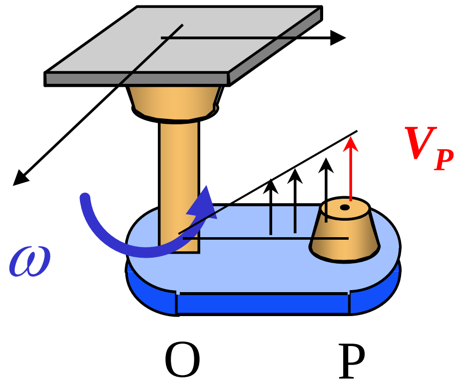
$$d\vec{L} = dm.\vec{v} \qquad d\vec{H}_O = \vec{O}P \times dm.\vec{v}$$

By direct mass integration:

$$\vec{L} = \int_{Body} \vec{v}dm \qquad \vec{H}_O = \int_{Body} \vec{O}P \times \vec{v}dm$$

INERTIAL MATRICES – simple movements

Plane rotating, with "O" belonging to the axis of rotation:



$$\vec{H}_O = \int_M \vec{OP} \times \left(\vec{\varpi} \times \vec{OP}\right) dm$$

Using vector components:

$$\vec{OP} = \begin{Bmatrix} x \\ y \\ z \end{Bmatrix} \text{ and } \vec{\varpi} = \begin{Bmatrix} \varpi_x \\ \varpi_y \\ \varpi_z \end{Bmatrix} \Rightarrow \vec{H}_O = \int_M \begin{Bmatrix} x \\ y \\ z \end{Bmatrix} \times \begin{Bmatrix} z\varpi_y - y\varpi_z \\ x\varpi_z - z\varpi_x \\ y\varpi_x - x\varpi_y \end{Bmatrix} dm$$

Introducing matrix formulation:

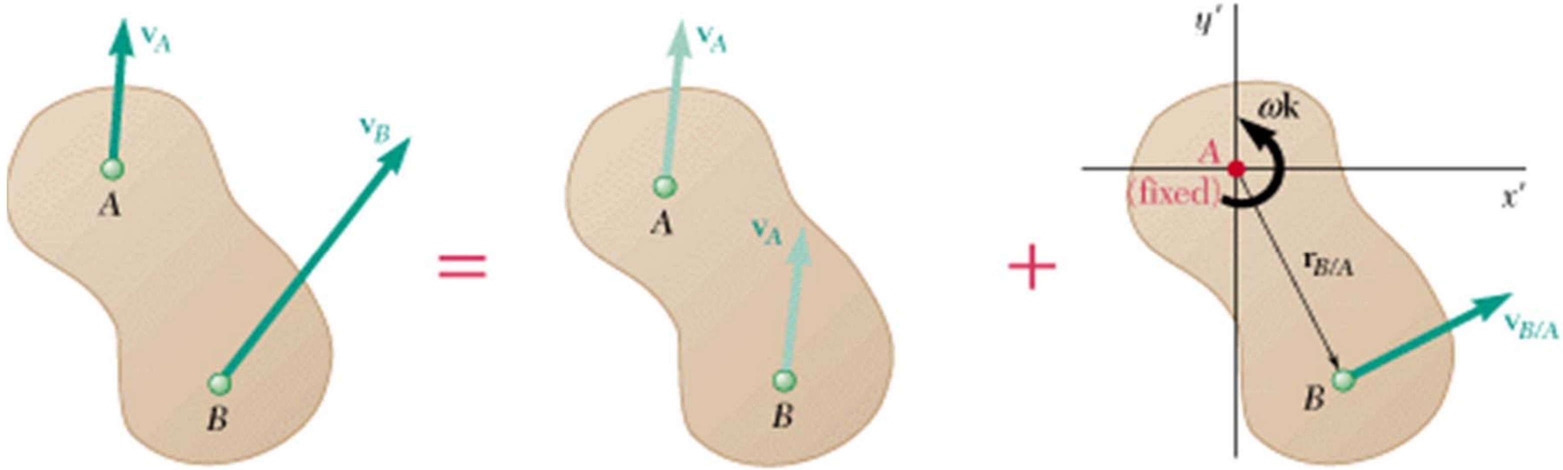
$$\vec{H}_O = \begin{bmatrix} \int_M (y^2+z^2)dm & -\int_M (xy)dm & -\int_M (xz)dm \\ & \int_M (x^2+z^2)dm & -\int_M (yz)dm \\ & & \int_M (y^2+x^2)dm \end{bmatrix} . \begin{Bmatrix} \varpi_x \\ \varpi_y \\ \varpi_z \end{Bmatrix}$$

Conclusion: The borne matrix is symmetric – **Inertial matrix**

INERTIAL MATRICES – General movements

In a more general movement:

All general movement may be decomposed in a summation of a translation movement and a rotation about the mass centre.

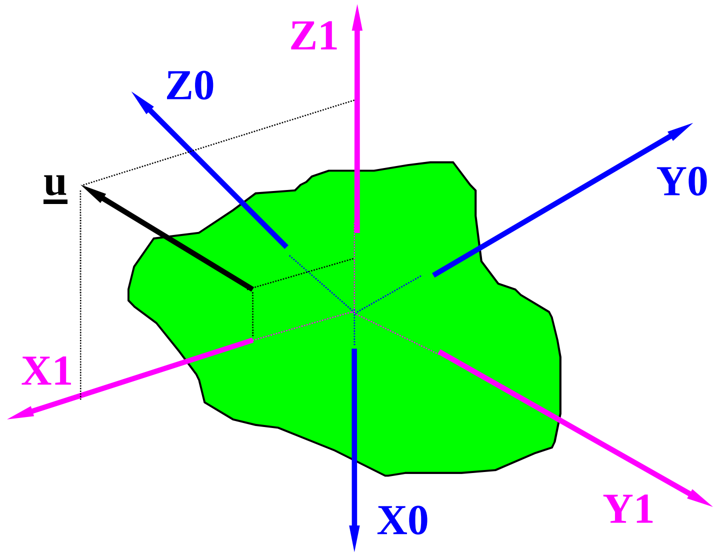


The part of the kinetic moment relative to the second decomposed movement may be calculated in the same way, as calculated to the plane rotating movement.

INERTIAL MATRICES – rotating referential

Rotating referential: Transformation matrix

$$\vec{u}\big|_{S1} = [T_{o->1}].\vec{u}\big|_{S0}$$



The transformation matrix will be composed of the direct cosines from each S0 axis over S1 system.

Knowing that:

$$\vec{u}\big|_{S0} = \begin{Bmatrix} x_0 \\ y_0 \\ z_0 \end{Bmatrix} \text{ and } \vec{u}\big|_{S1} = \begin{Bmatrix} x_1 \\ y_1 \\ z_1 \end{Bmatrix} \qquad T_{0->1} = \begin{bmatrix} Cos(X0,X1) & Cos(Y0,X1) & Cos(Z0,X1) \\ Cos(X0,Y1) & Cos(Y0,Y1) & Cos(Z0,Y1) \\ Cos(X0,Z1) & Cos(Y0,Z1) & Cos(Z0,Z1) \end{bmatrix}$$

Being this matrix orthogonal, then: $\left[T_{0->1}\right] = \left[T_{1->0}\right]^t$

Kinetic moment will be calculated: $\vec{H}_O\big|_{S1} = [T_{0->1}].\vec{H}_O\big|_{S0}$

INERTIAL MATRIX - change of referential

Angular momentum may be calculated according to:

$$\begin{aligned}\vec{H}_O\big|_{S1} &= \left[T_{0->1}\right].\vec{H}_O\big|_{S0} \\ &= \left[T_{0->1}\right].\left[I_O\big|_{S0}\right].\vec{\varpi}\big|_{S0} \\ &= \left[T_{0->1}\right].\left[I_O\big|_{S0}\right].\left[T_{0->1}\right]^t.\vec{\varpi}\big|_{S1}\end{aligned}$$

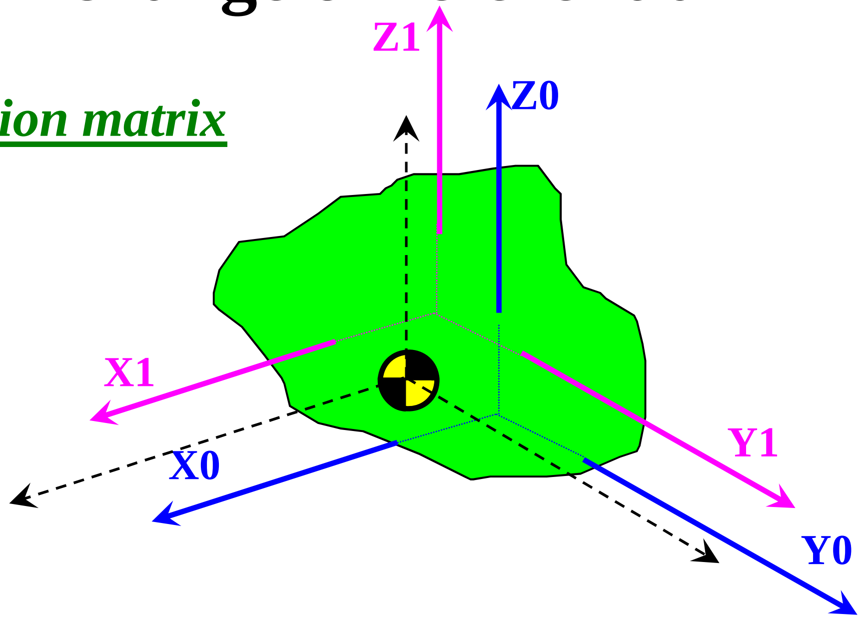
Inertial matrix may be calculated in a different coordinate system

$$\left[I_O\right]\big|_{S1} = \left[T_{0->1}\right].\left[I_O\right]\big|_{S0}.\left[T_{0->1}\right]^t$$

Note: Knowing the inertial matrix in a discrete point, is possible to calculate the moment relative to any axis passing through that point.

INERTIAL MATRIX - change of referential

Translation referential: Transformation matrix



Jacob Steiner (1796-1863)

By the Steiner theorem:

$$[I_O]_{S1} = [I_O]_{S0} - M \begin{bmatrix} \left(y_{CM_{S0}}{}^2 + z_{CM_{S0}}{}^2\right) & \dots & \dots \\ \dots & \dots & \dots \\ \dots & \dots & \dots \end{bmatrix} + M \begin{bmatrix} \left(\bar{y}^2 + \bar{z}^2\right) & \dots & \dots \\ \dots & \dots & \dots \\ \dots & \dots & \dots \end{bmatrix}$$

INERTIA PRINCIPAL DIRECTIONS

In general, in a solid rigid body, the kinetic moment will not have the same direction as the angular velocity vector. In the coincident cases, the directions are known as the inertia principal direction. In those cases:

$$[I_O].\vec{\varpi} = \lambda \vec{\varpi}$$

Gives origin to the following equation system:

$$\left([I_O] - \lambda [I_1]\right).\vec{\varpi} = \vec{0}$$

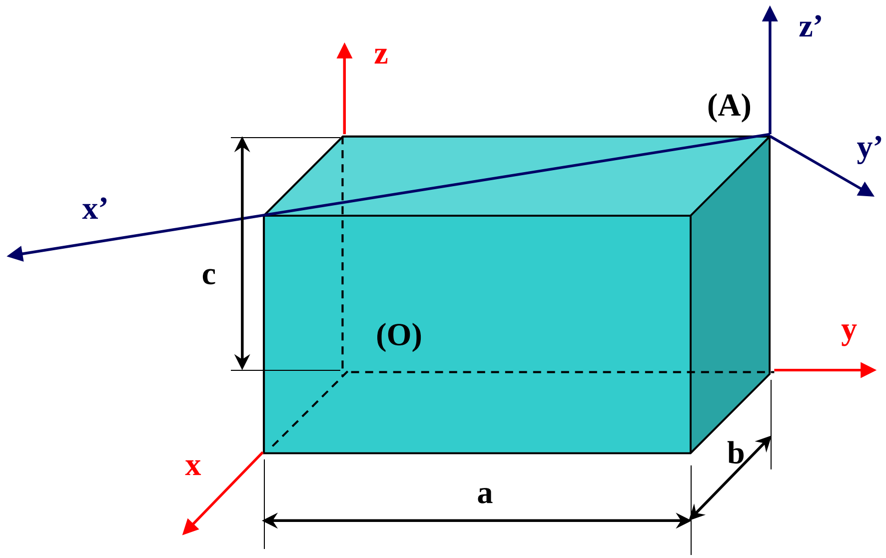
Being a homogeneous system, the only way to have solution different from zero is to establish the condition of determinant equal to zero:

$$\det\left([I_O] - \lambda [I_1]\right) = 0$$

Conclusion: A third order polynomial equation will result, being the three numerical solutions equal to the principal moments of inertia. For each principal moment λ we may expect an infinity of principal directions ω.

THEMATIC EXERCISE 10 – MASS MATRIX

- Calculate the mass matrix (inertial tensor) of the represented rectangular prism at point (O).
- Determine the mass matrix at point (A), relative to system S’, by referential transformation.
- Determine the inertial principal directions and the corresponding principal moments of inertia, at point (O).



•Exercise data:

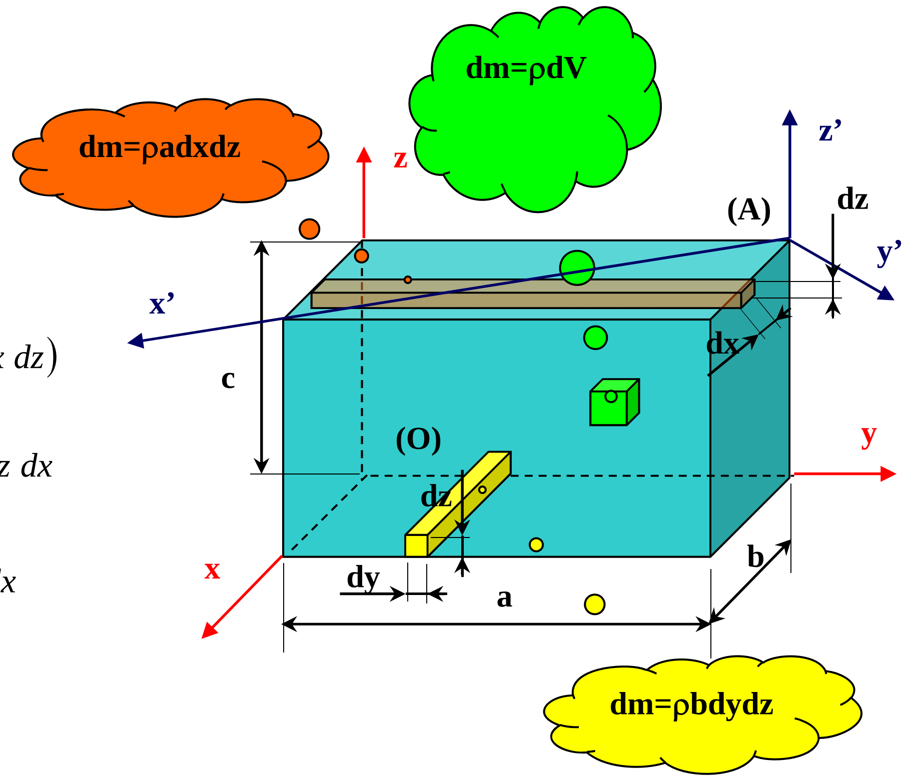
- m=12 mass unity [M]
- a=20 length unity [L]
- b=10 length unity [L]
- c=10 length unity [L]

THEMATIC EXERCISE -– MASS MATRIX

- Moments of inertia, point (O) system S:

$$I_{xx} = \int_m \left(y^2 + z^2\right) dm = \int_V \left(y^2 + z^2\right)\rho(b\,dy\,dz) = \int_V y^2 \rho(b\,dy\,dz) + \int_V z^2 \rho(b\,dy\,dz)$$
$$= \rho b \int_0^c \int_0^a y^2\,dy\,dz + \rho b \int_0^a \int_0^c z^2\,dz\,dy = \rho b \int_0^c \left[\frac{y^3}{3}\right]_0^a dz + \rho b \int_0^a \left[\frac{z^3}{3}\right]_0^c dy$$
$$= \rho b \int_0^c \frac{a^3}{3} dz + \rho b \int_0^a \frac{c^3}{3} dy = \rho b \frac{a^3}{3} c + \rho b \frac{c^3}{3} a = \rho abc\left(\frac{a^2}{3} + \frac{c^2}{3}\right) = m\left(\frac{a^2}{3} + \frac{c^2}{3}\right) = 2000\left[ML^2\right]$$

$$I_{yy} = \int_m \left(x^2 + z^2\right) dm = \int_V \left(x^2 + z^2\right)\rho(a\,dx\,dz) = \int_V x^2 \rho(a\,dx\,dz) + \int_V z^2 \rho(a\,dx\,dz)$$
$$= \rho a \int_0^c \int_0^b x^2\,dx\,dz + \rho a \int_0^b \int_0^c z^2\,dz\,dx = \rho a \int_0^c \left[\frac{x^3}{3}\right]_0^b dz + \rho a \int_0^b \left[\frac{z^3}{3}\right]_0^c dx$$
$$= \rho a \int_0^c \frac{b^3}{3} dz + \rho a \int_0^b \frac{c^3}{3} dx = \rho a \frac{b^3}{3} c + \rho a \frac{c^3}{3} b = \rho abc\left(\frac{b^2}{3} + \frac{c^2}{3}\right) = m\left(\frac{b^2}{3} + \frac{c^2}{3}\right) = 800\left[ML^2\right]$$



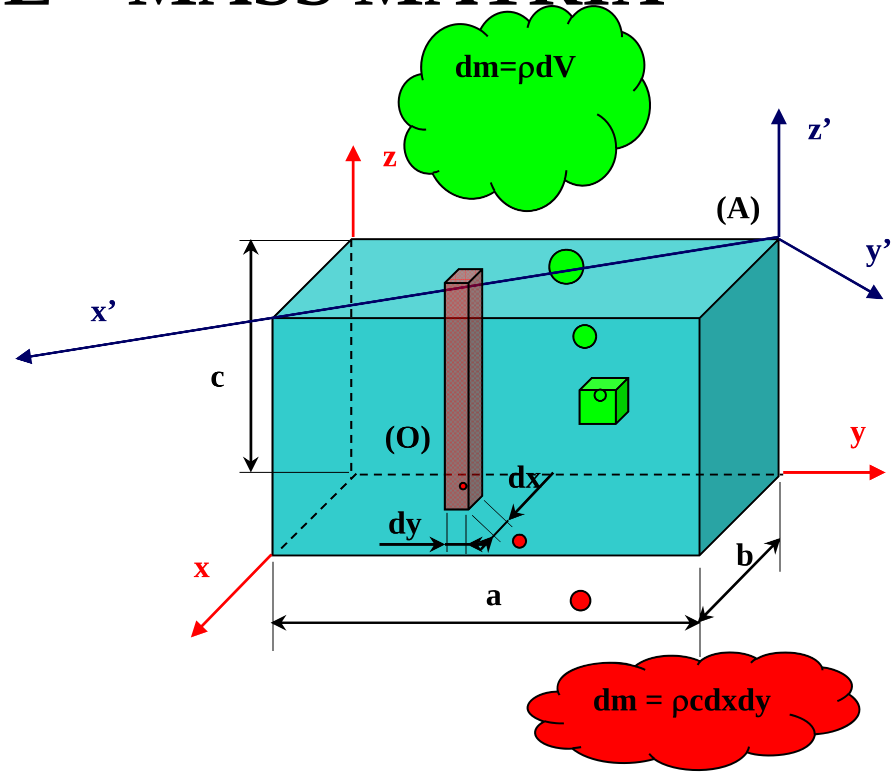
•Exercise data:

- •m=12 mass unity [M]
- •a=20 length unity [L]
- •b=10 length unity [L]
- •c=10 length unity [L]
- •m=ρ(abc)

THEMATIC EXERCISE – MASS MATRIX

- Moments of inertia, point (O) system S:

$$
\begin{aligned}
I_{zz} &= \int_m \left(x^2 + y^2\right) dm \\
&= \int_V \left(x^2 + y^2\right) \rho (c\ dx\ dy) \\
&= \int_V x^2 \rho (c\ dx\ dy) + \int_V y^2 \rho (c\ dx\ dy) \\
&= \rho c \int_0^a \int_0^b x^2\ dx\ dy + \rho c \int_0^b \int_0^a y^2\ dy\ dx \\
&= \rho c \int_0^a \left[\frac{x^3}{3}\right]_0^b dy + \rho c \int_0^b \left[\frac{y^3}{3}\right]_0^a dx \\
&= \rho c \int_0^a \frac{b^3}{3} dy + \rho c \int_0^b \frac{a^3}{3} dx \\
&= \rho c \frac{b^3}{3} a + \rho c \frac{a^3}{3} b \\
&= \rho abc \left(\frac{b^2}{3} + \frac{a^2}{3}\right) \\
&= m \left(\frac{b^2}{3} + \frac{a^2}{3}\right) \\
&= 2000 \left[ML^2\right]
\end{aligned}
$$



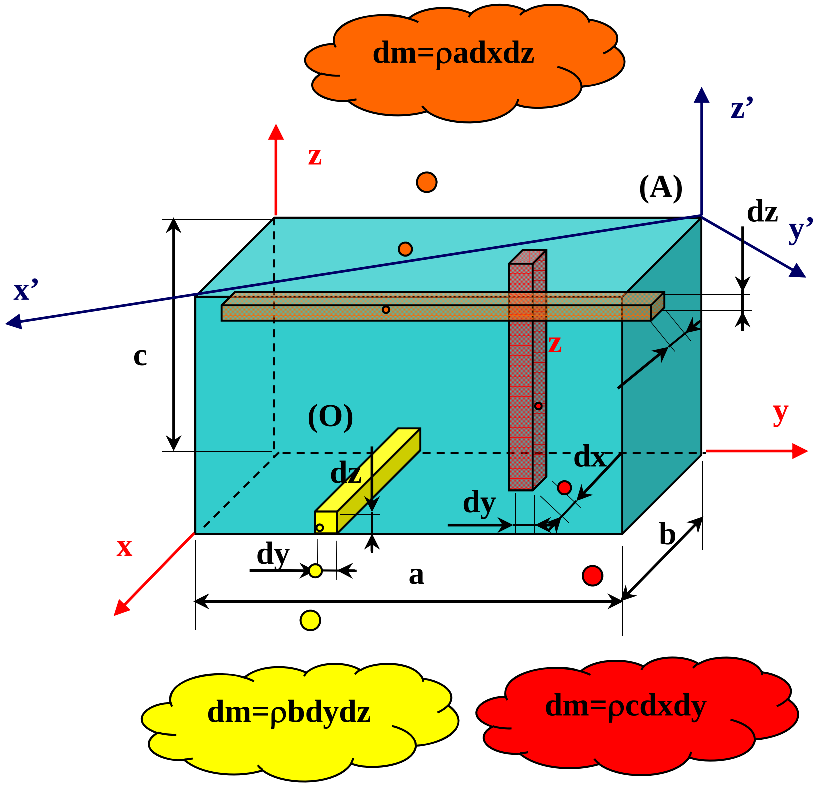
THEMATIC EXERCISE - – MASS MATRIX

- Products of inertia, system S:

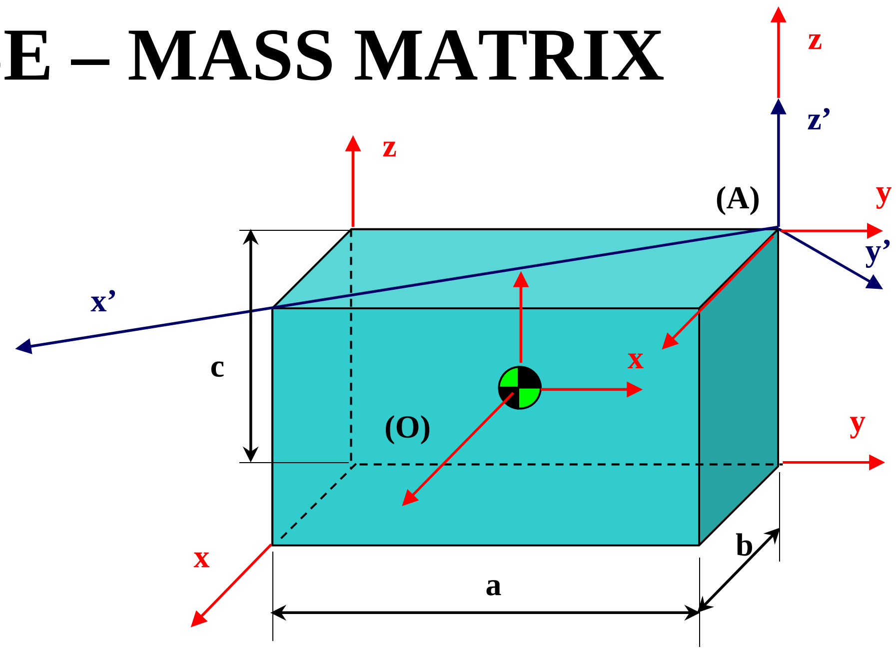
$$P_{xy} = \int_m (xy)\,dm = \int_V (xy)\,\rho(c\,dx\,dy) = \rho c\int_0^a y\int_0^b x\,dx\,dy = \rho c\int_0^a y\left[\frac{x^2}{2}\right]_0^b dy = \rho c\int_0^a \frac{b^2}{2}ydy = \rho c\frac{b^2}{2}\left[\frac{y^2}{2}\right]_0^a = \rho c\left(\frac{b^2}{2}\right)\left(\frac{a^2}{2}\right) = \rho cba\left(\frac{b}{2}\times\frac{a}{2}\right) = m\left(\frac{ab}{4}\right) = 600\left[ML^2\right]$$

$$P_{xz} = \int_m (xz)\,dm = \int_V (xz)\,\rho(a\,dx\,dz) = \rho a\int_0^c z\int_0^b x\,dx\,dz = \rho a\int_0^c z\left[\frac{x^2}{2}\right]_0^b dz = \rho a\int_0^c \frac{b^2}{2}zdz = \rho a\frac{b^2}{2}\left[\frac{z^2}{2}\right]_0^c = \rho a\left(\frac{b^2}{2}\right)\left(\frac{c^2}{2}\right) = \rho abc\left(\frac{b}{2}\times\frac{c}{2}\right) = m\left(\frac{bc}{4}\right) = 300\left[ML^2\right]$$

$$P_{yz} = \int_m (yz)\,dm = \int_V (yz)\,\rho(b\,dy\,dz) = \rho b\int_0^c z\int_0^a y\,dy\,dz = \rho b\int_0^c z\left[\frac{y^2}{2}\right]_0^a dz = \rho b\int_0^c \frac{a^2}{2}zdz = \rho b\frac{a^2}{2}\left[\frac{z^2}{2}\right]_0^c = \rho b\left(\frac{a^2}{2}\right)\left(\frac{c^2}{2}\right) = \rho bac\left(\frac{a}{2}\times\frac{c}{2}\right) = m\left(\frac{ac}{4}\right) = 600\left[ML^2\right]$$



THEMATIC EXERCISE – MASS MATRIX



- Mass matrix, point (O), system S:

$$[I_O]_S = \begin{bmatrix} I_{xx} & -P_{xy} & -P_{xz} \\ -P_{xy} & I_{yy} & -P_{yz} \\ -P_{xz} & -P_{yz} & I_{zz} \end{bmatrix} = \begin{bmatrix} 2000 & -600 & -300 \\ -600 & 800 & -600 \\ -300 & -600 & 2000 \end{bmatrix}$$

- Mass matrix, point (A), System S:
  - Parallel transposition, from (O) System S, to (A) system S.

$$[I_A]|_S = [I_O]|_S - m\begin{bmatrix} 125 & -50 & -25 \\ -50 & 50 & -50 \\ -25 & -50 & 125 \end{bmatrix} + m\begin{bmatrix} 125 & +50 & +25 \\ +50 & 50 & -50 \\ +25 & -50 & 125 \end{bmatrix}$$

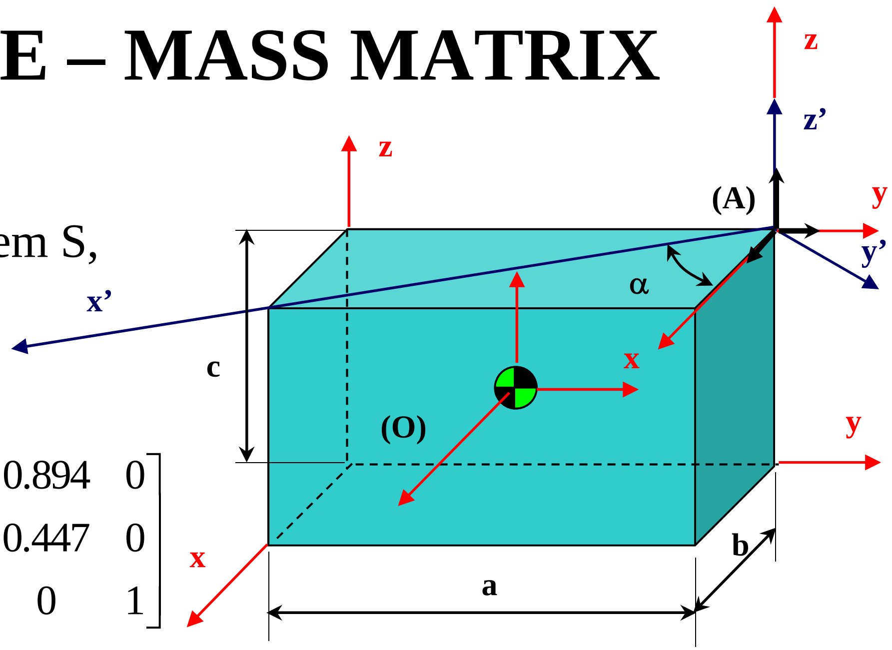
$$O\vec{G} = \begin{Bmatrix} 5 \\ 10 \\ 5 \end{Bmatrix} \qquad A\vec{G} = \begin{Bmatrix} 5 \\ -10 \\ -5 \end{Bmatrix}$$

$$[I_A]|_S = \begin{bmatrix} 2000 & -600 & -300 \\ -600 & 800 & -600 \\ -300 & -600 & 2000 \end{bmatrix} + 12\begin{bmatrix} 0 & 100 & 50 \\ 100 & 0 & 0 \\ 50 & 0 & 0 \end{bmatrix}$$

$$[I_A]|_S = \begin{bmatrix} 2000 & 600 & 300 \\ 600 & 800 & -600 \\ 300 & -600 & 2000 \end{bmatrix}$$

THEMATIC EXERCISE – MASS MATRIX

- Mass matrix, point (A), System S':
  - Rotation transposition, from point (A) System S, to point (A) system S'.



$$[I_A]\big|_{S'} = [T_{S->S'}][I_A]\big|_S [T_{S->S'}]^t$$

$$[I_A]\big|_{S'} = \begin{bmatrix} 0.447 & -0.894 & 0 \\ 0.894 & 0.447 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2000 & -600 & -300 \\ -600 & 800 & -600 \\ -300 & -600 & 2000 \end{bmatrix} \begin{bmatrix} 0.447 & 0.894 & 0 \\ -0.894 & 0.447 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

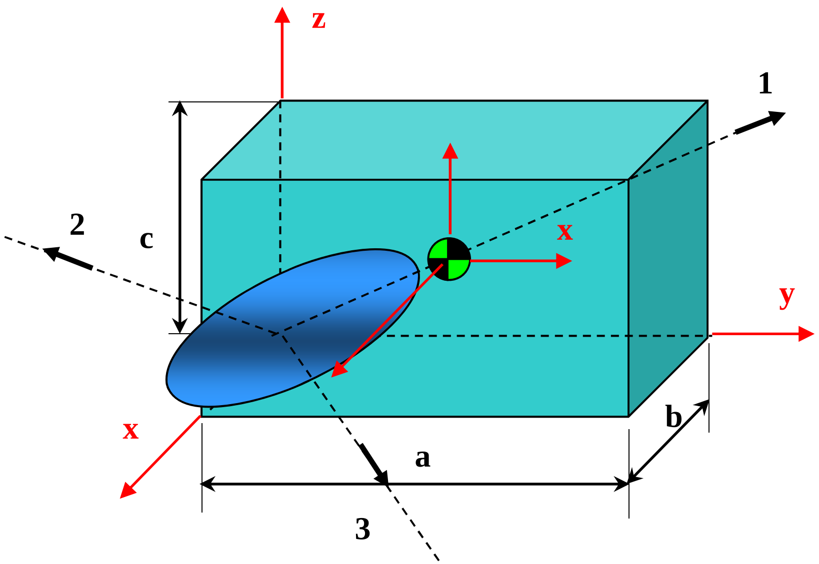
$$[I_A]\big|_{S'} = \begin{bmatrix} 560.0 & 120.0 & 670.82 \\ 120.0 & 2240.0 & 0 \\ 670.82 & 0 & 2000.0 \end{bmatrix}$$

$$[T_{S->S'}] = [\vec{i} \quad \vec{j} \quad \vec{k}] = \begin{bmatrix} \cos(\alpha) & -\sin(\alpha) & 0 \\ \sin(\alpha) & \cos(\alpha) & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0.447 & -0.894 & 0 \\ 0.894 & 0.447 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\alpha = arctg\left(\frac{a}{b}\right) = 63^o,4$$

THEMATIC EXERCISE – MASS MATRIX

- Principal moments of inertia
  - Calculated by determinant condition for indeterminate solutions



$$\det([I_O]-\lambda[I_1])=0$$

$$\Leftrightarrow \det\left(\begin{bmatrix}2000 & -600 & -300\\ -600 & 800 & -600\\ -300 & -600 & 2000\end{bmatrix}-\lambda\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\right)=0$$

$$\Leftrightarrow \det\left(\begin{bmatrix}2000-\lambda & -600 & -300\\ -600 & 800-\lambda & -600\\ -300 & -600 & 2000-\lambda\end{bmatrix}\right)=0$$

- The characteristic polynomial:

$$\Leftrightarrow -\lambda^3 + A_1\lambda^2 + A_2\lambda + A_3 = 0$$

$$A_1 = I_{xx} + I_{yy} + I_{zz} = 4800$$

$$A_2 = -\left(I_{xx}I_{yy} + I_{yy}I_{zz} + I_{zz}I_{xx}\right) + P_{xy}^2 + P_{yz}^2 + P_{zx}^2 = -6.39\times10^6$$

$$A_3 = I_{xx}I_{yy}I_{zz} - I_{xx}P_{yz}^2 - I_{yy}P_{xz}^2 - I_{zz}P_{xy}^2 - 2P_{xy}P_{yz}P_{xz} = 1.472\times10^9$$

Numerical Solution HP48GX:

->Solve-> polynomial-> $a_3x^3 + a_2x^2 + a_1x + a_0 = 0$

$$\begin{cases}\lambda_1 = 289.53\\ \lambda_2 = 2210.47\\ \lambda_3 = 2300.00\end{cases}$$

THEMATIC EXERCISE – MASS MATRIX

- Principal directions:
  - Calculated with the indeterminate homogeneous system.

$$([I_O]-\lambda[I_1]).\vec{\varpi}=\vec{0}$$

$$\begin{bmatrix} 2000-\lambda_i & -600 & -300 \\ -600 & 800-\lambda_i & -600 \\ -300 & -600 & 2000-\lambda_i \end{bmatrix}\begin{Bmatrix} w_{ix} \\ w_{iy} \\ w_{iz} \end{Bmatrix}=\begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

PIS (Possible Indeterminate System)

- 1st principal direction:
  - Specify (w_{ix}=1), λi=λ1 and extract two equations from the system above

$$\begin{bmatrix} 2000-\lambda_i & -600 & -300 \\ -600 & 800-\lambda_i & -600 \\ -300 & -600 & 2000-\lambda_i \end{bmatrix}\begin{Bmatrix} w_{ix} \\ w_{iy} \\ w_{iz} \end{Bmatrix}=\begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

$$\begin{bmatrix} 800-\lambda_1 & -600 \\ -600 & 2000-\lambda_1 \end{bmatrix}\begin{Bmatrix} w_{iy} \\ w_{iz} \end{Bmatrix}=\begin{Bmatrix} 600 \\ 300 \end{Bmatrix} \Leftrightarrow \begin{bmatrix} 510.47 & -600 \\ -600 & 1710.47 \end{bmatrix}\begin{Bmatrix} w_{iy} \\ w_{iz} \end{Bmatrix}=\begin{Bmatrix} 600 \\ 300 \end{Bmatrix}$$

$$\begin{Bmatrix} w_{iy} \\ w_{iz} \end{Bmatrix}=\begin{Bmatrix} 2.35 \\ 1 \end{Bmatrix}$$

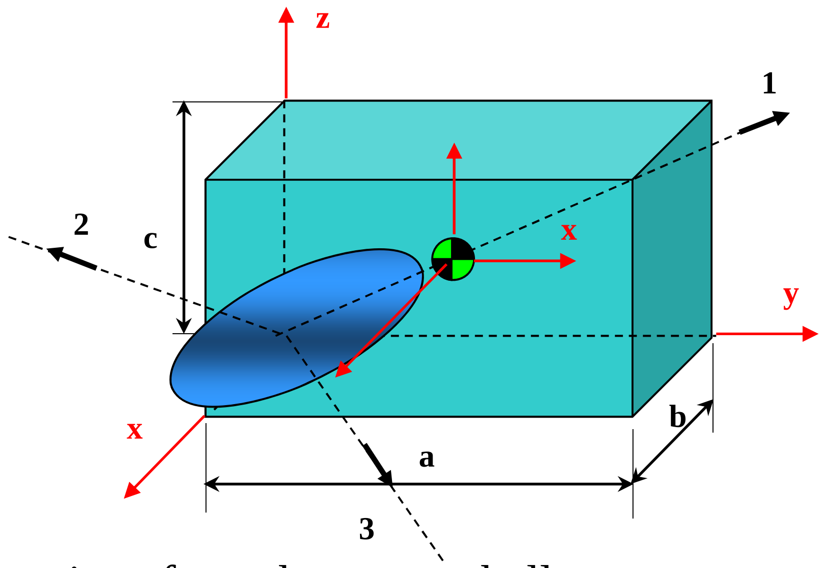
$$\begin{Bmatrix} w_{1x} \\ w_{1y} \\ w_{1z} \end{Bmatrix}=\begin{Bmatrix} 1 \\ 2.35 \\ 1 \end{Bmatrix}\Rightarrow \left\|\begin{Bmatrix} \hat{w}_{1x} \\ \hat{w}_{1y} \\ \hat{w}_{1z} \end{Bmatrix}\right\|=\begin{Bmatrix} 0.364 \\ 0.857 \\ 0.364 \end{Bmatrix}$$

THEMATIC EXERCISE – MASS MATRIX

- 2nd principal direction:
  - Specify (w_{ix}=1), λi=λ2 and extract two equations from the system bellow

$$\begin{bmatrix} 2000-\lambda_i & -600 & -300 \\ -600 & 800-\lambda_i & -600 \\ -300 & -600 & 2000-\lambda_i \end{bmatrix} \begin{Bmatrix} w_{ix} \\ w_{iy} \\ w_{iz} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

$$\left\| \begin{Bmatrix} \hat{w}_{2x} \\ \hat{w}_{2y} \\ \hat{w}_{2z} \end{Bmatrix} \right\| = \begin{Bmatrix} -0.606 \\ 0.515 \\ -0.606 \end{Bmatrix}$$

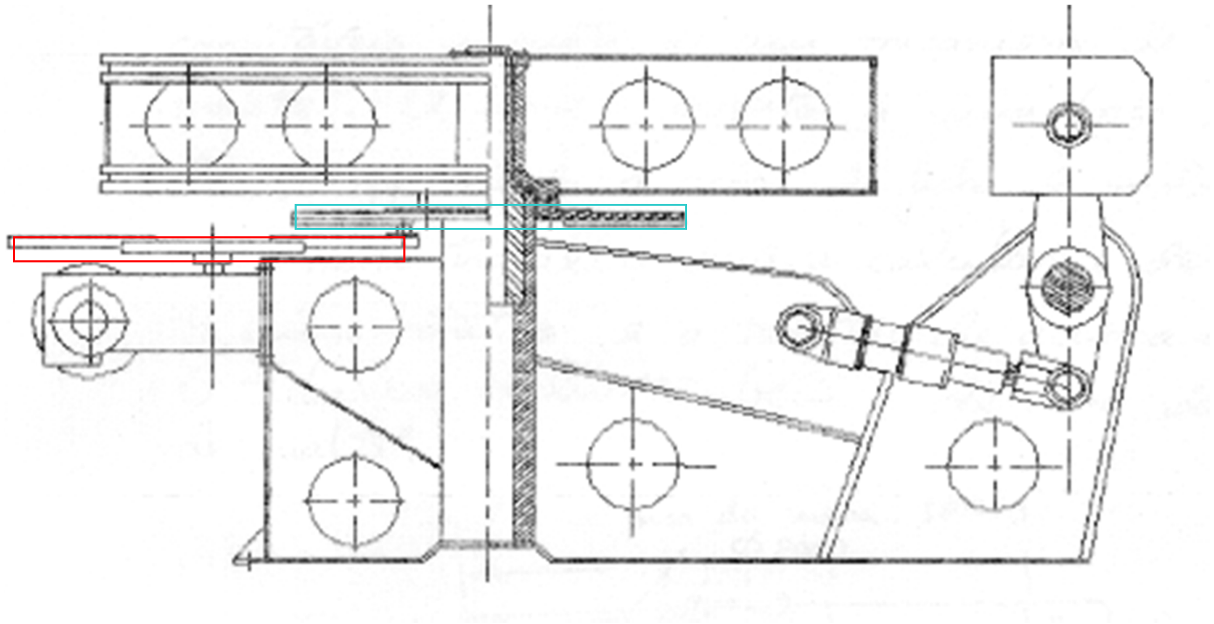


- 3rd principal direction:
  - Specify (w_{ix}=1), λi=λ3 and extract two equations from the system bellow

$$\begin{bmatrix} 2000-\lambda_i & -600 & -300 \\ -600 & 800-\lambda_i & -600 \\ -300 & -600 & 2000-\lambda_i \end{bmatrix} \begin{Bmatrix} w_{ix} \\ w_{iy} \\ w_{iz} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

$$\left\| \begin{Bmatrix} \hat{w}_{3x} \\ \hat{w}_{3y} \\ \hat{w}_{3z} \end{Bmatrix} \right\| = \begin{Bmatrix} 0.7071 \\ 0 \\ 0.7071 \end{Bmatrix}$$

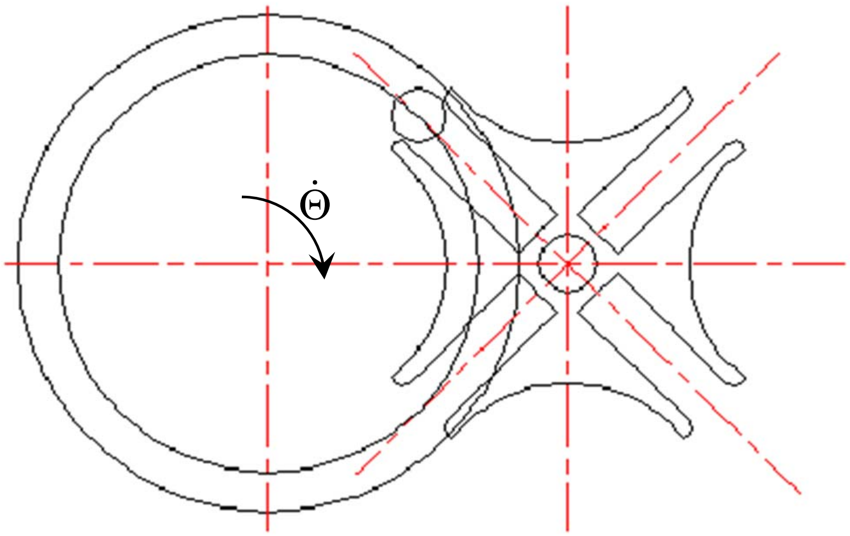
STUDY CASE



A rotative transfer machine, for shoes industry, with four different working points will be working with a special mechanism – *Malta crossing system*. The rotating table will be submitted to a radial force of 8000 [N] and its own body load should not pass through 1800 [kg]. The external dimension should not be grater than 2000[mm]. Its productivity factor should be grater than 13 shoes per minute.

STUDY CASE – solution

System power dimensioning:



1. The required power should be calculated by the product of the maximum binary required and the angular velocity.
2. The required maximum binary may be calculated by the dynamic momentum (time derivative of the kinetic momentum or angular momentum).
3. The electric motor power will be calculated by the product of the motor out binary and the angular velocity. This angular velocity is connected to the productivity solution.

$$\dot{\Theta} = 1{,}415(rad/s)$$

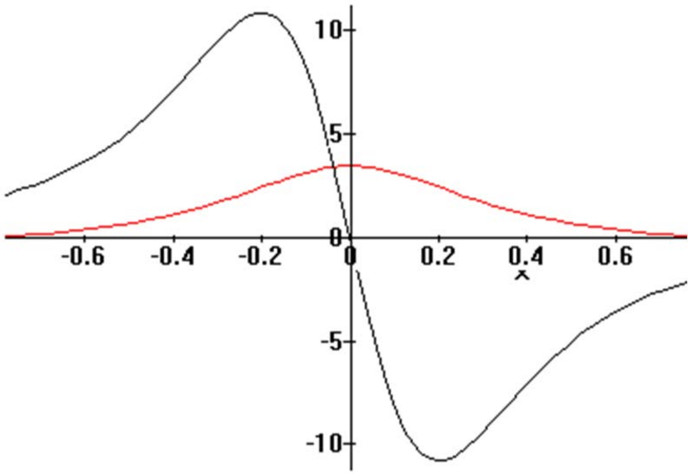
$$\vec{K}_{CM} = \dot{\vec{H}}_{CM} \quad , \quad \vec{H}_{CM} = \begin{bmatrix} & & \\ & I & \\ & & \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ \dot{\psi} \end{Bmatrix}$$

STUDY CASE – solution

Kinematic analysis: Angular velocity and acceleration

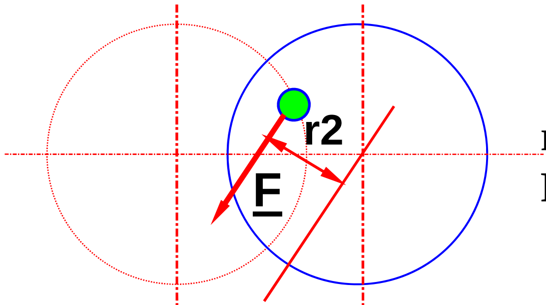
The maximum value of acceleration will provide the maximum binary. This value will be expected to θ=-11.7º, being equal to **10,786 (rad/s²)**

$$\sum \vec{M}_{CM} = \vec{K}_{CM} \quad \Leftrightarrow \quad B_{máx} = I_{zz}\ddot{\psi}$$



The inertial moment may be calculated by:

$$Izz \approx M \frac{\left(\phi^2{}_{ext} - \phi^2{}_{\text{int}}\right)}{8} = 87{,}75(\text{kg.m}^2)$$



r2=0,204(m)
F=4643 (N)

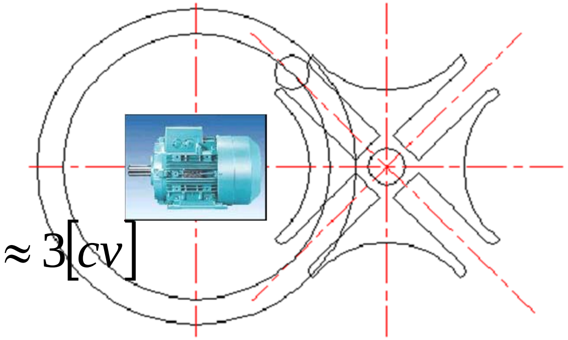
Knowing that:

$$\vec{K}_{CM} = \sum \vec{M}^{\vec{F}} = 946{,}46[Nm]$$

3rd Newton's law:

$$B_{motor} = 1579{,}5[Nm]$$

$$Power = B_{motor}.\dot{\Theta} = 1579{,}5 \times 1{,}415 = 2235[W] \approx 3[cv]$$

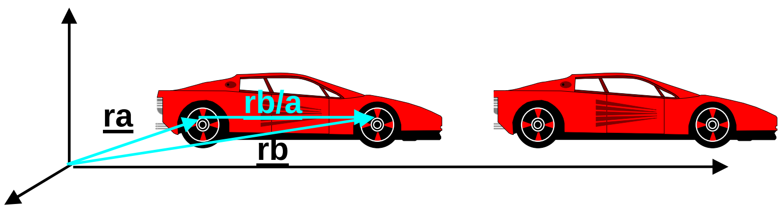


Final step: motor catalogue.

KINEMATICS OF RIGID BODIES

1- Translation
2- Fix point rotation
3- General plane motion
4- Three-dimensional movement around a fix axis
5- General motion

1- Translation



Reference position

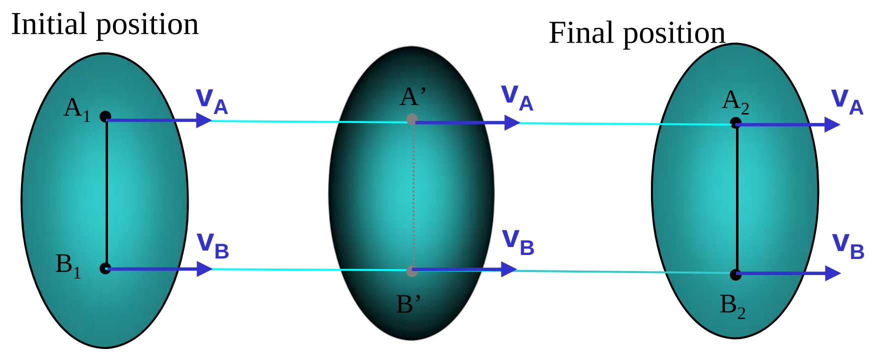
$$\vec{r}_A + \vec{r}_{B/A} = \vec{r}_B$$

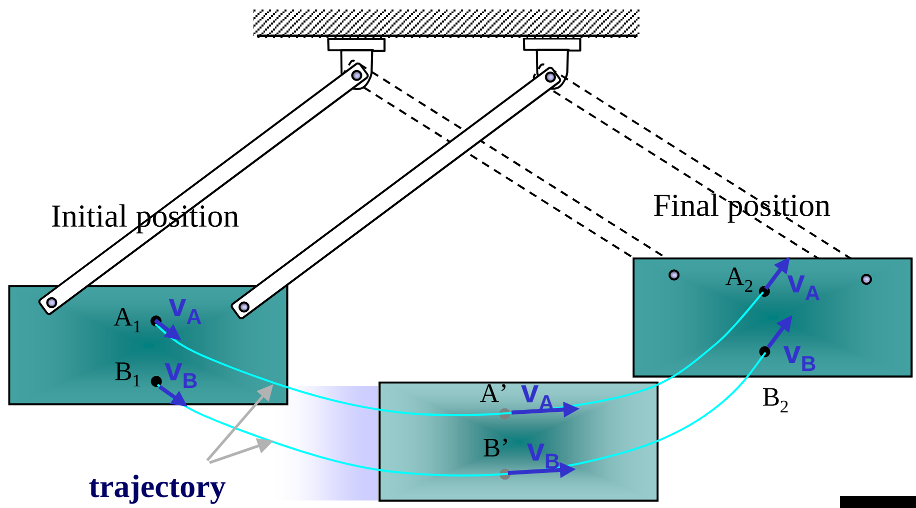
Differentiating in relation to time:

$$\vec{v}_A + \vec{0} = \vec{v}_B$$

Differentiating one more time:

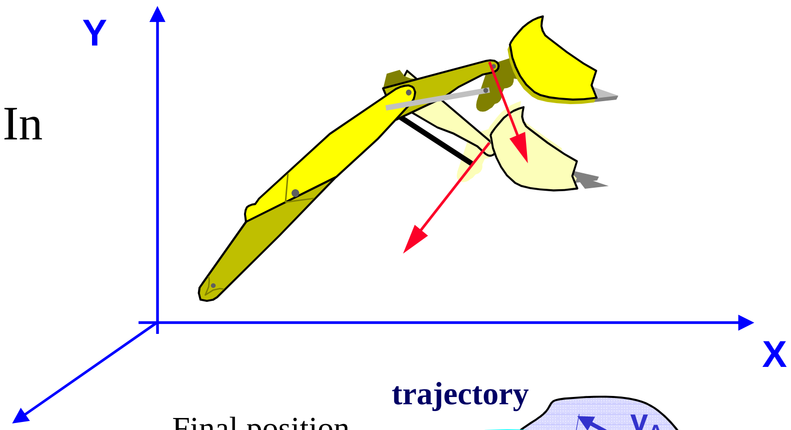
$$\vec{a}_A = \vec{a}_B$$





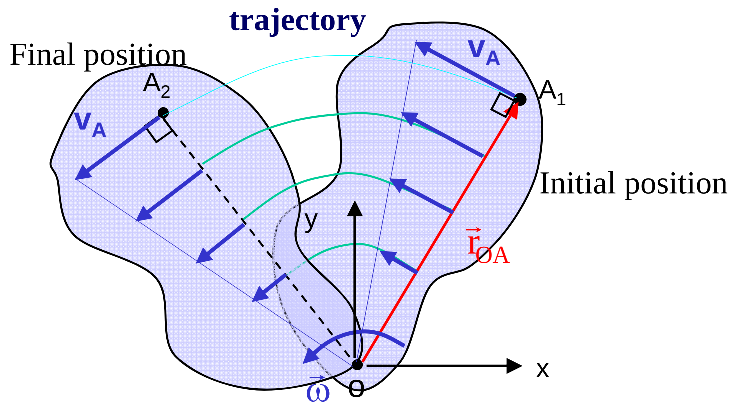
FIX POINT ROTATION

2- Fix point rotation



Vector velocity is always tangent to the trajectory. In intrinsic coordinates we can write:

$$v = \frac{ds}{dt}$$



Linear velocity results from the external product definition

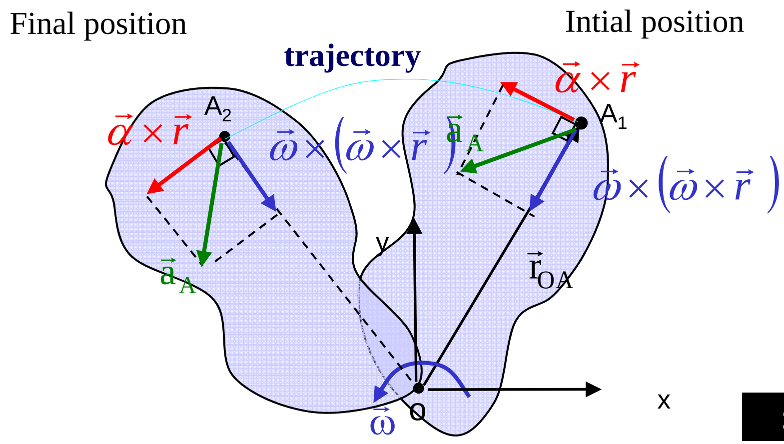
$$\vec{v} = \vec{w} \times \vec{r}$$

Angular velocity parallel to the fixed axis rotation

$$\vec{w} = \dot{\theta}\vec{k}$$

Angular acceleration $\dot{w}$ is parallel to the fixed axis rotation:

$$\vec{a} = \dot{\vec{w}} \times \vec{r} + \vec{w} \times (\vec{w} \times \vec{r})$$



Note: Movement may be effectively discovery by one of two possibilities:

1- $\theta = \theta(t)$

2- $\ddot{\theta} = \ddot{\theta}(\theta, \dot{\theta})$

FIX POINT ROTATION

2- Fix point rotation : Equations

Uniform rotation

$$\theta = \theta_0 + \dot{\theta} t$$

Uniform accelerated rotation

$$\dot{\theta} = \dot{\theta}_0 + \alpha t$$

$$\theta = \theta_0 + \dot{\theta} t + \ddot{\theta} t^2 / 2$$

Relative velocity

$$\vec{V}_{B/A} = w\vec{k} \times \vec{r}_{B/A}$$

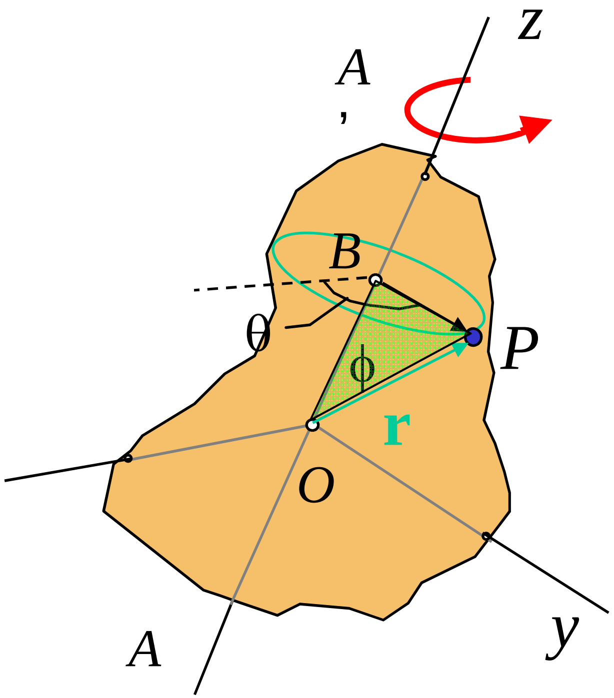
Conclusion:

1- General expression, valid for two points belonging to the same rigid body.

$$\vec{V}_B = \vec{V}_A + \vec{w} \times A\vec{B}$$

2- Angular velocity is independent from the reference point.

RIGID BODY – kinematics: position and velocity



In rigid body ***translation***, *all points of the body have the same velocity and the same acceleration at any given instant.*

Considering the ***rotation*** of a rigid body about a fixed axis, the position of the body is defined by the angle θ that the line *BP*, drawn from the axis of rotation to a point *P* of the body, forms with a fixed plane. The magnitude of the velocity of *P* is:

$$v = \frac{ds}{dt} = r\dot{\theta}\sin\phi$$

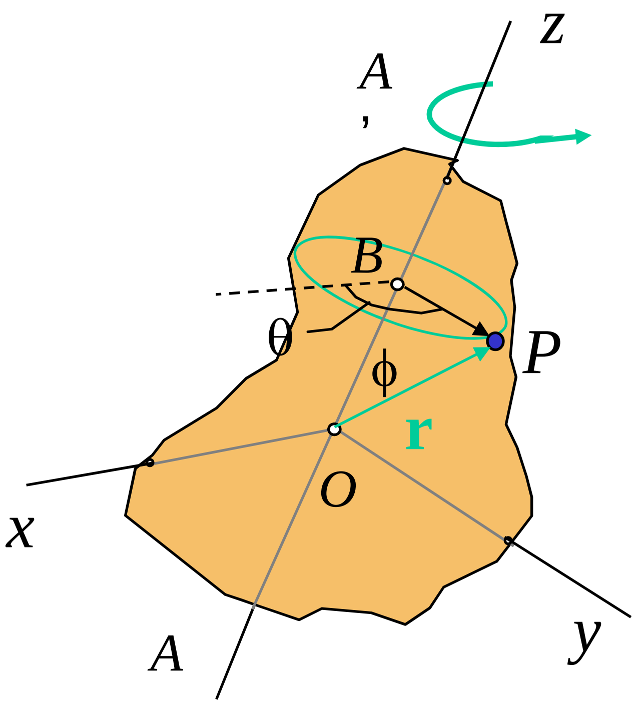
where $\dot{\theta}$ is the time derivative of θ.

The velocity of *P* is expressed as $$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \boldsymbol{\omega} \times \mathbf{r}$$

where the vector $\boldsymbol{\omega} = \omega\mathbf{k} = \dot{\theta}\mathbf{k}$

ω is directed along the fixed axis of rotation and represents the ***angular velocity*** of the body.

RIGID BODY – kinematics: acceleration



$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \boldsymbol{\omega} \times \mathbf{r}$$

$$\boldsymbol{\omega} = \omega\mathbf{k} = \dot{\theta}\mathbf{k}$$

Denoting by $\boldsymbol{\alpha}$ the time derivative $d\boldsymbol{\omega}/dt$ of the angular velocity, we express the acceleration of *P* as:

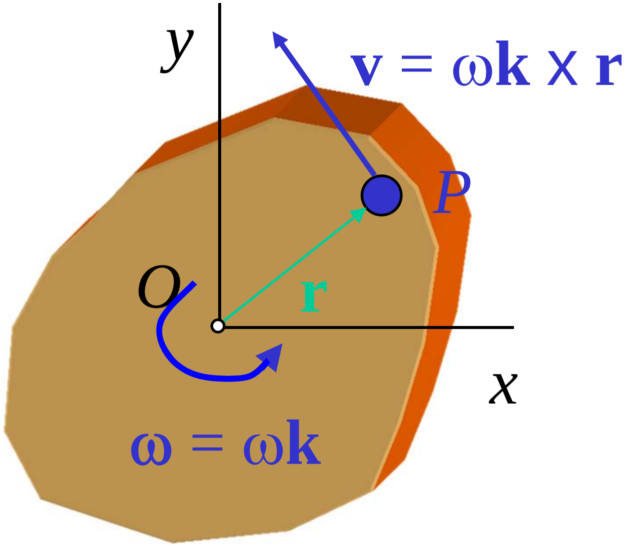
$$\mathbf{a} = \boldsymbol{\alpha} \times \mathbf{r} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})$$

differentiating $\boldsymbol{\omega}$ and recalling that **k** is constant in magnitude and direction, we find that:

$$\boldsymbol{\alpha} = \alpha\mathbf{k} = \dot{\omega}\mathbf{k} = \ddot{\theta}\mathbf{k}$$

The vector $\boldsymbol{\alpha}$ represents the ***angular acceleration*** of the body and is directed along the fixed axis of rotation.

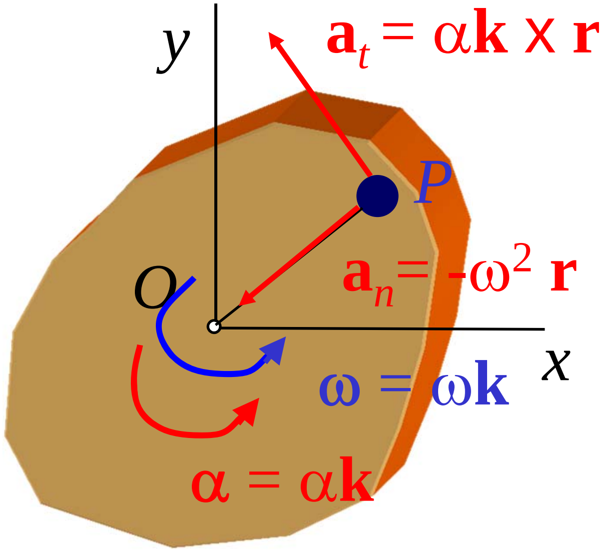
PLANE ROTATION



Consider the motion of a representative slab located in a plane perpendicular to the axis of rotation of the body. The angular velocity is perpendicular to the slab, so the velocity of point *P* of the slab is:

$$\mathbf{v} = \omega \mathbf{k} \times \mathbf{r}$$

where **v** is contained in the plane of the slab. The acceleration of point *P* can be resolved into tangential and normal components, respectively equal to:



$$\mathbf{a}_t = \alpha \mathbf{k} \times \mathbf{r} \qquad a_t = r\alpha$$

$$\mathbf{a}_n = -\omega^2 \mathbf{r} \qquad a_n = r\omega^2$$

ANGULAR VELOCITY AND ACCELERATION

The ***angular velocity*** and ***angular acceleration*** of the slab can be expressed as

$$\omega = \frac{d\theta}{dt} = \dot{\theta}$$

$$\alpha = \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2} = \ddot{\theta}$$

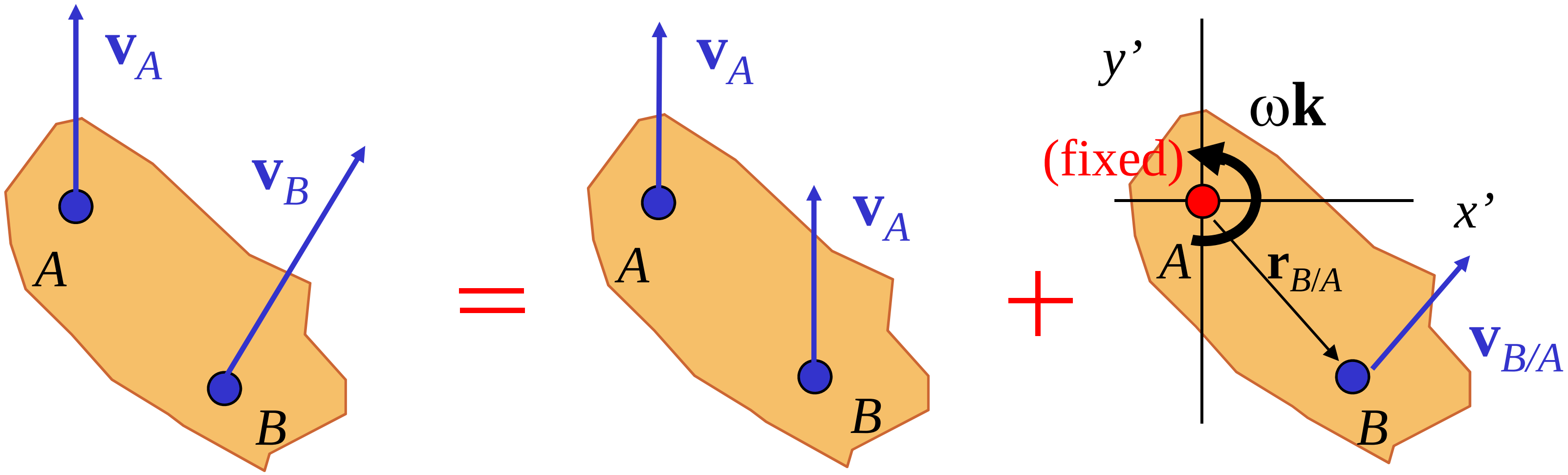
or

$$\omega = \alpha \frac{d\theta}{d\omega}$$

Two particular cases of rotation are frequently encountered: ***uniform rotation*** and ***uniformly accelerated rotation***. Problems involving either of these motions can be solved by using equations similar to those for uniform rectilinear motion and uniformly accelerated rectilinear motion of a particle, where *x*, *v*, and *a* are replaced by θ, ω, and α.

GENERAL PLANE MOTION

3- General plane motion



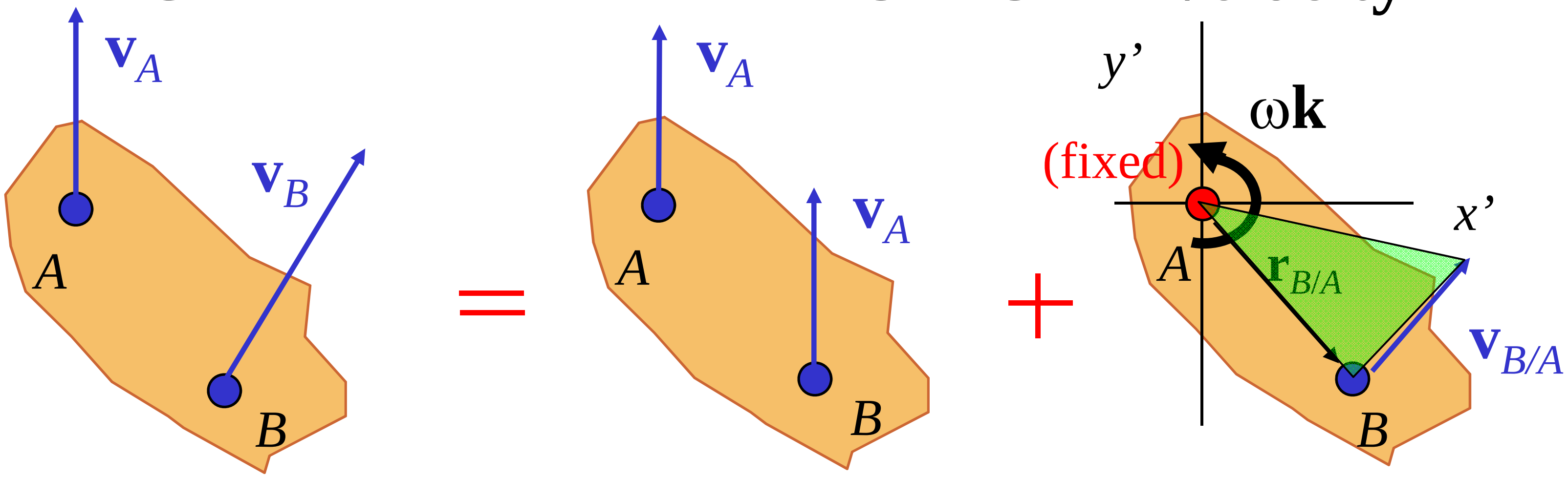
Plane motion = Translation with *A* + Rotation about *A*

The ***most general plane motion*** of a rigid slab can be considered as the ***sum of a translation and a rotation***. The slab shown can be assumed to translate with point *A*, while simultaneously rotating about *A*. It follows that the velocity of any point *B* of the slab can be expressed as:

$$\mathbf{v}_B = \mathbf{v}_A + \mathbf{v}_{B/A}$$

where $\mathbf{v}_A$ is the velocity of *A* and $\mathbf{v}_{B/A}$ is the relative velocity of *B* with respect to *A*.

GENERAL PLANE MOTION – velocity



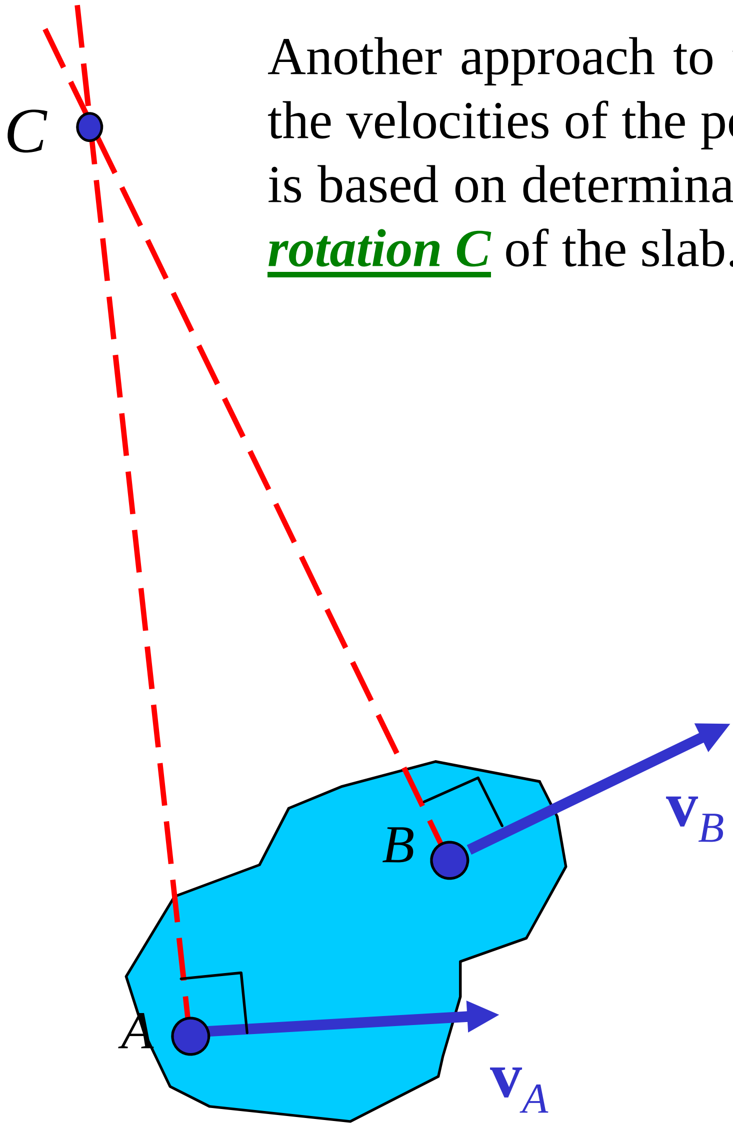
Denoting by $\mathbf{r}_{B/A}$ the position of *B* relative to *A*, we note that

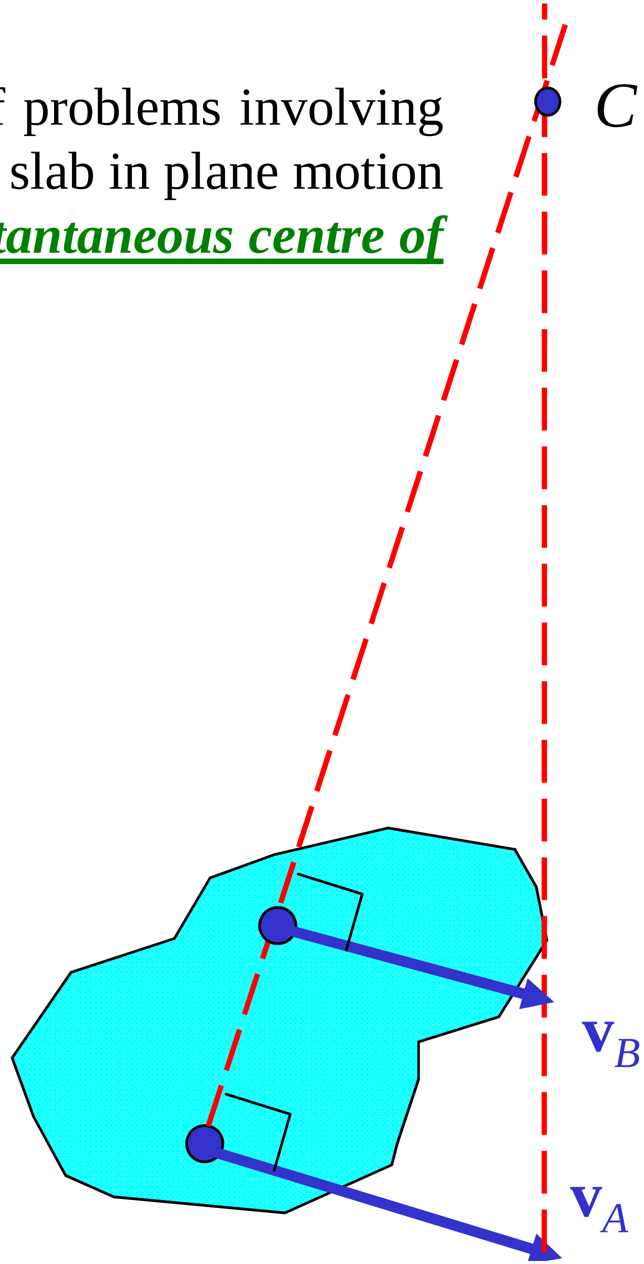
$$\mathbf{v}_{B/A} = \omega\mathbf{k} \times \mathbf{r}_{B/A} \qquad v_{B/A} = (r_{B/A})\omega = \mathbf{r}\omega$$

The fundamental equation relating the absolute velocities of points *A* and *B* and the relative velocity of *B* with respect to *A* can be expressed in the form of a vector diagram and used to solve problems involving the motion of various types of mechanisms.

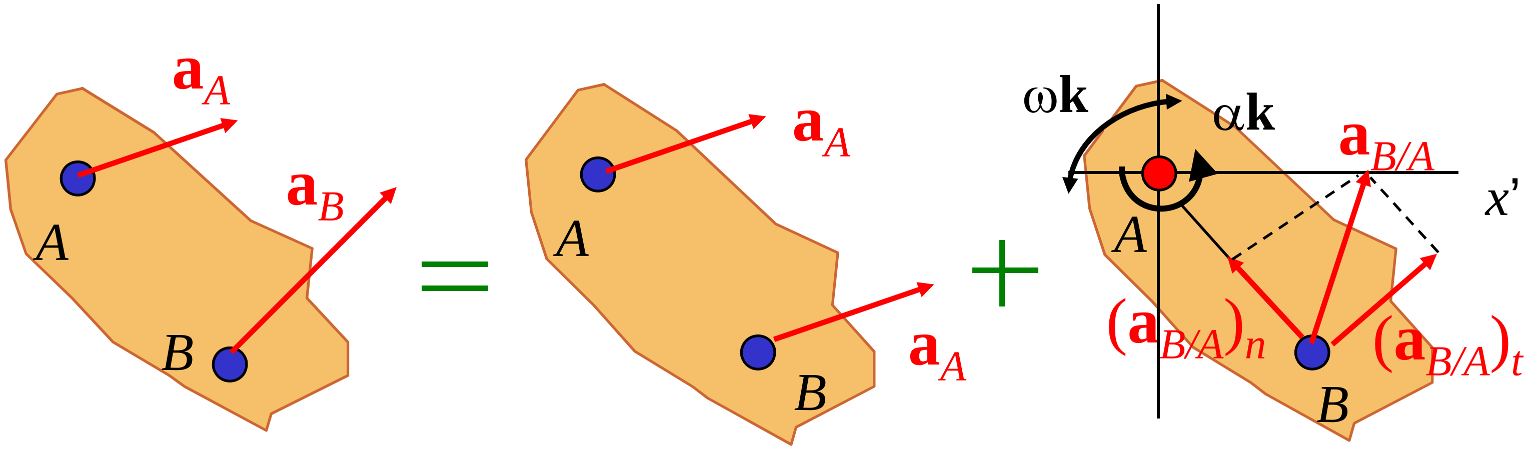
INSTANTANEOUS CENTRE OF ROTATION

Another approach to the solution of problems involving the velocities of the points of a rigid slab in plane motion is based on determination of the ***instantaneous centre of rotation C*** of the slab.





GENERAL PLANE MOTION - acceleration



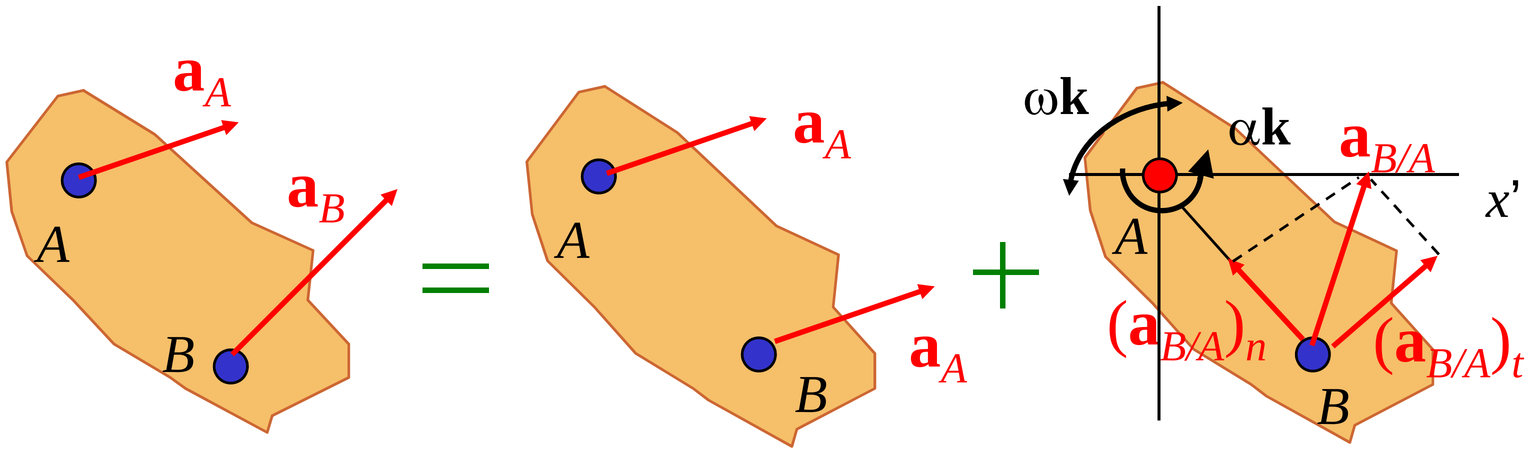
Plane motion = Translation with A + Rotation about A

The fact that any plane motion of a rigid slab can be considered the sum of a translation of the slab with reference to point A and a rotation about A is used to relate the absolute accelerations of any two points A and B of the slab and the relative acceleration of B with respect to A.

$$\mathbf{a}_B = \mathbf{a}_A + \mathbf{a}_{B/A}$$

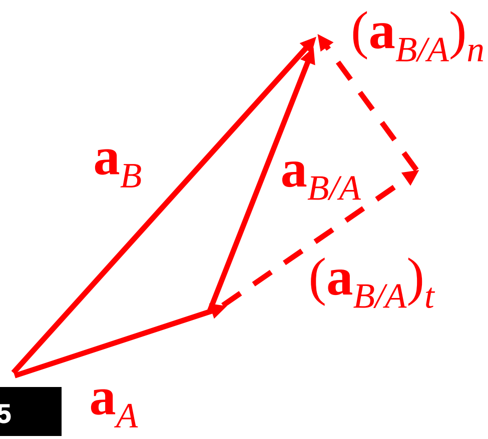
where $\mathbf{a}_{B/A}$ consists of a <u>*normal component*</u> $(\mathbf{a}_{B/A})_n$ of magnitude $r\omega^2$ directed toward A, and a <u>*tangential component*</u> $(\mathbf{a}_{B/A})_t$ of magnitude $r\alpha$ perpendicular to the line AB.

GENERAL PLANE MOTION – acceleration



Plane motion = Translation with A + Rotation about A

The fundamental equation relating the absolute accelerations of points A and B and the relative acceleration of B with respect to A can be expressed in the form of a vector diagram and used to determine the accelerations of given points of various mechanisms.

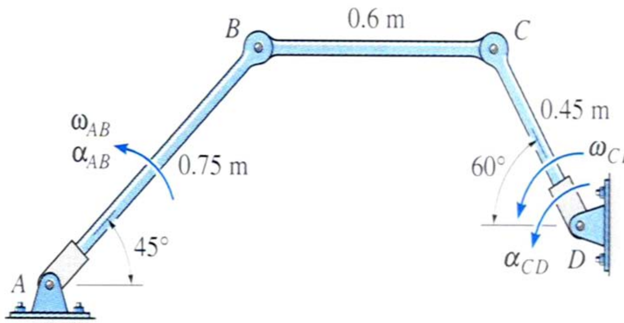


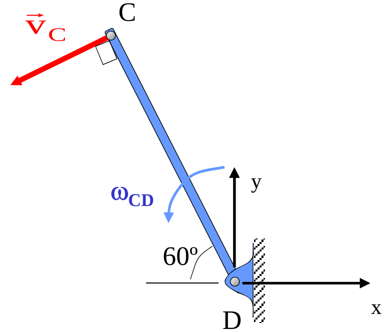
Important note:

The instantaneous centre of rotation C cannot be used for the determination of accelerations, since point C , in general, does *not* have zero acceleration.

THEMATIC EXERCISE 11

The "four body" mechanism represented in the left figure has three bars, connected by two distinct points B and C. For the represented specific time, the bar CD presents an angular acceleration of α_{CD}=5 rad/s2 and angular velocity equal to w_{CD}=2 rad/s, both anticlockwise. Determine the angular velocity and acceleration of the bar AB.





Velocity analysis of bar CD (fixed plane rotation):

$$\vec{v}_C = \vec{v}_D + \vec{v}_{C/D} = \vec{v}_D + \vec{\omega}_{CD} \times \vec{r}_{DC}$$

$$\vec{v}_C = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} + \begin{Bmatrix} 0 \\ 0 \\ 2 \end{Bmatrix} \times \begin{Bmatrix} -0.225 \\ 0.39 \\ 0 \end{Bmatrix} = \begin{Bmatrix} -0.8 \\ -0.45 \\ 0 \end{Bmatrix} [m/s]$$

$$\vec{r}_{DC} = \begin{Bmatrix} -0.45\cos(60) \\ 0.45\sin(60) \\ 0 \end{Bmatrix} = \begin{Bmatrix} -0.225 \\ 0.39 \\ 0 \end{Bmatrix} [m]$$

Velocity analysis of bar BC (general plane rotation):

$$\vec{v}_B = \vec{v}_C + \vec{v}_{B/C} = \vec{v}_C + \vec{\omega}_{BC} \times \vec{r}_{CB}$$

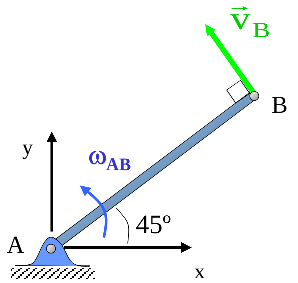
$$\vec{v}_B = \begin{Bmatrix} -0.8 \\ -0.45 \\ 0 \end{Bmatrix} + \begin{Bmatrix} 0 \\ 0 \\ -\omega_{BC} \end{Bmatrix} \times \begin{Bmatrix} -0.6 \\ 0 \\ 0 \end{Bmatrix} = \begin{Bmatrix} -0.8 \\ -0.45 + 0.6\omega_{BC} \\ 0 \end{Bmatrix} [m/s]$$

$$\vec{r}_{CB} = \begin{Bmatrix} -0.6 \\ 0 \\ 0 \end{Bmatrix} m \qquad \vec{v}_C = \begin{Bmatrix} -0.8 \\ -0.45 \\ 0 \end{Bmatrix} m/s$$

$$\vec{\omega}_{BC} = \begin{Bmatrix} 0 \\ 0 \\ -\omega_{BC} \end{Bmatrix} rad/s$$

THEMATIC EXERCISE (cont.)

Being the motion of the bar AB considered as a fixed point plane rotation, a velocity and acceleration analysis may be done:



$$\vec{v}_B = \vec{v}_A + \vec{v}_{B/A} = \vec{v}_A + \vec{\omega}_{AB} \times \vec{r}_{AB}$$

$$\vec{r}_{AB} = \begin{Bmatrix} 0.75 * \cos(45) \\ 0.75 * \sin(45) \\ 0 \end{Bmatrix} = \begin{Bmatrix} 0.53 \\ 0.53 \\ 0 \end{Bmatrix} m$$

$$\vec{v}_B = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} + \begin{Bmatrix} 0 \\ 0 \\ \omega_{AB} \end{Bmatrix} \times \begin{Bmatrix} 0.53 \\ 0.53 \\ 0 \end{Bmatrix} = \begin{Bmatrix} -0.53\,\omega_{AB} \\ 0.53\,\omega_{AB} \\ 0 \end{Bmatrix} [m/s]$$

The velocity of point B calculated by this two expression should be equal.

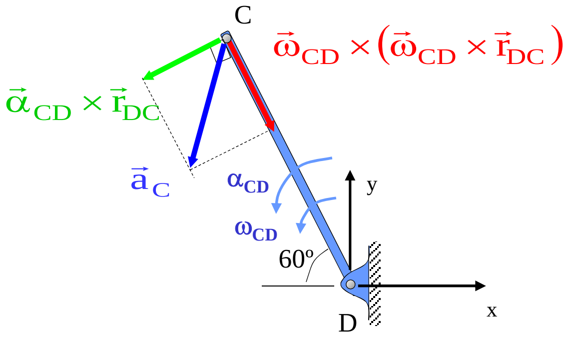
$$\begin{Bmatrix} -0.53 * \omega_{AB} \\ 0.53 * \omega_{AB} \\ 0 \end{Bmatrix} = \begin{Bmatrix} -0.8 \\ -0.45 + 0.6 * \omega_{BC} \\ 0 \end{Bmatrix} \Leftrightarrow \begin{matrix} \omega_{AB} = 1.51\ rad/s \\ \omega_{BC} = 2.083\ rad/s \end{matrix}$$

$$\vec{\omega}_{BC} = \begin{Bmatrix} 0 \\ 0 \\ -2.083 \end{Bmatrix} rad/s$$

$$\vec{\omega}_{AB} = \begin{Bmatrix} 0 \\ 0 \\ 1.51 \end{Bmatrix} rad/s$$

Acceleration analysis of bar CD (fixed plane rotation):

$$\vec{a}_C = \vec{a}_D + \vec{\alpha}_{CD} \times \vec{r}_{DC} + \vec{\omega}_{CD} \times (\vec{\omega}_{CD} \times \vec{r}_{DC}) =$$
$$= \vec{a}_D + (\vec{a}_{C/D})_t + (\vec{a}_{C/D})_n$$



$$\vec{\omega}_{CD} = \begin{Bmatrix} 0 \\ 0 \\ 2 \end{Bmatrix} rad/s \qquad \vec{\alpha}_{CD} = \begin{Bmatrix} 0 \\ 0 \\ 5 \end{Bmatrix} rad/s^2$$

THEMATIC EXERCISE (cont.)

$$\vec{a}_C = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} + \begin{Bmatrix} 0 \\ 0 \\ 5 \end{Bmatrix} \times \begin{Bmatrix} -0.225 \\ 0.39 \\ 0 \end{Bmatrix} + \begin{Bmatrix} 0 \\ 0 \\ 2 \end{Bmatrix} \times \left(\begin{Bmatrix} 0 \\ 0 \\ 2 \end{Bmatrix} \begin{Bmatrix} -0.225 \\ 0.39 \\ 0 \end{Bmatrix} \right) =$$

$$= \begin{Bmatrix} -1.95 \\ -1.125 \\ 0 \end{Bmatrix} + \begin{Bmatrix} 0.9 \\ -1.6 \\ 0 \end{Bmatrix} = \begin{Bmatrix} -1.05 \\ -2.725 \\ 0 \end{Bmatrix} m/s^2$$

Acceleration analysis of bar BC (general plane rotation):

$$\vec{a}_B = \vec{a}_C + \vec{\alpha}_{BC} \times \vec{r}_{CB} + \vec{\omega}_{BC} \times (\vec{\omega}_{BC} \times \vec{r}_{CB})$$

$$\vec{a}_B = \begin{Bmatrix} -1.05 \\ -2.725 \\ 0 \end{Bmatrix} + \begin{Bmatrix} 0 \\ 0 \\ -\alpha_{BD} \end{Bmatrix} \times \begin{Bmatrix} -0.6 \\ 0 \\ 0 \end{Bmatrix} + \begin{Bmatrix} 0 \\ 0 \\ -2.083 \end{Bmatrix} \times \left(\begin{Bmatrix} 0 \\ 0 \\ -2.083 \end{Bmatrix} \begin{Bmatrix} -0.6 \\ 0 \\ 0 \end{Bmatrix} \right) =$$

$$= \begin{Bmatrix} -1.05 \\ -2.725 \\ 0 \end{Bmatrix} + \begin{Bmatrix} 0 \\ 0.6 * \alpha_{BD} \\ 0 \end{Bmatrix} + \begin{Bmatrix} 2.6 \\ 0 \\ 0 \end{Bmatrix} = \begin{Bmatrix} 1.55 \\ -2.725 + 0.6 * \alpha_{BD} \\ 0 \end{Bmatrix} m/s^2$$

Acceleration analysis of bar AB (fixed plane rotation):

$$\vec{a}_B = \vec{a}_A + \vec{a}_{B/A}$$

$$\vec{a}_B = \vec{a}_A + \vec{\alpha}_{AB} \times \vec{r}_{AB} + \vec{\omega}_{AB} \times (\vec{\omega}_{AB} \times \vec{r}_{AB})$$

$$\vec{\omega}_{AB} = \begin{Bmatrix} 0 \\ 0 \\ 1.51 \end{Bmatrix} rad/s$$

THEMATIC EXERCISE (cont.)

$$\vec{a}_B = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} + \begin{Bmatrix} 0 \\ 0 \\ \alpha_{AB} \end{Bmatrix} \times \begin{Bmatrix} 0.53 \\ 0.53 \\ 0 \end{Bmatrix} + \begin{Bmatrix} 0 \\ 0 \\ 1.51 \end{Bmatrix} \times \left(\begin{Bmatrix} 0 \\ 0 \\ 1.51 \end{Bmatrix} \begin{Bmatrix} 0.53 \\ 0.53 \\ 0 \end{Bmatrix} \right) =$$

$$= \begin{Bmatrix} -0.53 * \alpha_{AB} \\ 0.53 * \alpha_{AB} \\ 0 \end{Bmatrix} + \begin{Bmatrix} -1.208 \\ -1.208 \\ 0 \end{Bmatrix} = \begin{Bmatrix} -1.208 - 0.53 * \alpha_{AB} \\ -1.208 + 0.53 * \alpha_{AB} \\ 0 \end{Bmatrix} m/s^2$$

The acceleration of point B calculated by this two expression should be equal, then:

$$\begin{Bmatrix} -1.208 - 0.53 * \alpha_{AB} \\ -1.208 + 0.53 * \alpha_{AB} \\ 0 \end{Bmatrix} = \begin{Bmatrix} 1.55 \\ -2.725 + 0.6 * \alpha_{BD} \\ 0 \end{Bmatrix}$$

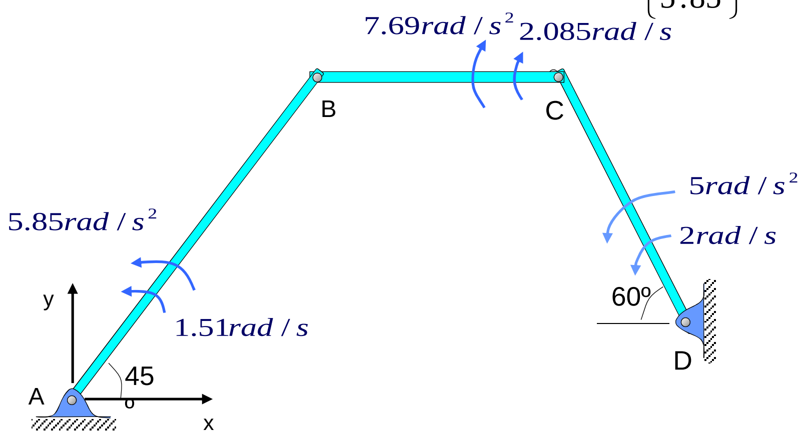
$$x: -1.208 - 0.53 * \alpha_{AB} = 1.55$$

$$y: -1.208 + 0.53 * \alpha_{AB} = -2.725 + 0.6 * \alpha_{BD}$$

$$\alpha_{AB} = 5.85 \, rad/s^2$$

$$\alpha_{BD} = 7.69 \, rad/s^2$$

$$\vec{\alpha}_{AB} = \begin{Bmatrix} 0 \\ 0 \\ 5.85 \end{Bmatrix} rad/s^2 \qquad \vec{\alpha}_{BC} = \begin{Bmatrix} 0 \\ 0 \\ -7.69 \end{Bmatrix} rad/s^2$$



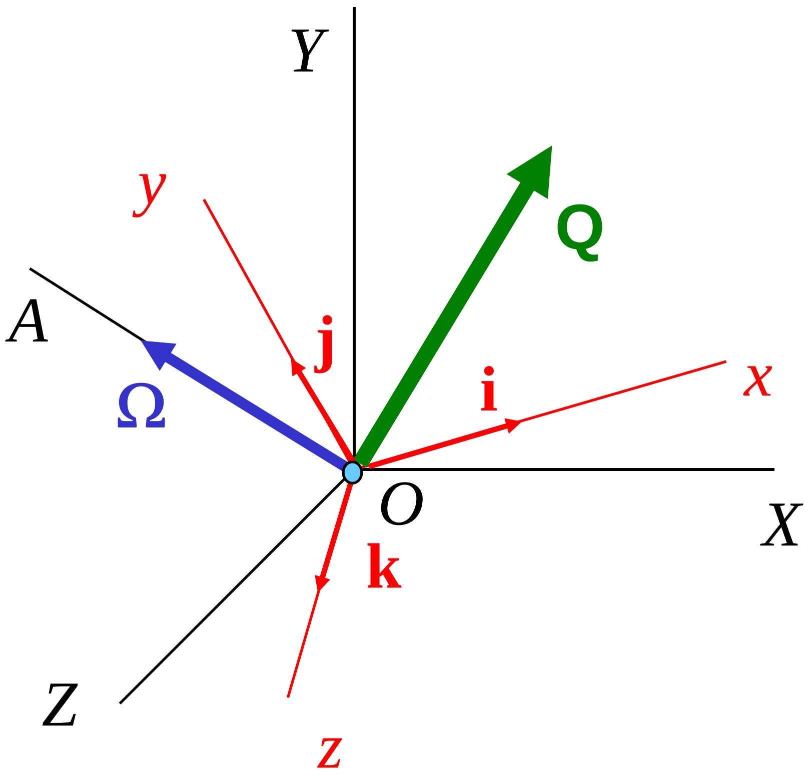
VECTOR DIFFERENTIATION REGARDING A MOVING COORDINATE SYSTEM

•The rate of change of a vector is the same with respect to a fixed frame of reference and with respect to a frame in translation.

•The rate of change of a vector with respect to a rotating frame of reference is different.

The rate of change of a general vector **Q** with respect a fixed frame *OXYZ* and with respect to a frame *Oxyz* rotating with an angular velocity **Ω** is:

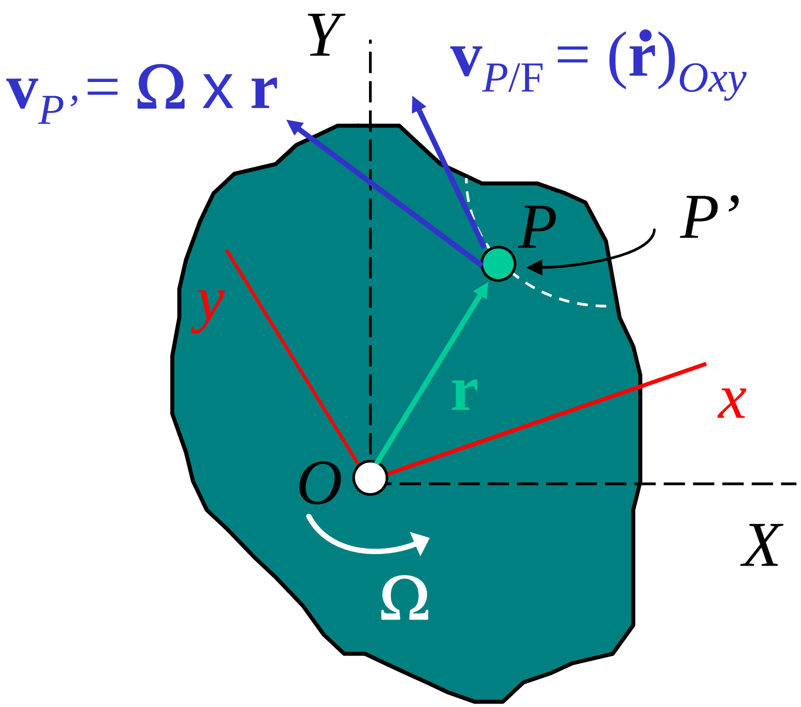
$$(\dot{\mathbf{Q}})_{OXYZ} = (\dot{\mathbf{Q}})_{Oxyz} + \boldsymbol{\Omega} \times \mathbf{Q}$$



The first part represents the rate of change of **Q** with respect to the rotating frame *Oxyz* and the second part, **Ω** x **Q**, is induced by the rotation of the frame *Oxyz*.

ANGULAR VELOCITY OF THE MOVING REFERENCE SYSTEM.

PLANE MOVEMENT (velocity)



Consider the two-dimensional analysis of a particle *P* , moving with respect to a frame F rotating with an angular velocity **Ω** about a fixed axis. The absolute velocity of *P* can be expressed as:

$$\mathbf{v}_P = \mathbf{v}_{P'} + \mathbf{v}_{P/F}$$

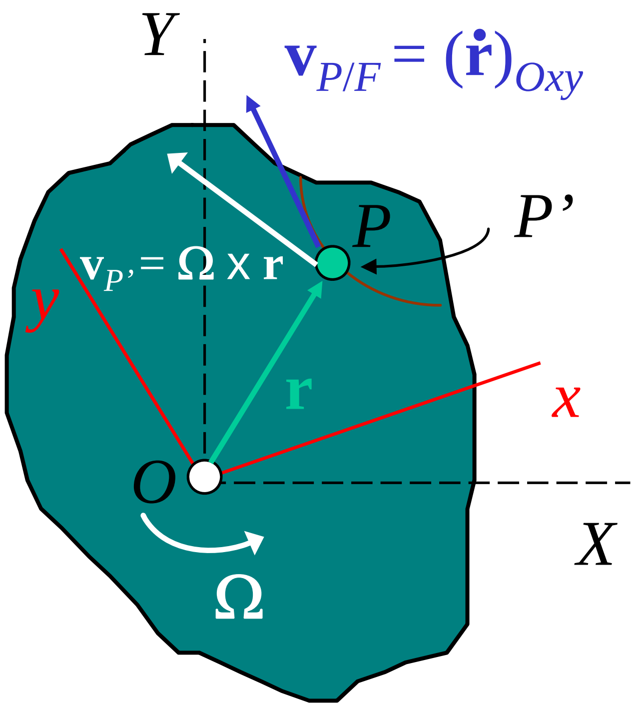
Where: $\mathbf{v}_P$ = absolute velocity of particle *P*

$\mathbf{v}_{P'}$ = velocity of point *P'* of moving frame F coinciding with *P*

$\mathbf{v}_{P/F}$ = velocity of *P* relative to moving frame F

The same expression for $\mathbf{v}_P$ is obtained if the frame is in translation rather than rotation.

PLANE MOVEMENT (acceleration)



When the frame is in rotation, the expression for the acceleration of P contains an additional term $\mathbf{a}_c$ called the *complementary acceleration* or *Coriolis acceleration*.

$$\mathbf{a}_P = \mathbf{a}_{P'} + \mathbf{a}_{P/F} + \mathbf{a}_c$$

Where: $\mathbf{a}_P$ = absolute acceleration of particle P

$\mathbf{a}_{P'}$ = acceleration of point P' of moving frame F coinciding with P

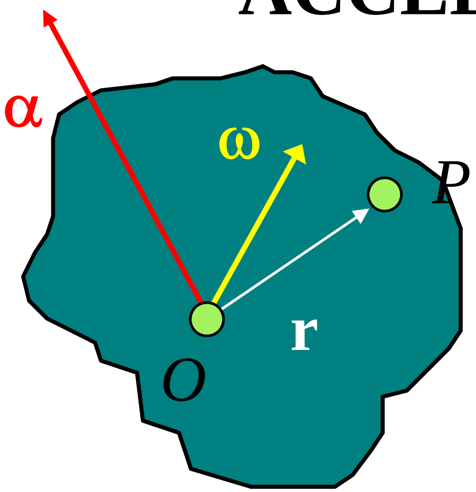
$\mathbf{a}_{P/F}$ = acceleration of P relative to moving frame F

$$\mathbf{a}_c = 2\Omega \times (\dot{\mathbf{r}})_{Oxy} = 2\Omega \times \mathbf{v}_{P/F}$$

= complementary, or Coriolis, acceleration

Since $\mathbf{\Omega}$ and $\mathbf{v}_{P/F}$ are perpendicular to each other in the case of plane motion, the Coriolis acceleration has a magnitude $a_c = 2\Omega v_{P/F}$. ***Its direction is obtained by rotating the vector $\mathbf{v}_{P/F}$ through 90º in the sense of rotation of the moving frame***. The Coriolis acceleration can be used to analyze the motion of mechanisms which contain parts sliding on each other.

ACCELERATION – general equations



In three dimensions (3D), the most general displacement of a rigid body with a fixed point *O* is equivalent to a rotation of the body about an axis through *O*. The angular velocity ω and the ***instantaneous axis of rotation*** of the body at a given instant can be defined. The velocity of a point *P* of the body can be expressed as:

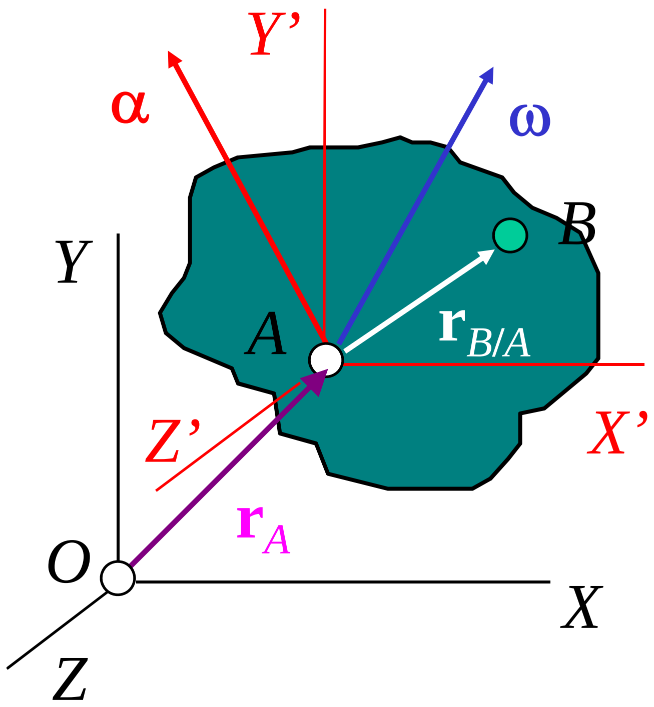
$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \boldsymbol{\omega} \times \mathbf{r}$$

Differentiating this expression, the acceleration is

$$\mathbf{a} = \boldsymbol{\alpha} \times \mathbf{r} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})$$

Since the direction of ω changes from instant to instant, the angular acceleration α is, in general, not directed along the instantaneous axis of rotation.

GENERAL MOTION



The ***most general motion of a rigid body in space is equivalent, at any given instant, to the sum of a translation and a rotation.*** ***Considering two particles A and B of the body***

$$\mathbf{v}_B = \mathbf{v}_A + \mathbf{v}_{B/A}$$

where $\mathbf{v}_{B/A}$ is the velocity of B relative to a frame $AX'Y'Z'$ attached to A and of fixed orientation. Denoting by $\mathbf{r}_{B/A}$ the position vector of B relative to A, we write:

$$\mathbf{v}_B = \mathbf{v}_A + \boldsymbol{\omega} \times \mathbf{r}_{B/A}$$

where $\boldsymbol{\omega}$ is the angular velocity of the body at the instant considered. The acceleration of B is, by similar reasoning

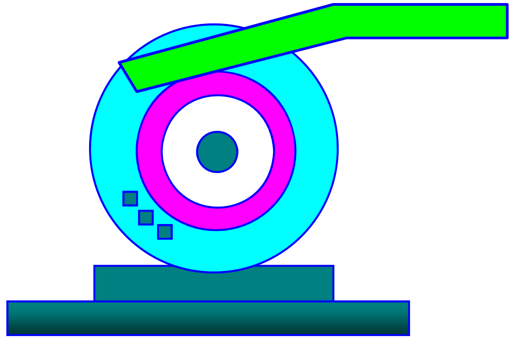
$$\mathbf{a}_B = \mathbf{a}_A + \mathbf{a}_{B/A} \qquad \text{or} \qquad \mathbf{a}_B = \mathbf{a}_A + \boldsymbol{\alpha} \times \mathbf{r}_{B/A} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}_{B/A})$$

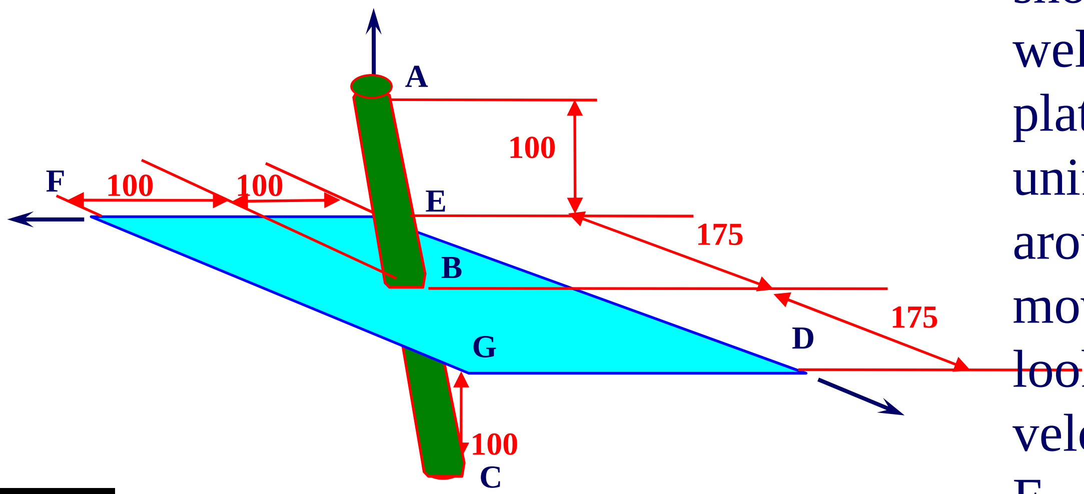
PROPOSED EXERCISES

EP 15.4 – A small rotating pulley is attached to a electric motor with nominal speed equal to 1800 [r.p.m.]. When the electric motor is turned on, all the assembly will take the regime speed after 5 [s]. When the motor is off, the system takes 90 [s] to stop. If a uniform accelerated motion is considered, calculate the number of motor revolutions in both conditions:

a) For normal operations conditions

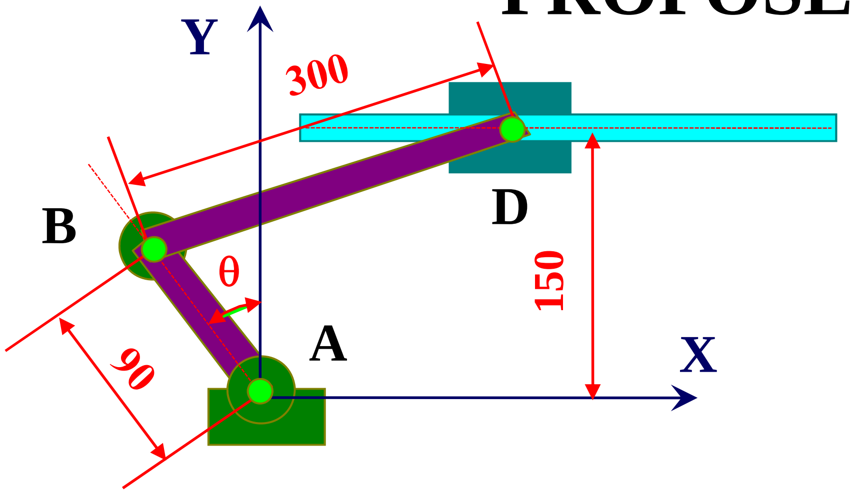
b) For stoping after turning off.





EP 15.8 - The element rigid body shown in the left figure is made with a welded axis ABC to the rectangular plate DEFH. The assembly turns with a uniform angular velocity 9 [rad/s] around the ABC axis. Knowing that the movement is anticlockwise, when looking from "C", determine the velocity and acceleration of the vertices F.

PROPOSED EXERCISES



EP 15.44- The crank AB has a constant angular velocity of 200 [r.p.m.] in the anticlockwise sense. Determine the angular velocity of the bar BD and the speed of the cursor D, when:

a) θ=0º

b) θ=90º

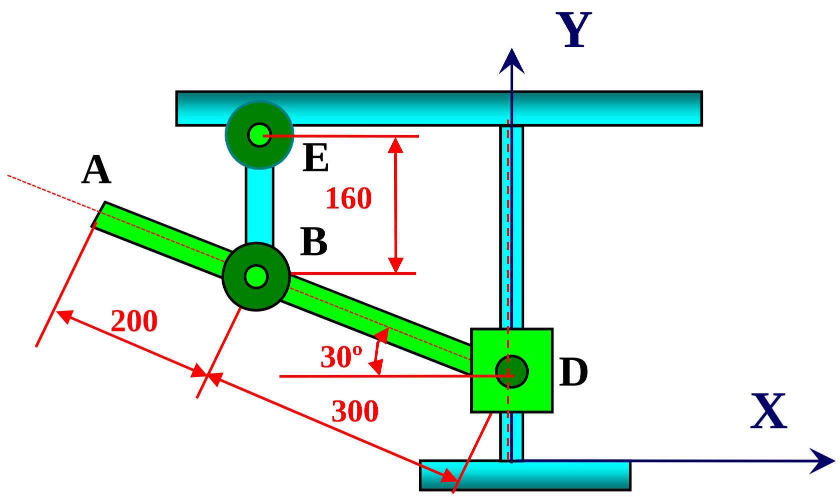
c) θ=180º

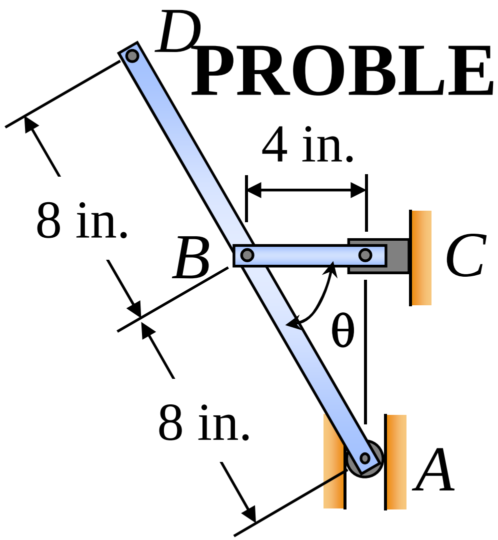
EP 15.60 – Knowing that the cursor velocity is equal to 1.8 [m/s] from bottom to top, determine for the illustrated configuration:

a) The angular velocity from the element AD

b) The velocity of point B

c) The velocity of point A





PROBLEM 15.248 – Thematic exercise 12

Knowing that at the instant shown crank *BC* has a constant angular velocity of 45 rpm clockwise, determine the acceleration :
(a)Of point *A*;
(b)Of point *D*.

1. Determine velocities in a body rotating about a fixed axis:

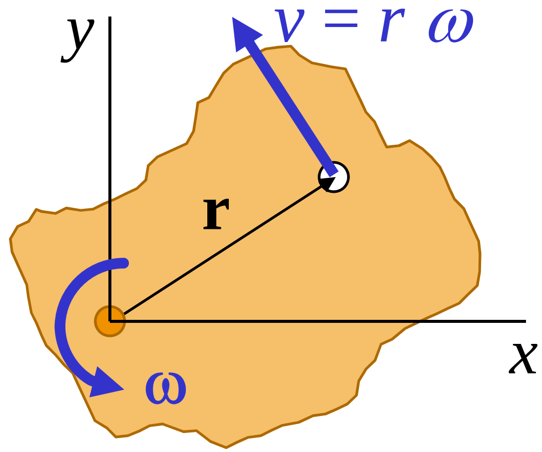
In vector form the velocity in the body is given by:

$$\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r}$$

Where **v** , **ω** , and **r** are the velocity of the point, the angular velocity of the body, and the position vector from the axis to the point. The magnitude of the velocity is given by:

$$v = r\,\omega$$

where *v*, *r*, and *ω* are the magnitudes of the corresponding vectors.



Calculating the velocity of point *B* in crank *BC*.

$$\omega_{BC} = (45\text{ rpm})\left(\frac{2\,\pi}{60}\,\frac{\text{rad}}{\text{rev}}\,\frac{\text{min}}{\text{s}}\right) = 1.5\pi\text{ rad/s}$$

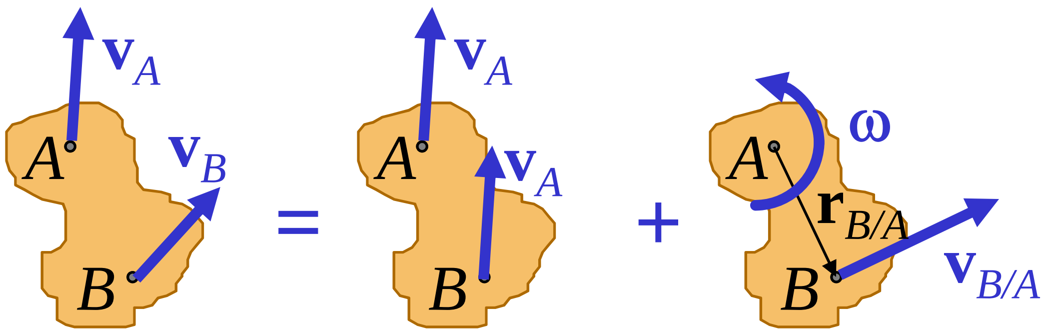
$$v_B = \omega_{BC}\, r_{B/C} = (1.5\pi\text{ rad/s})(4\text{ in}) = 18.85\text{ in/s}$$

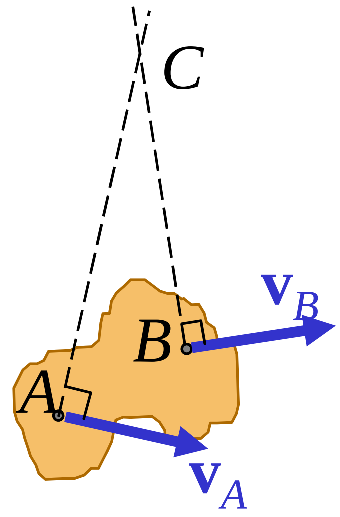
$$\mathbf{v}_B = 18.85\text{ in/s} \uparrow$$

PROBLEM 15.248 (solution)

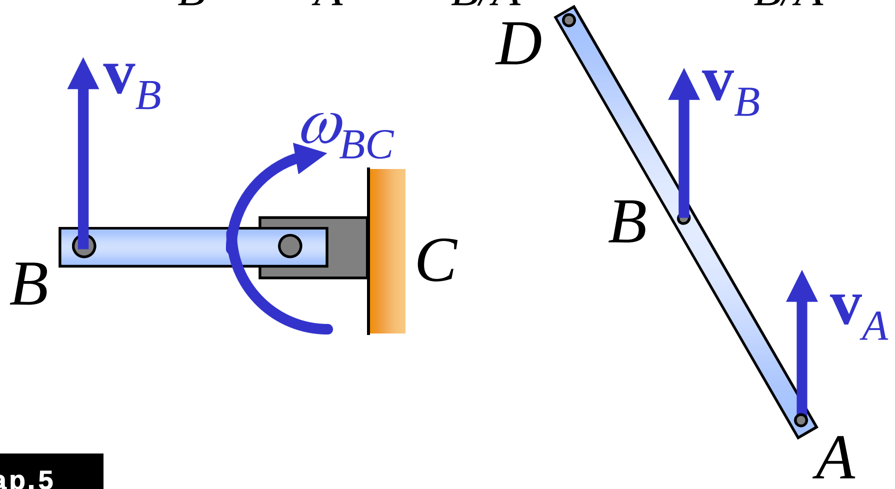
2. Determine velocities in a body under general plane motion:

Velocities can be determined either by method of instantaneous center of rotation, or by considering the motion of the body as the sum of a translation and a rotation.





$$\mathbf{v}_B = \mathbf{v}_A + \mathbf{v}_{B/A} \qquad \mathbf{v}_{B/A} = \boldsymbol{\omega} \times \mathbf{r}_{B/A}$$

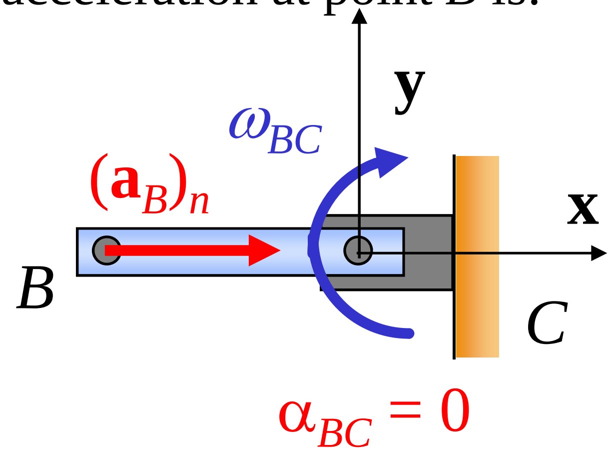


Since point A is forced to move in the vertical direction, and the direction of the velocity of point B is up, the angular velocity of bar AD is zero (<u>in that instant</u>): $w_{AD} = 0$

PROBLEM 15.248 (solution)

3. Determine accelerations in a body rotating about a fixed axis.

Calculating the acceleration of point *B* in crank *BC*. Since the angular velocity of crank *BC* is constant, $\alpha_{BC} = 0$ and $(a_B)_t = 0$. The normal component of the acceleration at point *B* is:



$$\vec{a}_B = \cancel{\vec{a}_C} + \cancel{\dot{\vec{w}} \times \vec{CB}} + \vec{w} \times \left(\vec{w} \times \vec{CB}\right)$$

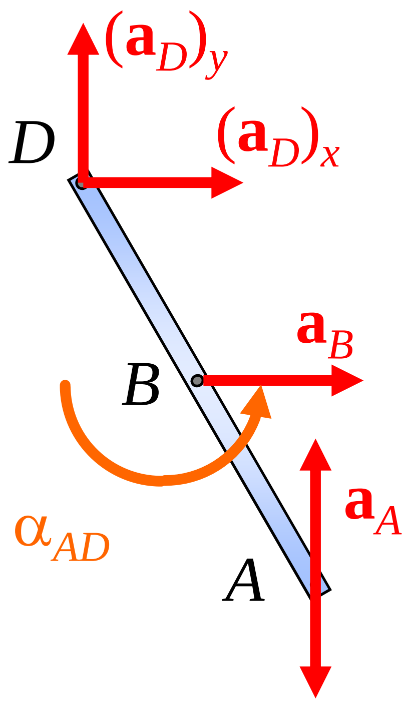
$$\vec{a}_B = \vec{w} \times \left(\vec{w} \times \vec{CB}\right)$$

$$= \begin{Bmatrix} 0 \\ 0 \\ -w_{BC} \end{Bmatrix} \times \left(\begin{Bmatrix} 0 \\ 0 \\ -w_{BC} \end{Bmatrix} \times \begin{Bmatrix} -4 \\ 0 \\ 0 \end{Bmatrix} \right) = \begin{Bmatrix} 0 \\ 0 \\ -w_{BC} \end{Bmatrix} \times \begin{Bmatrix} 0 \\ 4w_{BC} \\ 0 \end{Bmatrix}$$

$$= \begin{Bmatrix} 4w^2{}_{BC} \\ 0 \\ 0 \end{Bmatrix} = \begin{Bmatrix} 88{,}83 \\ 0 \\ 0 \end{Bmatrix} \left[in/s^2\right]$$

PROBLEM 15.248 (solution)

4. Determine accelerations in a body under general plane motion:



Notes:
- The acceleration $\mathbf{a}_A$ must be vertical;
- α_{AD} is assumed CCW and w_{AD} vanish in that instant.

1) $$\vec{a}_A = \vec{a}_B + \dot{\vec{w}} \times \vec{BA} + \vec{w} \times \left(\vec{w} \times \vec{BA}\right)$$

$$\begin{Bmatrix} 0 \\ a_{Ay} \\ 0 \end{Bmatrix} = \begin{Bmatrix} 88.83 \\ 0 \\ 0 \end{Bmatrix} + \begin{Bmatrix} 0 \\ 0 \\ \dot{w}_z \end{Bmatrix} \times \begin{Bmatrix} 8Cos(\theta) \\ -8Sin(\theta) \\ 0 \end{Bmatrix} + \begin{Bmatrix} 0 \\ 0 \\ w_z \end{Bmatrix} \times \left(\begin{Bmatrix} 0 \\ 0 \\ w_z \end{Bmatrix} \times \begin{Bmatrix} 8Cos(\theta) \\ -8Sin(\theta) \\ 0 \end{Bmatrix} \right)$$

$$\begin{cases} 0 = 88.83 + 8\dot{w}_z Sin(\theta) - 8w_z^2 Cos(\theta) \\ a_{Ay} = 8\dot{w}_z Cos(\theta) + 8w_z^2 Sin(\theta) \\ 0 = 0 \end{cases}$$

2) $$\vec{v}_A = \vec{v}_B + \left(\vec{w} \times \vec{BA}\right)$$

$$\begin{Bmatrix} 0 \\ v_{Ay} \\ 0 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 18.85 \\ 0 \end{Bmatrix} + \begin{Bmatrix} 0 \\ 0 \\ w_z \end{Bmatrix} \times \begin{Bmatrix} 8Cos(\theta) \\ -8Sin(\theta) \\ 0 \end{Bmatrix} \Leftrightarrow \begin{Bmatrix} 0 \\ v_{Ay} \\ 0 \end{Bmatrix} = \begin{Bmatrix} 8wSin(\theta) \\ 8wCos(\theta) \\ 0 \end{Bmatrix}$$

From system 1) $\dot{w}_z$=-12.82 [rad/s^2] and from system 2) a_{Ay}=-51.28 [in/s^2]

PROBLEM 15.248 (solution)

4. Determine accelerations in a body under general plane motion:

$$\vec{a}_D = \vec{a}_B + \dot{\vec{w}} \times \vec{BD} + \vec{w} \times \left(\vec{w} \times \vec{BD}\right)$$

$$\begin{Bmatrix} a_{Dx} \\ a_{Dy} \\ 0 \end{Bmatrix} = \begin{Bmatrix} 88.83 \\ 0 \\ 0 \end{Bmatrix} + \begin{Bmatrix} 0 \\ 0 \\ \dot{w}_z \end{Bmatrix} \times \begin{Bmatrix} -8Cos(\theta) \\ 8Sin(\theta) \\ 0 \end{Bmatrix} + \begin{Bmatrix} 0 \\ 0 \\ w_z \end{Bmatrix} \times \left(\begin{Bmatrix} 0 \\ 0 \\ w_z \end{Bmatrix} \times \begin{Bmatrix} -8Cos(\theta) \\ 8Sin(\theta) \\ 0 \end{Bmatrix} \right)$$

$$\begin{cases} a_{Dx} = 88.83 - 8\dot{w}_z Sin(\theta) + 8w_z^2 Cos(\theta) \\ a_{Dy} = -8\dot{w}_z Cos(\theta) - 8w_z^2 Sin(\theta) \\ 0 = 0 \end{cases}$$

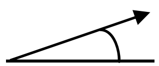
Knowing that $\dot{w}_z$=-12.82 [rad/s^2] and w_z=0 [rad/s], we conclude:

$(a_D)_x$ = 88.82 - (8)(- 12.82) sin 60^o

$(a_D)_x$ = 177.7 in/s^2

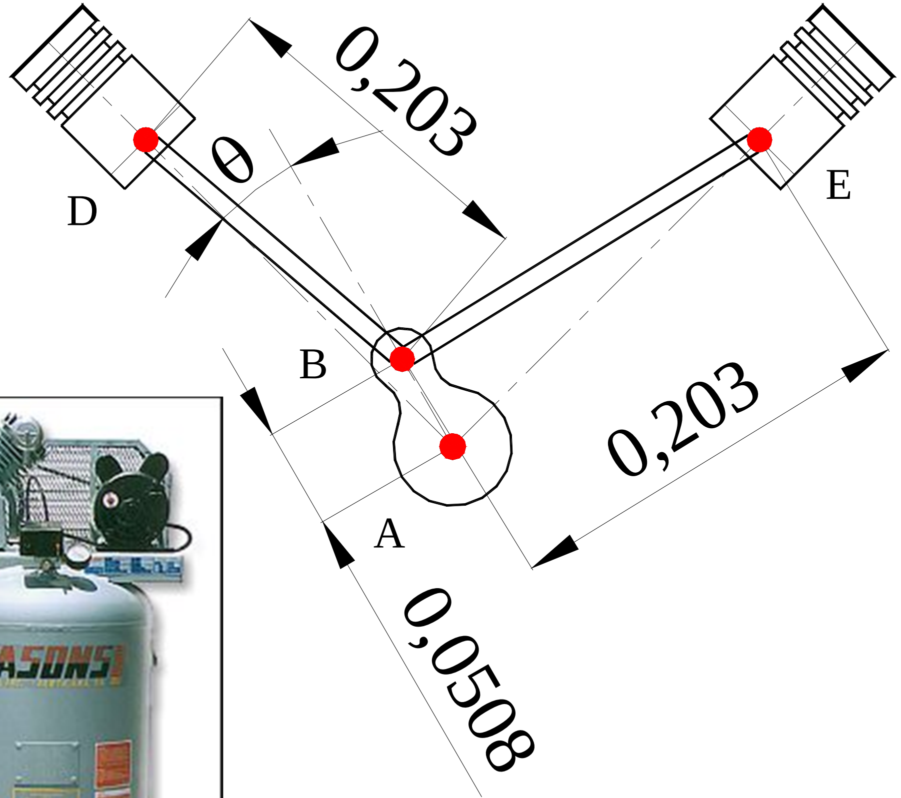
$(a_D)_y$ = 0 - (8)(- 12.82) cos 60^o

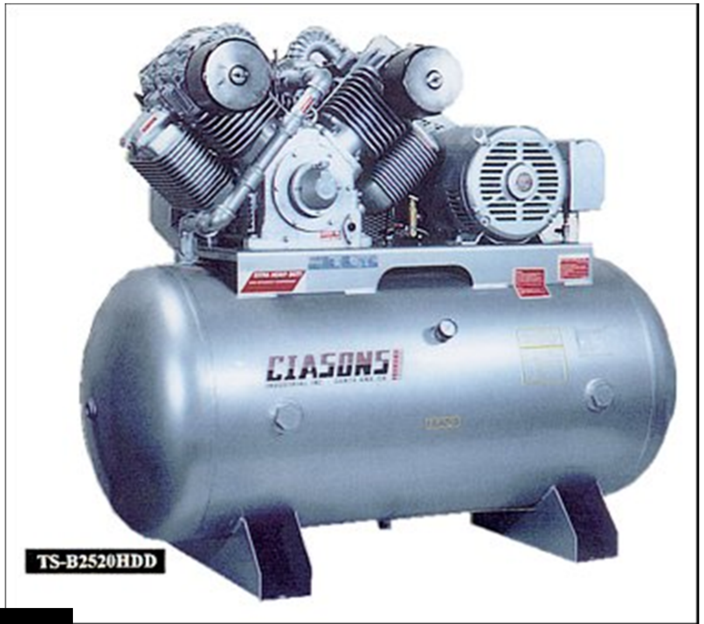
$(a_D)_y$ = 51.3 in/s^2 **a**$_D$ = 184.9 in/s^2 16.1^o

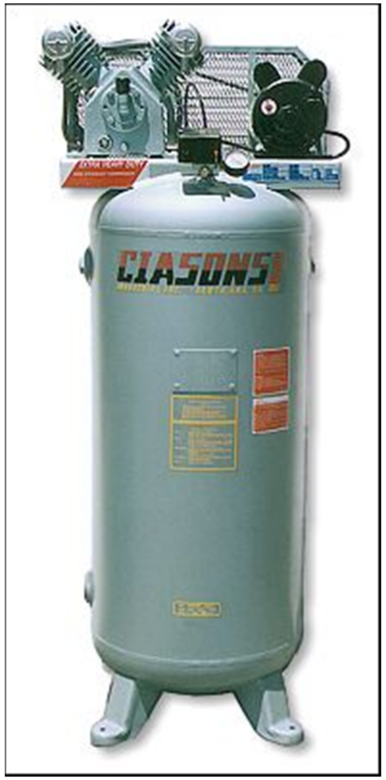


PROBLEM EP 15.90

The two cylinders air compressor presents two 0.203 [m] long arms BD and BE. Knowing that the length of arm *AB* is equal to 0.0508 [m], and that it rotates at a constant rate ω = 1800 [rpm], in the clockwise sense. Determine the pistons acceleration when $\theta = 0^o$.







PROBLEM EP 15.90 - ANALYTICAL SOLUTION

1. Determine VELOCITY of point B:

$$\vec{v}_B = \vec{v}_A + \vec{w}_1 \times \vec{AB} = \vec{0} + \begin{Bmatrix} 0 \\ 0 \\ -188.5 \end{Bmatrix} \times \begin{Bmatrix} 0.0508\sin(\theta) \\ 0.0508\cos(\theta) \\ 0 \end{Bmatrix} = \begin{Bmatrix} 9.576\cos(\theta) \\ -9.576\sin(\theta) \\ 0 \end{Bmatrix}$$

2. Determine VELOCITY of point D and ANGULAR VELOCITY of arm BD:

$$\vec{v}_D = \vec{v}_B + \vec{w}_2 \times \vec{BD} \Leftrightarrow \begin{Bmatrix} 0 \\ v_{Dy} \\ 0 \end{Bmatrix} = \begin{Bmatrix} 9.576\cos(\theta) \\ -9.576\sin(\theta) \\ 0 \end{Bmatrix} + \begin{Bmatrix} 0 \\ 0 \\ w_{2z} \end{Bmatrix} \times \begin{Bmatrix} -0.203\sin(\alpha) \\ 0.203\cos(\alpha) \\ 0 \end{Bmatrix} \Leftrightarrow \begin{cases} w_{2z} = \dfrac{9.576\cos(\theta)}{0.203\sqrt{1-0.25^2\sin^2(\theta)}} \\ v_{Dy} = -9.576\sin(\theta) - \dfrac{9.576\cos(\theta)}{\sqrt{1-0.25^2\sin^2(\theta)}} 0.25\sin(\theta) \\ 0 = 0 \end{cases}$$

$$\theta = 0^{\circ} \Rightarrow \begin{cases} w_{2z} = \dfrac{9.576}{0.203} = 47.17[rad/s] \\ v_{Dy} = 0[m/s] \end{cases}$$

3. Determine ACCELERATION of point B

$$\vec{a}_B = \vec{a}_A + \dot{\vec{w}}_1 \times \vec{AB} + \vec{w}_1 \times \left(\vec{w}_1 \times \vec{AB}\right) =$$

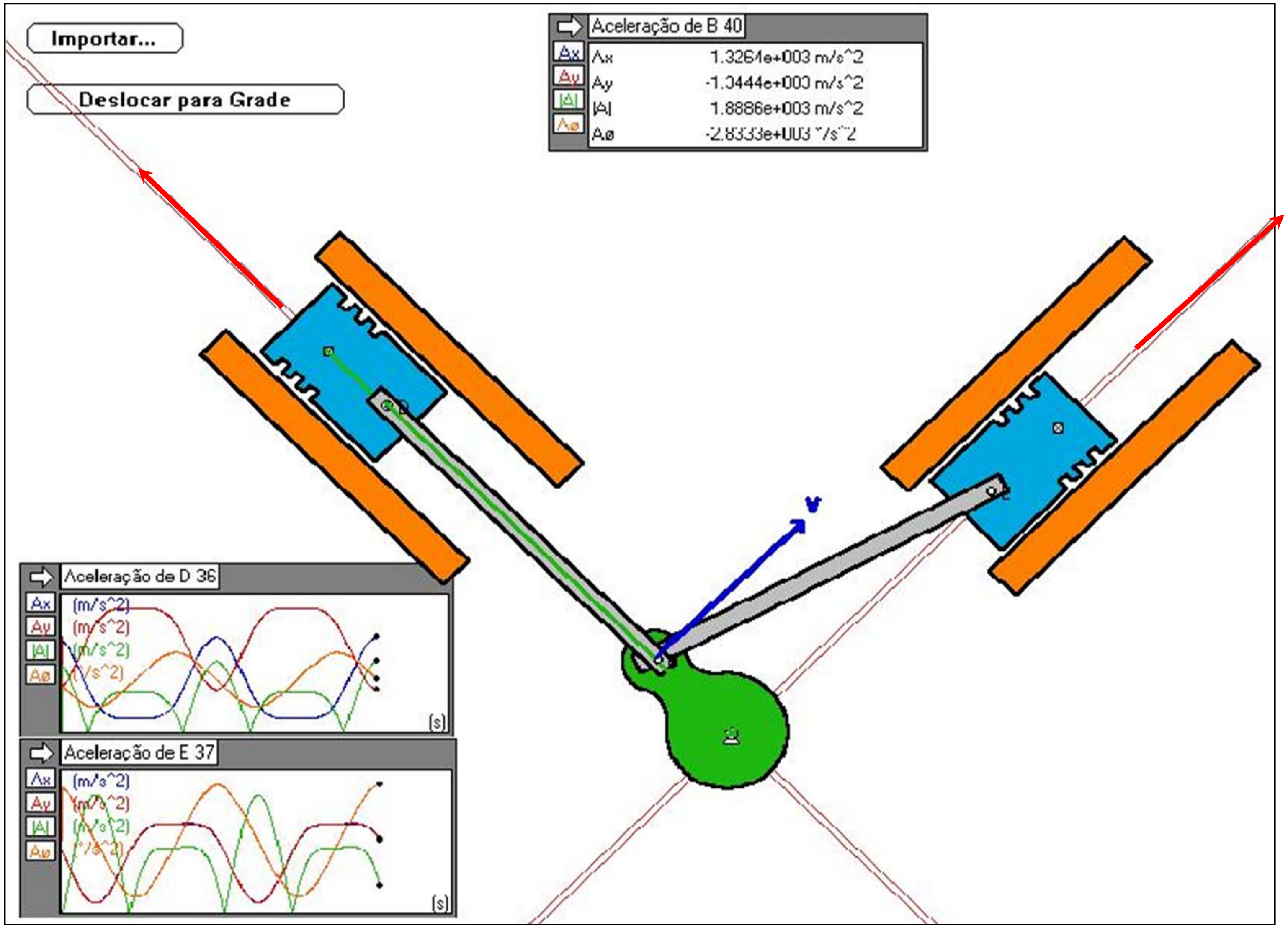
$$\vec{a}_B = \vec{0} + \begin{Bmatrix} 0 \\ 0 \\ -188.5 \end{Bmatrix} \times \left[\begin{Bmatrix} 0 \\ 0 \\ -188.5 \end{Bmatrix} \times \begin{Bmatrix} 0.0508\sin(\theta) \\ 0.0508\cos(\theta) \\ 0 \end{Bmatrix} \right] = \begin{Bmatrix} -1805.0\sin(\theta) \\ -1805.0\cos(\theta) \\ 0 \end{Bmatrix}$$

$$\theta = 0^{\circ} \Rightarrow \vec{a}_B = \begin{Bmatrix} 0 \\ -1805.0 \\ 0 \end{Bmatrix} [m/s^2]$$

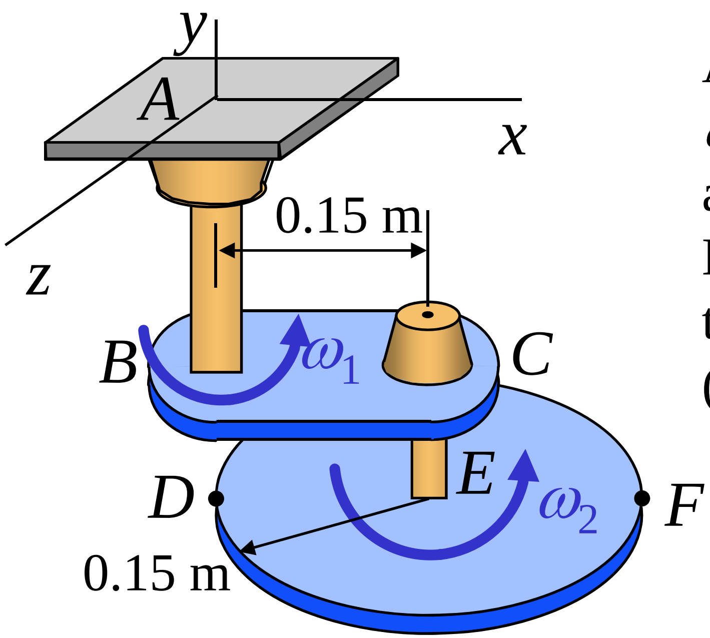
$$\vec{a}_D = \vec{a}_B + \dot{\vec{w}}_2 \times \vec{BD} + \vec{w}_2 \times \left(\vec{w}_2 \times \vec{BD}\right) =$$

$$\begin{Bmatrix} 0 \\ a_{Dy} \\ 0 \end{Bmatrix} = \begin{Bmatrix} -1805.0\sin(\theta) \\ -1805.0\cos(\theta) \\ 0 \end{Bmatrix} + \begin{Bmatrix} 0 \\ 0 \\ \dot{w}_2 \end{Bmatrix} \times \begin{Bmatrix} -0.203\sin(\alpha) \\ 0.203\cos(\alpha) \\ 0 \end{Bmatrix} + \begin{Bmatrix} 0 \\ 0 \\ w_2 \end{Bmatrix} \times \left[\begin{Bmatrix} 0 \\ 0 \\ w_2 \end{Bmatrix} \times \begin{Bmatrix} -0.203\sin(\alpha) \\ 0.203\cos(\alpha) \\ 0 \end{Bmatrix} \right] \Leftrightarrow \begin{cases} a_{Dy} = -1805\cos(\theta) - 0.203\dot{w}_2\sin(\alpha) - 0.203w_2^2\cos(\alpha) \\ \dot{w}_2 = -\dfrac{1805\sin(\theta) - 0.203w_2^2\sin(\alpha)}{0.203w_2^2\cos(\alpha)} \end{cases}$$

PROBLEM EP 15.90
NUMERICAL SOLUTION



PROBLEM 15-250



A disk of 0.15-m radius rotates at a constant rate ω_2 with respect to plate *BC*, which itself rotates at the constant rate ω_1 about the *y* axis. Knowing that $\omega_1 = \omega_2 = 3$ rad/s, determine, for the position shown the velocity and acceleration (a) of point *D*, (b) of point *F*.

1. Determination of the velocities in general motion of a rigid body:

$$\mathbf{v}_B = \mathbf{v}_A + \mathbf{w} \times \mathbf{r}_{B/A}$$

Where $\mathbf{v}_B$ is the velocity of point *B*, $\mathbf{v}_A$ is the (known) velocity of point *A*, **w** is the angular velocity of the body with respect to a fixed frame of reference, and $\mathbf{r}_{B/A}$ is the position vector of *B* relative to *A*.

$$\omega = \omega_1 + \omega_2 = 3\,\mathbf{j} + 3\,\mathbf{j} = 6\,\mathbf{j} \ \ [\text{rad/s}]$$

(a) Point *D*:

$$\mathbf{v}_D = \mathbf{v}_C + \omega \times CD$$

$$\mathbf{v}_D = (3\,\mathbf{j}) \times (0.15\,\mathbf{i}) + (6\,\mathbf{j}) \times (-0.15\,\mathbf{i})$$

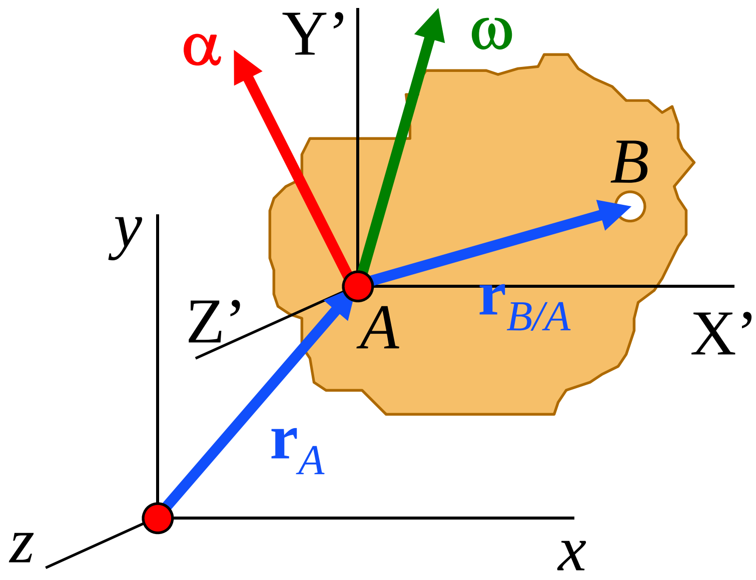
$$\mathbf{v}_D = 0.45\,\mathbf{k} \ \ [\text{m/s}]$$

PROBLEM 15-250 - solution

2. Determine accelerations in general motion of a rigid body:

$$\mathbf{a}_B = \mathbf{a}_A + \alpha \times \mathbf{AB} + \mathbf{w} \times (\mathbf{w} \times \mathbf{AB})$$

Where $\mathbf{a}_B$ is the acceleration of point *B*, $\mathbf{a}_A$ is the (known) acceleration of point *A*, α and $\mathbf{w}$ are the angular acceleration and angular velocity of the body with respect to a fixed reference frame, and **AB** is the position vector of *B* relative to *A*.



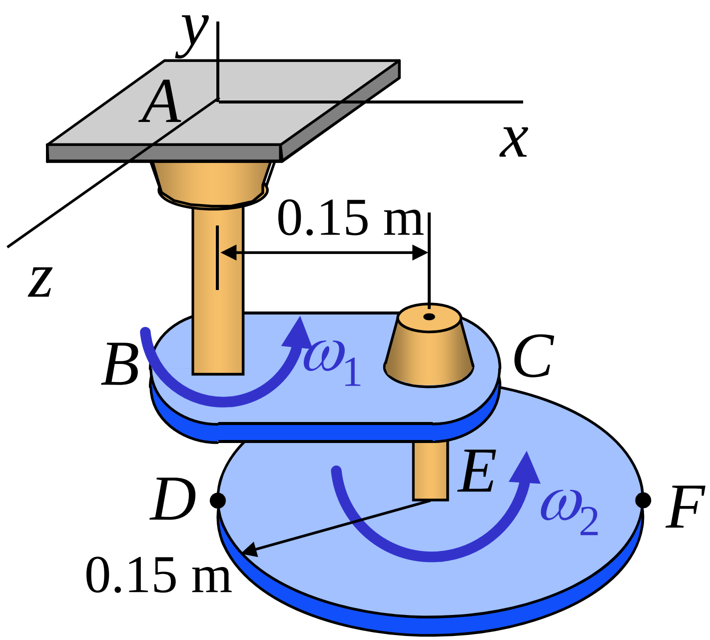
$$\mathbf{a}_D = \mathbf{a}_C + \alpha \times \mathbf{r}_{D/E} + \omega \times (\omega \times \mathbf{r}_{D/E})$$

$$\mathbf{a}_D = \omega_1 \times (\omega_1 \times \mathbf{r}_{C/B}) + \omega \times (\omega \times \mathbf{r}_{D/E})$$

$$\mathbf{a}_D = (3\,\mathbf{j}) \times (3\,\mathbf{j} \times 0.15\,\mathbf{i}) + (6\,\mathbf{j}) \times [(6\,\mathbf{j}) \times (-0.15\,\mathbf{i})]$$

$$\mathbf{a}_D = -1.35\,\mathbf{i} + 5.4\,\mathbf{i} \quad \Leftrightarrow \quad \mathbf{a}_D = 4.05\,\mathbf{i} \ \ \text{m/s}^2$$

PROBLEM 15-250 - solution



(b) Point *F*:

$\mathbf{v}_F = \mathbf{v}_C + \boldsymbol{\omega} \times \text{CF}$

$\mathbf{v}_F = (3\ \mathbf{j}) \times (0.15\ \mathbf{i}) + (6\ \mathbf{j}) \times (0.15\ \mathbf{i})$

$\mathbf{v}_F = -1.35\ \mathbf{k}\ \ [\text{m/s}]$

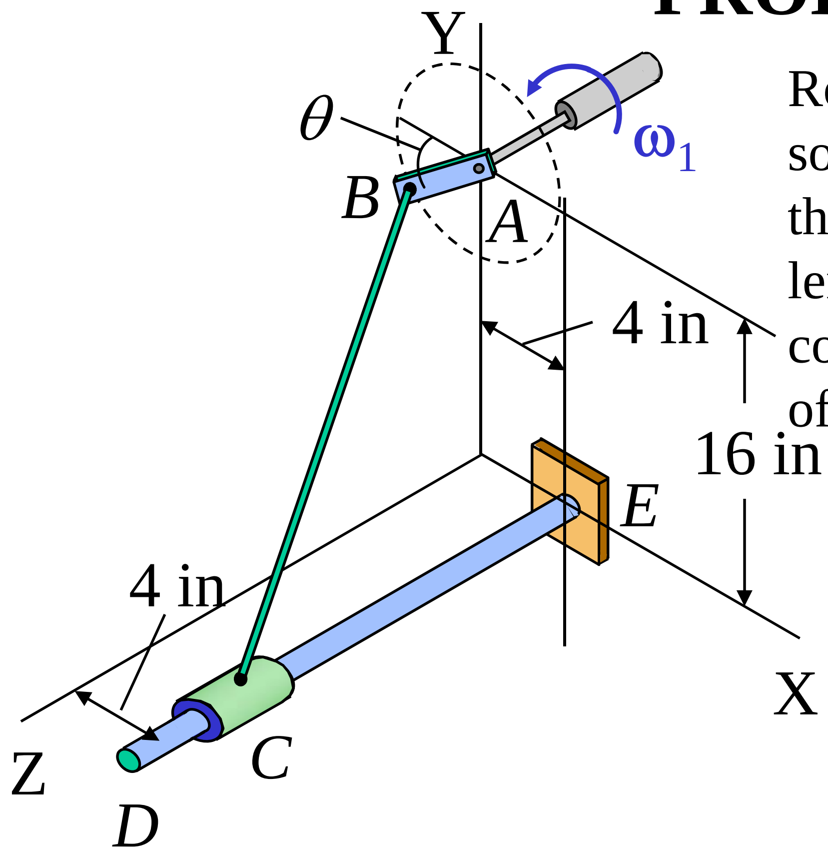
$\mathbf{a}_F = \mathbf{a}_C + \boldsymbol{\alpha} \times \mathbf{EF} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{EF})$

$\mathbf{a}_F = \boldsymbol{\omega}_1 \times (\boldsymbol{\omega}_1 \times \mathbf{BC}) + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{EF})$

$\mathbf{a}_F = (3\ \mathbf{j}) \times (3\ \mathbf{j} \times 0.15\ \mathbf{i}) + (6\ \mathbf{j}) \times [(6\ \mathbf{j}) \times (0.15\ \mathbf{i})]$

$\mathbf{a}_F = -1.35\ \mathbf{i} - 5.4\ \mathbf{i}$ ⇔ $\mathbf{a}_F = -6.75\ \mathbf{i}\ \ \text{m/s}^2$

PROBLEM 15-256



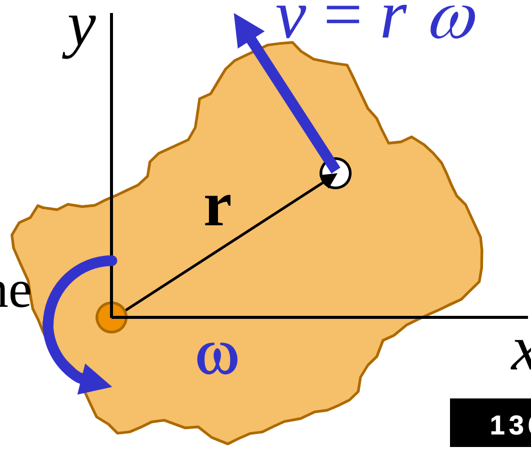
Rod *BC* of length 24 in. is connected by ball-and-socket joints to a rotating arm *AB* and to a collar *C* that slides on the fixed rod *DE*. Knowing that length of arm *AB* is 4 in. and that it rotates at a constant rate ω_1 = 10 [rad/s], determine the velocity of collar *C* when $\theta = 90^o$.

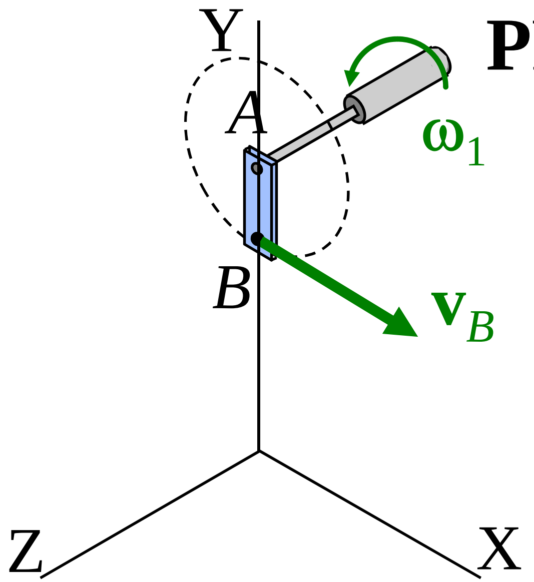
1. Determine velocities in a body rotating about a fix axis:

In vector form, the velocity of a point in the body is given by:

v = **w** x **r**

Where **v** , **w** , and **r** are the velocity of the point, the angular velocity of the body, and the position vector from the axis to the point.





PROBLEM 15-256 - solution

Determine the velocity of point B when $\theta = 90^o$:

$\boldsymbol{\omega}_1 = 10\ \mathbf{k}$ [rad/s]

$\mathbf{r}_{B/A} = \mathbf{AB} = -4\ \mathbf{j}$ [in]

$\mathbf{v}_B = \boldsymbol{\omega}_1 \times \mathbf{AB}$

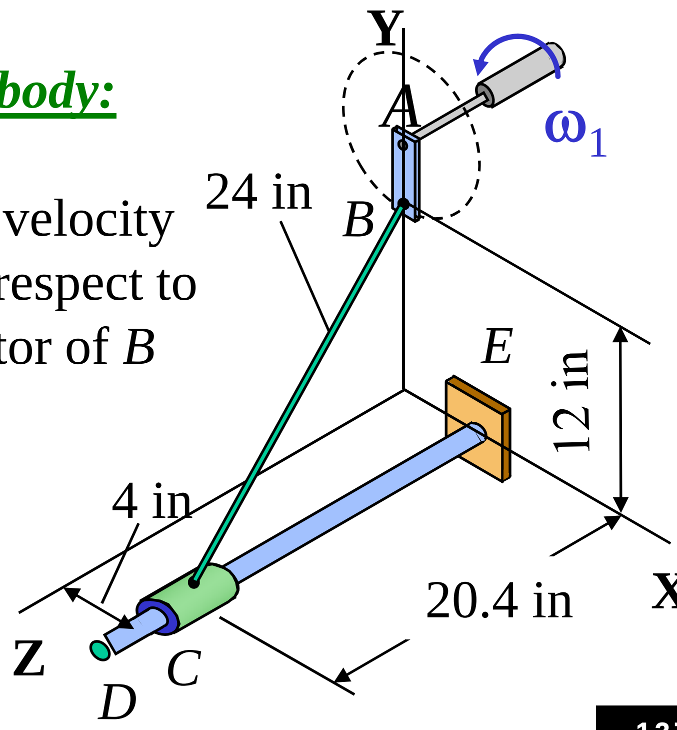
$\mathbf{v}_B = 10\ \mathbf{k} \times (-4\ \mathbf{j})$

$\mathbf{v}_B = (40 \text{ in/s})\ \mathbf{i}$

2. Determine velocities in general motion of a rigid body:

$$\mathbf{v}_B = \mathbf{v}_A + \boldsymbol{\omega} \times \mathbf{r}_{B/A}$$

Where $\mathbf{v}_B$ is the velocity of point B, $\mathbf{v}_A$ is the (known) velocity of point A, **w** is the angular velocity of the body with respect to a fixed frame of reference, and $\mathbf{r}_{B/A}$ is the position vector of B relative to A.



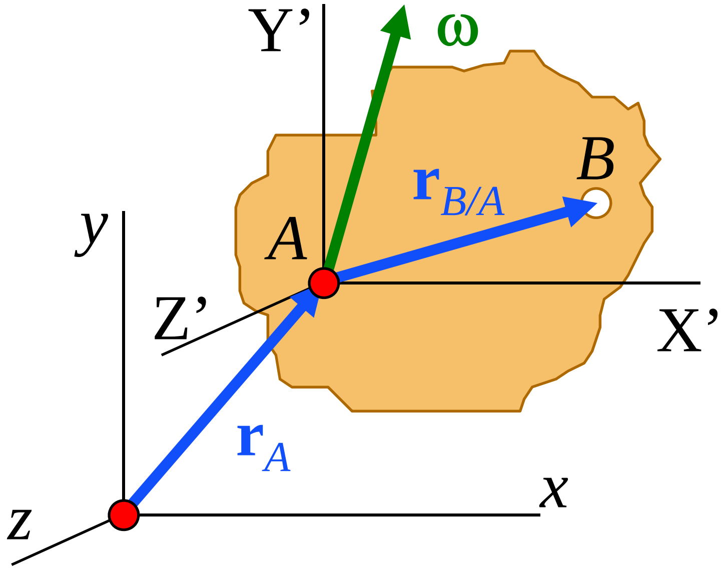
$\mathbf{v}_B = (40 \text{ in/s})\ \mathbf{i}$

$\mathbf{v}_C = v_C\ \mathbf{k}$

$\boldsymbol{\omega} = \omega_x\ \mathbf{i} + \omega_y\ \mathbf{j} + \omega_z\ \mathbf{k}$

$\mathbf{r}_{C/B} = 4\ \mathbf{i} - 12\ \mathbf{j} + 20.4\ \mathbf{k}$

PROBLEM 15-256 - solution



$\mathbf{v}_C = \mathbf{v}_B + \boldsymbol{\omega} \times \mathbf{r}_{C/B}$

$$v_C\,\mathbf{k} = (\,40 \text{ in/s})\,\mathbf{i} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \omega_x & \omega_y & \omega_z \\ 4 & -12 & 20.4 \end{vmatrix}$$

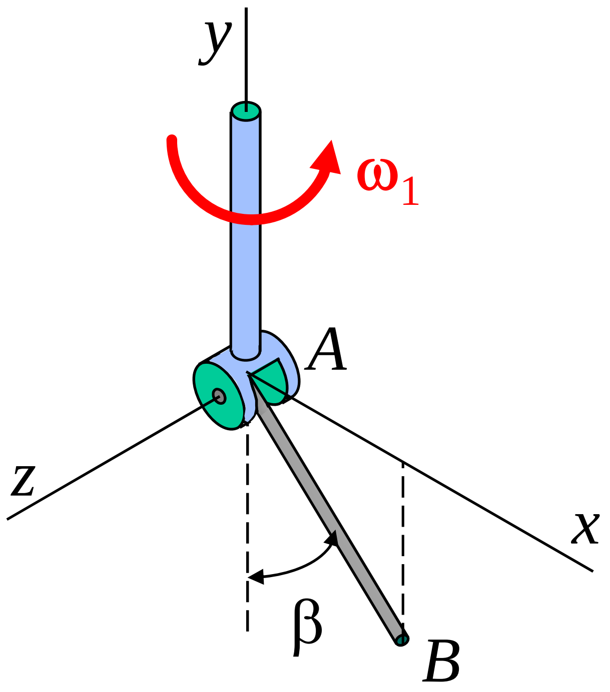
$$v_C\,\mathbf{k} = (40)\,\mathbf{i} + (20.4\omega_y + 12\omega_z)\mathbf{i} + (-20.4\omega_x + 4\omega_z)\mathbf{j} + (-12\omega_x - 4\omega_y)\mathbf{k}$$

Equate coefficients of **i, j, k** :

$$\begin{cases} 0 = 40 + 20.4\omega_y + 12\omega_z \\ 0 = -20.4\omega_x + 4\omega_z \\ v_C = -12\omega_x - 4\omega_y \end{cases}$$

Solve for v_C : (First eliminate w_z and then eliminate $(3w_x + w_y)$.)

$v_C = 7.84$ [in/s] $\qquad \mathbf{v}_C = 7.84\,\mathbf{k}$ [in/s]

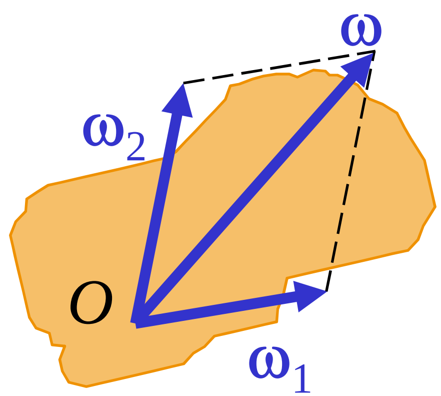


PROBLEM 15-259

Rod *AB* of length 125 mm is attached to a vertical rod that rotates about the *y* axis at the constant rate ω_1 = 5 rad/s. Knowing that the angle formed by rod *AB* and the vertical is increasing at the constant rate $d\beta/dt$ = 3 rad/s, determine the velocity and acceleration of end *B* of the rod when $\beta = 30^o$.

1. To determine the velocity and acceleration of a point of a body rotating about a fixed point (Determine the angular velocity w of the body):

The angular velocity *w* with respect to a fixed frame of Reference is often obtained by adding two component angular velocities w_1 and w_2.

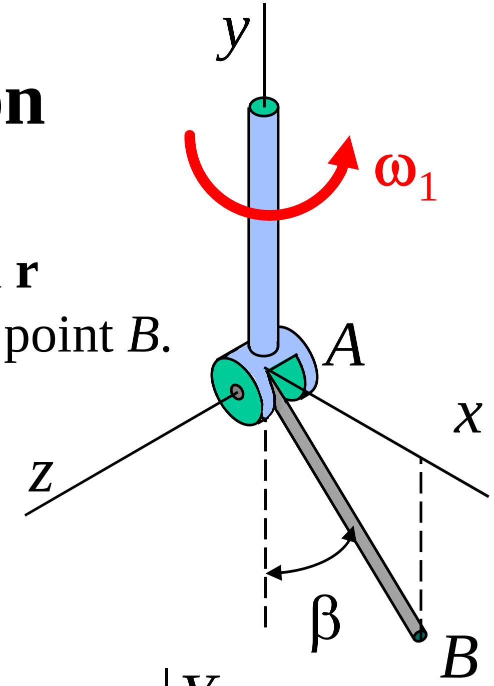


$$\boldsymbol{\omega}_1 = 5\,\mathbf{j}\ \text{ rad/s}$$

$$\omega_2 = d\beta/dt = 3\text{ rad/s},\qquad \boldsymbol{\omega}_2 = 3\,\mathbf{k}\ \text{ rad/s}$$

$$\boldsymbol{\omega} = \boldsymbol{\omega}_1 + \boldsymbol{\omega}_2 = 5\,\mathbf{j} + 3\,\mathbf{k}\ \ \text{rad/s}$$

PROBLEM 15-259 - solution



1b. Compute the velocity of a point of the body:

The velocity **v** of point *B* in the body is given by: **v** = **w** x **r**
where **r** is the position vector connecting the fixed point *O* to point *B*.

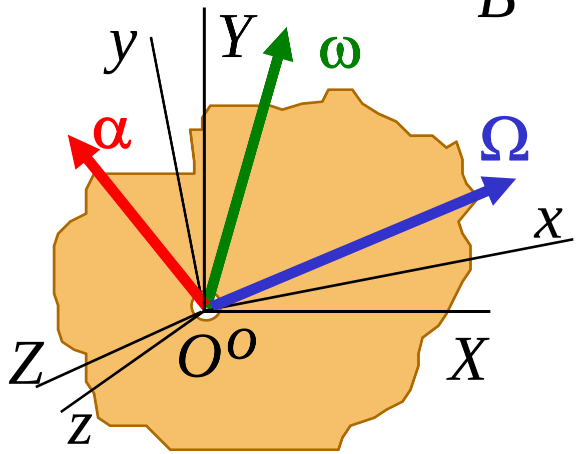
The velocity of end B:

For $\beta = 30^o$ and $r_{B/A} = 0.125$

$\mathbf{r}_{B/A} = 0.125 \sin 30^o\, \mathbf{i} - 0.125 \cos 30^o\, \mathbf{j} = 0.0625\, \mathbf{i} - 0.1083\, \mathbf{j}$

$$\mathbf{v}_B = \boldsymbol{\omega} \times \mathbf{r}_{B/A} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 5 & 3 \\ 0.0625 & -0.1083 & 0 \end{vmatrix}$$

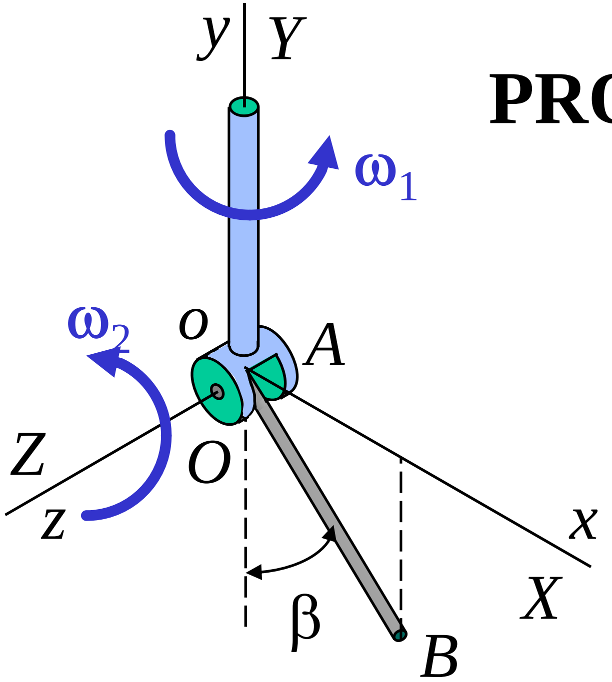
$\mathbf{v}_B = 0.325\, \mathbf{i} + 0.188\, \mathbf{j} - 0.313\, \mathbf{k}$ m/s



1c. Determine the angular acceleration a of the body:

$\boldsymbol{\alpha}$ is the rate of change $(\dot{\mathbf{w}})_{OXYZ}$ of the vector **w** w.r.t. a fixed frame of reference *OXYZ*, $(\dot{\mathbf{w}})_{oxyz}$ is the rate of change of **w** w.r.t. a rotating frame of reference *oxyz*, $\boldsymbol{\Omega}$ is the angular velocity of the rotating frame.

$$\alpha = (\dot{w})_{OXYZ} = (\dot{w})_{oxyz} + \Omega \times w$$



PROBLEM 15-259 - solution

Frame *OXYZ* is fixed.
Frame *oxyz* is attached to the vertical rod and rotates with constant angular velocity $\mathbf{w}_1$.

Consequently: $\dot{\mathbf{w}}_1 = 0$ and $\mathbf{\Omega} = \mathbf{w}_1$

$$\alpha = \dot{\omega} = \dot{\omega}_1 + \dot{\omega}_2 = 0 + \dot{\omega}_2$$

$$\alpha = (\dot{\omega}_2)_{OXYZ} = (\dot{\omega}_2)_{oxyz} + \mathbf{\Omega} \times \omega_2$$

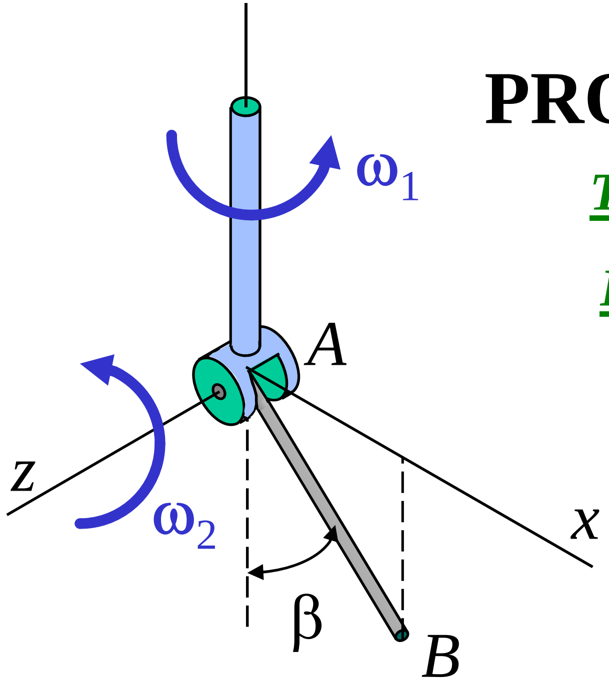
$$\alpha = 0 + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 5 & 0 \\ 0 & 0 & 3 \end{vmatrix}$$

$$\alpha = 15\,\mathbf{i} \quad \text{rad/s}^2$$

1d. Compute the acceleration of a point of a body:

$$\mathbf{a}_B = \alpha \times \mathbf{r}_{B/A} + \omega \times (\omega \times \mathbf{r}_{B/A})$$

Where $\mathbf{a}_B$ is the acceleration of point *B*, $\mathbf{a}_A$ is the (known) acceleration of point *A*, α and ω are the angular acceleration and angular velocity of the body with respect to a fixed frame of reference, and $\mathbf{r}_{B/A}$ is the position vector of *B* relative to *A*.



PROBLEM 15-259 – solution

The acceleration of end B:

Recall: $\omega = 5\,\mathbf{j} + 3\,\mathbf{k}$ [rad/s]

$\alpha = 15\,\mathbf{i}$ [rad/s^2]

$\mathbf{r}_{B/A} = 0.0625\,\mathbf{i} - 0.1083\,\mathbf{j}$

$\mathbf{v}_B = \omega \times \mathbf{r}_{B/A} = 0.325\,\mathbf{i} + 0.188\,\mathbf{j} - 0.313\,\mathbf{k}$ [m/s]

$$\mathbf{a}_B = \alpha \times \mathbf{r}_{B/A} + \omega \times (\omega \times \mathbf{r}_{B/A})$$

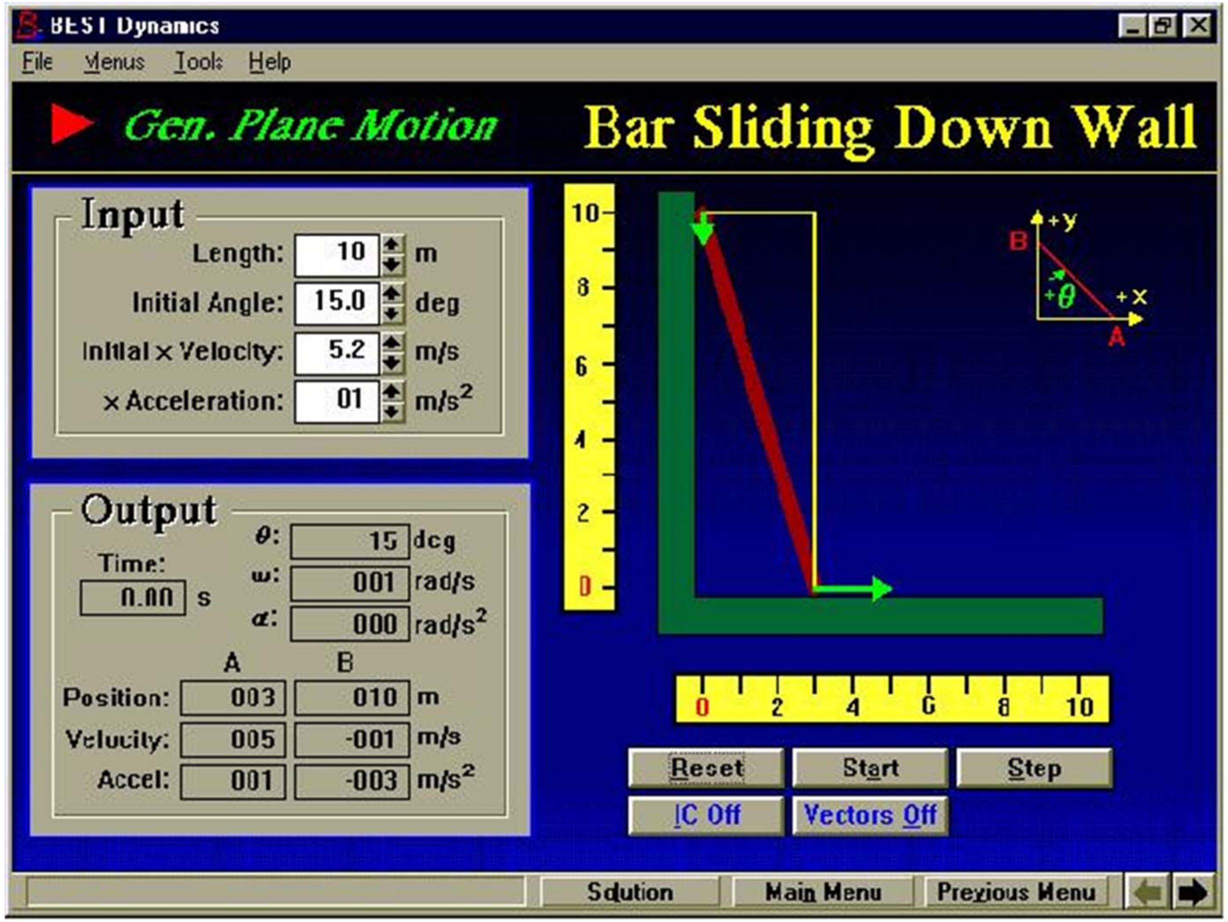
$$\mathbf{a}_B = \alpha \times \mathbf{r}_{B/A} + \omega \times \mathbf{v}_B$$

$$\mathbf{a}_B = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 15 & 0 & 0 \\ 0.0625 & -0.108 & 0 \end{vmatrix} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 5 & 3 \\ 0.325 & 0.188 & -0.313 \end{vmatrix}$$

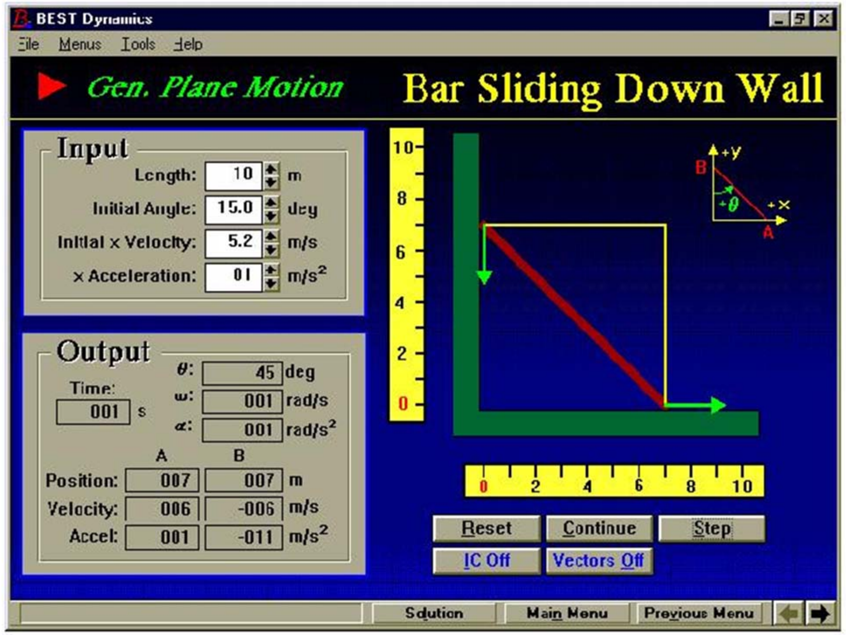
$\mathbf{a}_B = -2.13\,\mathbf{i} + 0.97\,\mathbf{j} - 3.25\,\mathbf{k}$ [m/s^2]

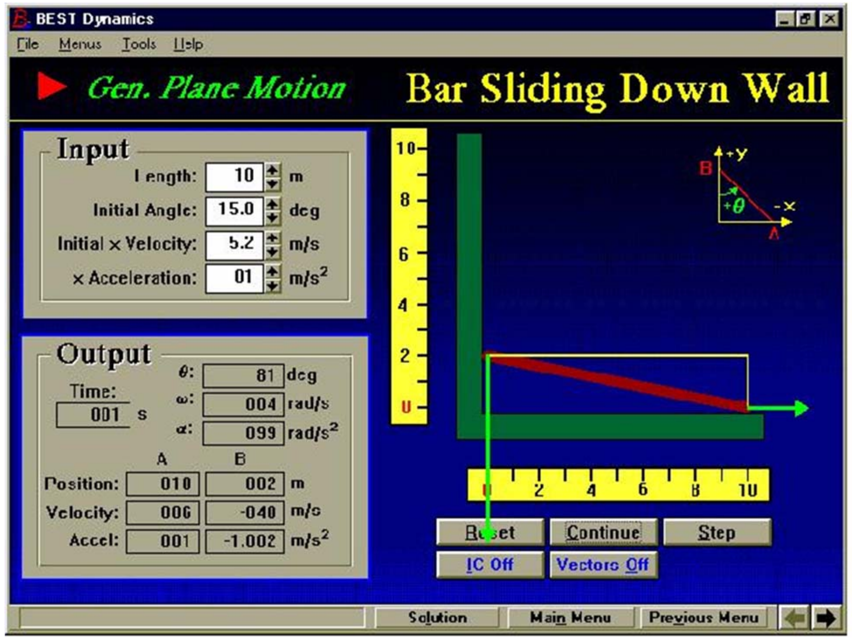
SLIDING BAR EXERCISE

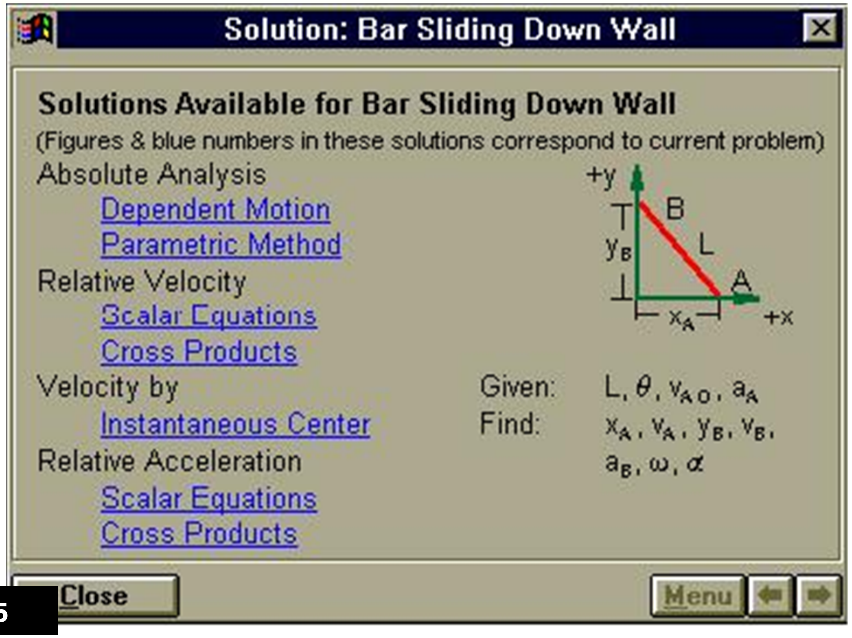
The bar sliding down the wall is represented in the figure, using translation rigid joints at the extremities. For a given initial data, calculate the velocity and acceleration of the extreme points.

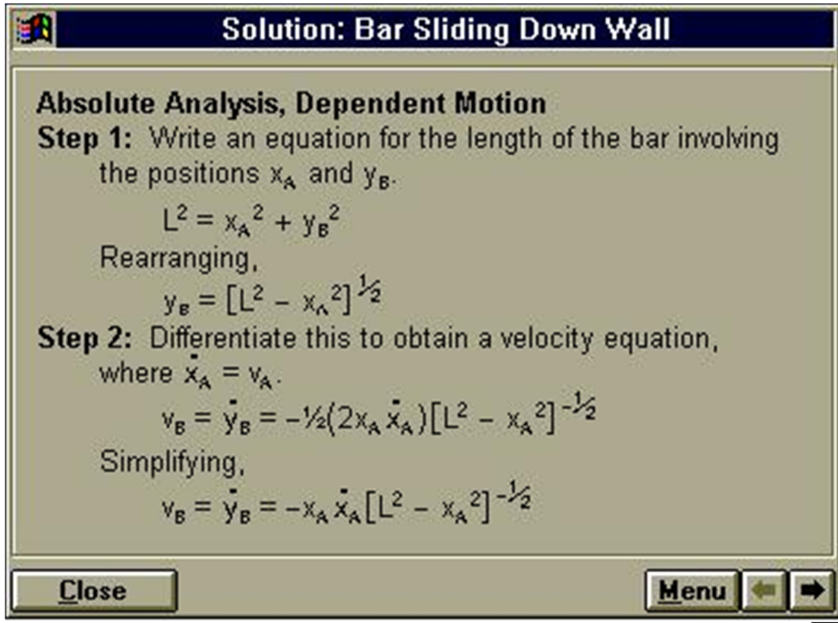


SLIDING BAR EXERCISE - solution









SLIDING BAR EXERCISE - solution

Solution: Bar Sliding Down Wall

Absolute Analysis, Parametric Method

Step 1: Write position equations in terms of the parameter, θ.

$$x_A = L \sin\theta \qquad y_B = L\cos\theta$$

Step 2: Differentiate using the chain rule to obtain velocities.

$$v_A = \dot{x}_A = L\,\omega\cos\theta \qquad v_B = \dot{y}_B = -L\,\omega\sin\theta$$

Step 3: Differentiate again to obtain acceleration equations.

$$a_A = \ddot{x}_A = L\,\alpha\cos\theta - L\,\omega^2\sin\theta$$

$$a_B = \ddot{y}_B = -L\,\alpha\sin\theta - L\,\omega^2\cos\theta$$

Close | Menu

Solution: Bar Sliding Down Wall

Relative Velocity, Scalar Equations

Step 1: Write a relative velocity equation, noting that $\vec{v}_A$ and $\vec{v}_B$ are constrained to act horizontally and vertically, respectively. Sketch pictures under each term of the equation.

$$\vec{v}_B \quad = \quad \vec{v}_A \quad + \quad \vec{v}_{B/A}$$

Close | Menu

Solution: Bar Sliding Down Wall

Relative Velocity, Cross Products

Step 1: Write a relative velocity equation, expressing the relative term as $\vec{v}_{B/A} = \vec{\omega}_{AB} \times \vec{r}_{AB}$. $(r_{AB} = L)$

$$\vec{v}_B = \vec{v}_A + \vec{\omega}_{AB} \times \vec{r}_{AB}$$

where: $\vec{r}_{AB} = r_{AB}\left(-\cos 47°\,\hat{\imath} + \sin 47°\,\hat{\jmath}\right)$

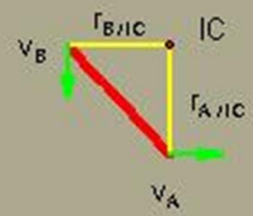
$\vec{\omega}_{AB} = \omega_{AB}\,\hat{k}$

Close | Menu

Solution: Bar Sliding Down Wall

Velocities from Instantaneous Center

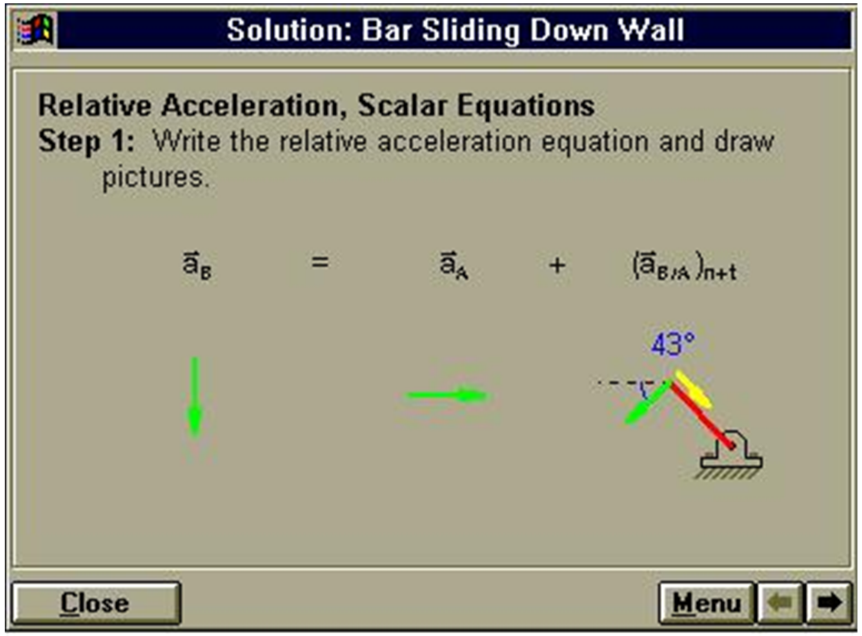
Step 1: Draw a picture of link AB, showing v_A and v_B. Note that these velocities are constrained to act horizontally and vertically.

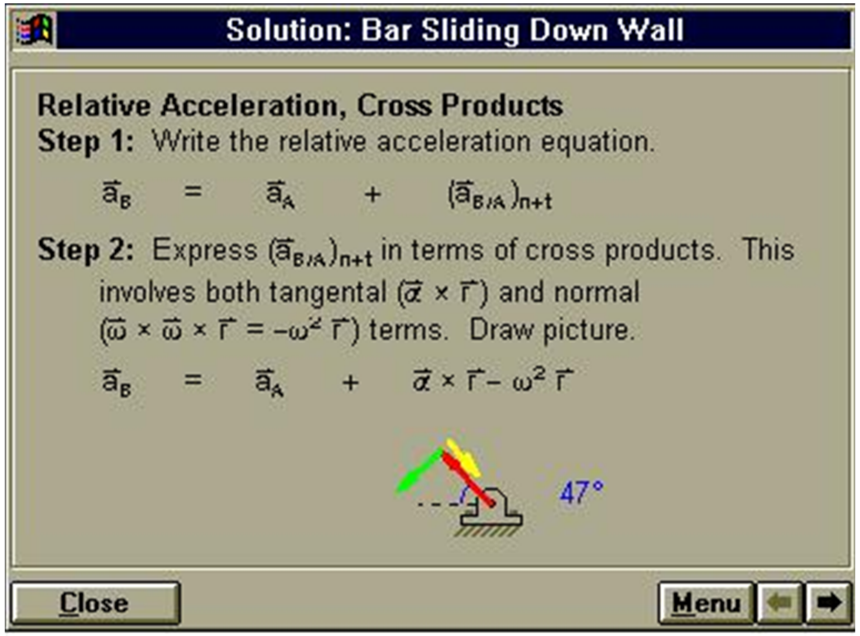


Step 2: Locate the instantaneous center (IC) by drawing construction lines perpendicular to velocities v_A and v_B.

Close | Menu

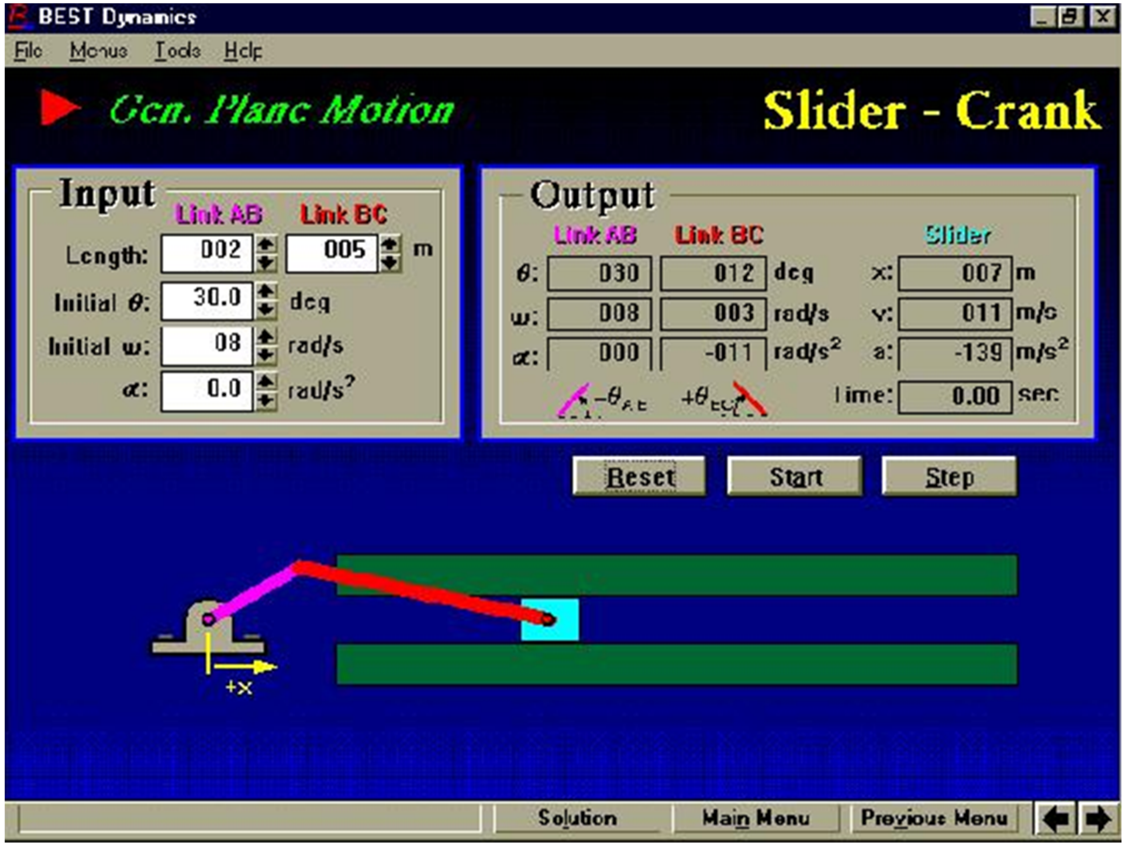
SLIDING BAR EXERCISE – solution



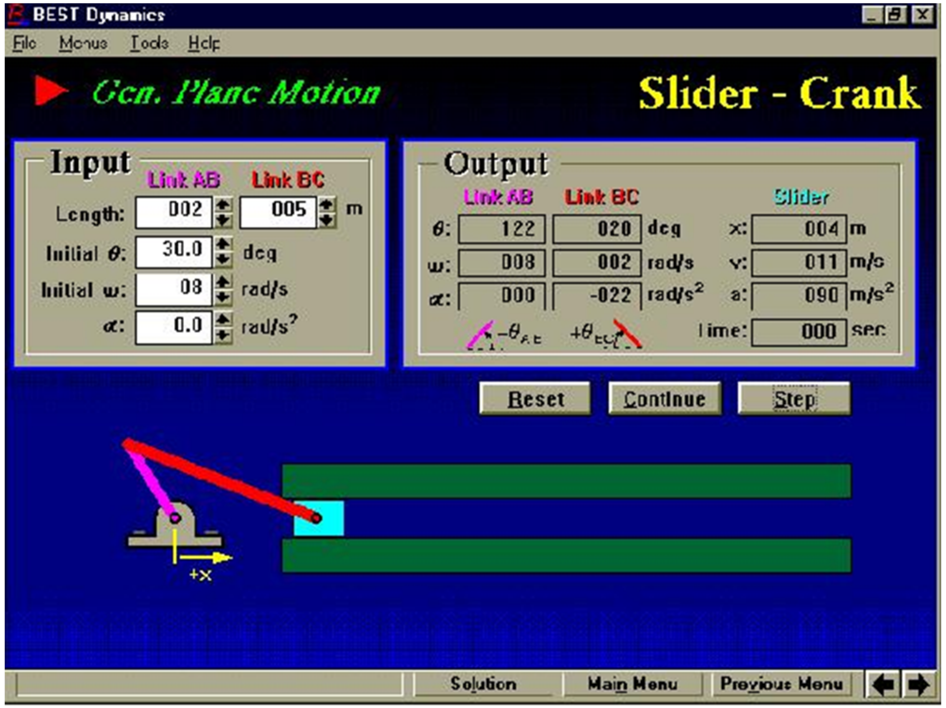


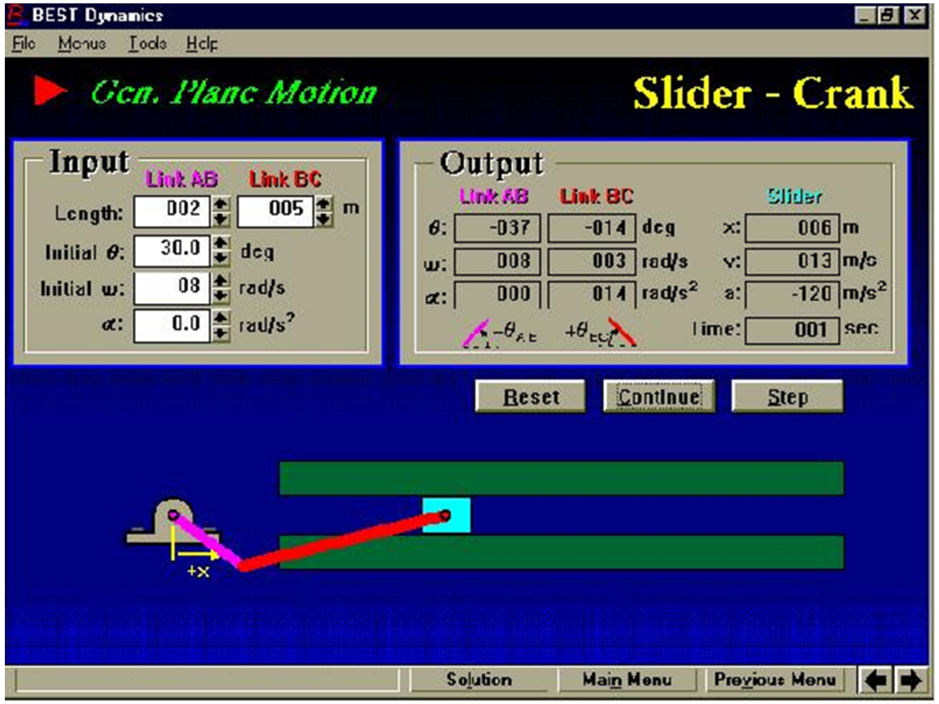
SLIDER CRANK MECHANISM

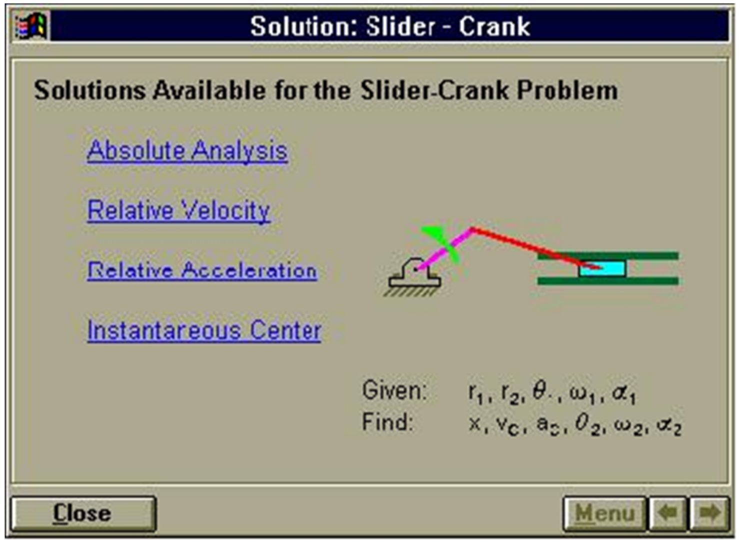
The slider crank mechanism is represented in the figure, using rotational pin joints at the left extremities and translation joint in the right point. For a given initial data, calculate the relative velocity and acceleration of the crank body and the instantaneous centre of rotation.



SLIDER CRANK MECHANISM - solution

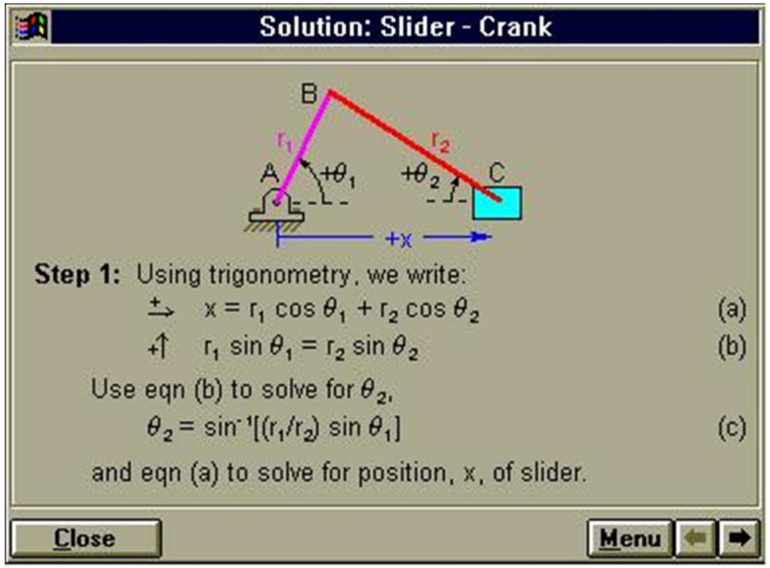




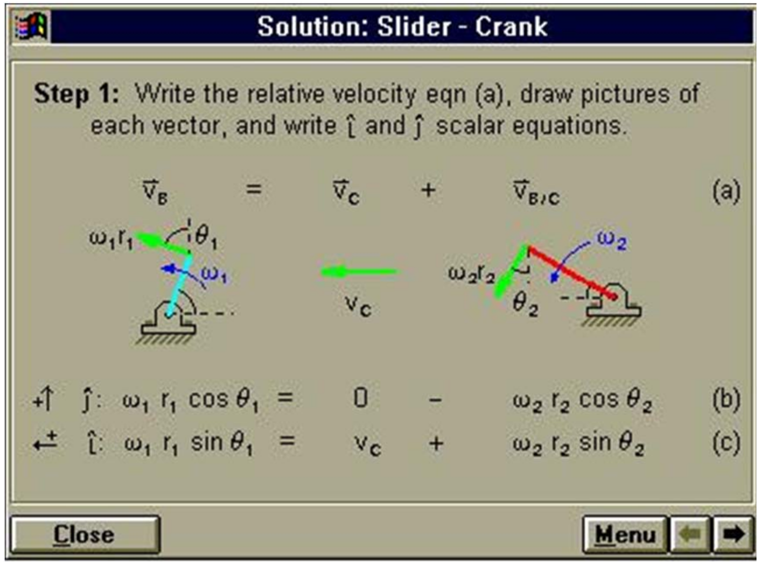


SLIDER CRANK MECHANISM – solution

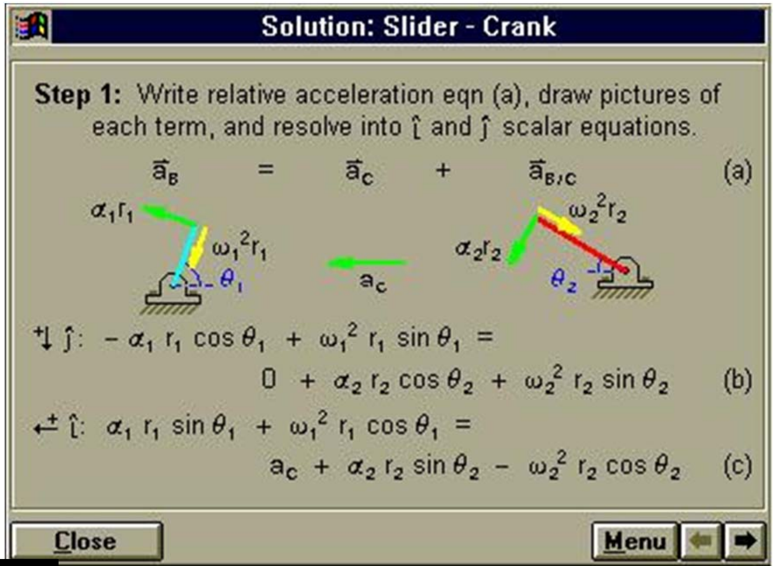
Absolute analysis



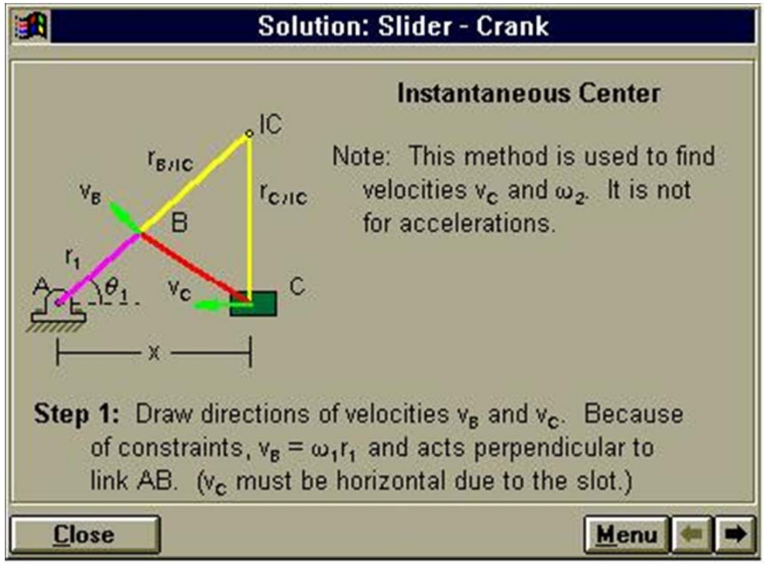
Relative velocity



Relative acceleration

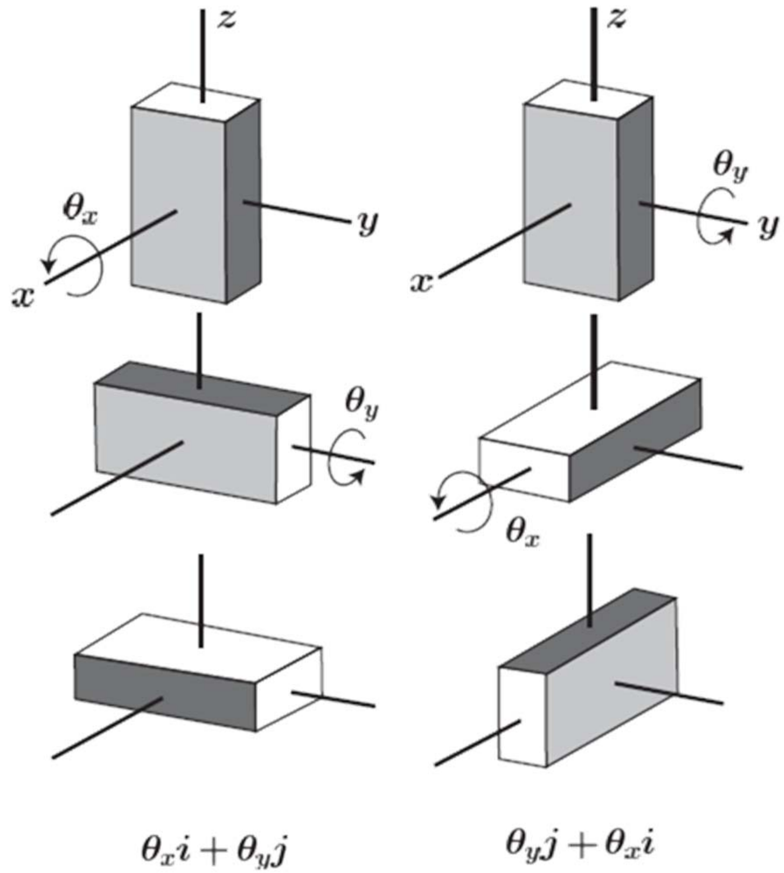


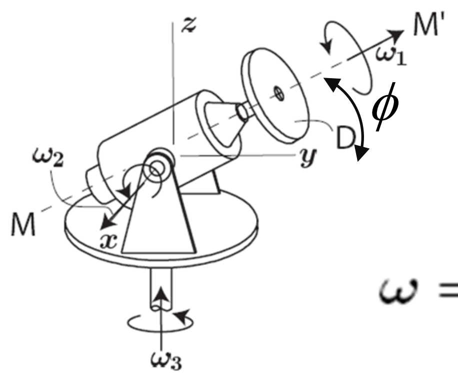
Instantaneous centre of rotation



REVISIONS ABOUT ROTATION

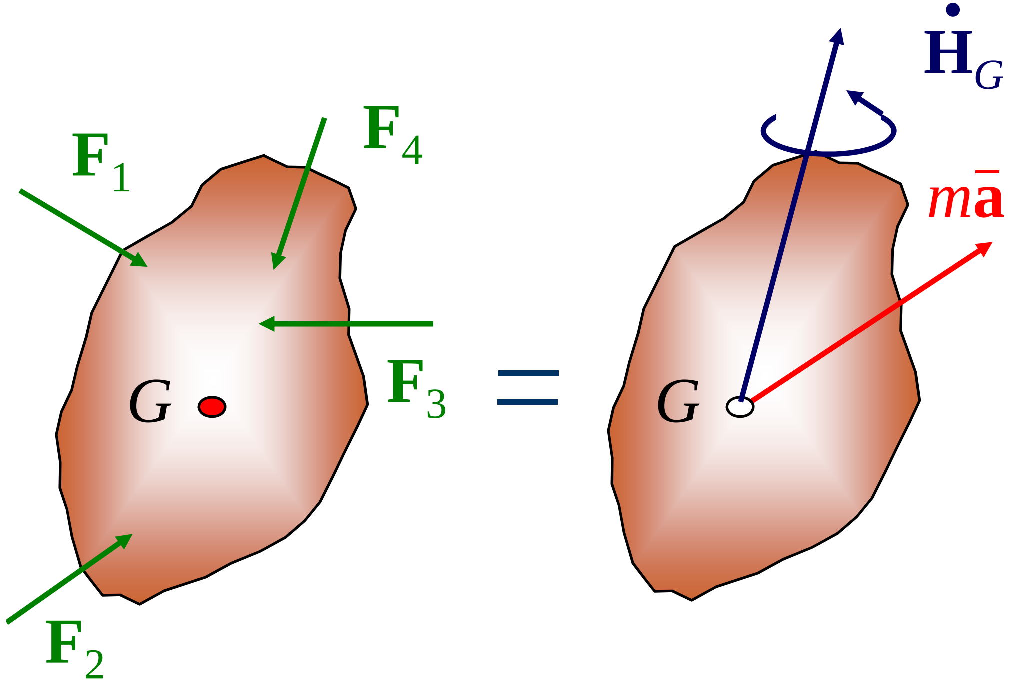
- Rotation About a Fixed Point “O“:
  - θ1+θ2 is different from θ2+θ1.
  - finite rotations cannot be treated as vectors, since they do not satisfy simple vector operations such as the parallelogram vector addition law.
  - infinitesimal rotations indeed behave as vectors.
  - angular velocities can be added vectorially, ex: W=w1+w2.





$$\boldsymbol{\omega} = \omega_2\, \boldsymbol{i} + \omega_1 \cos\phi\; \boldsymbol{j} + (\omega_1 \sin\phi + \omega_3)\; \boldsymbol{k}$$

PLANE MOTION OF RIGID BODIES: FORCES AND ACCELERATIONS



The relations existing between the forces acting on a rigid body, the shape and mass of the body, and the motion produced are studied as the ***kinetics of rigid bodies***.

In general, our analysis is restricted to the *plane motion of rigid slabs* and rigid bodies symmetrical with respect to the reference plane.

The two equations for the motion of a system of particles apply to the most general case of the motion of a rigid body. The first equation defines the motion of the mass centre *G* of the body.

$$\Sigma \mathbf{F} = m\bar{\mathbf{a}}$$

where m is the mass of the body, and $\bar{\mathbf{a}}$ the acceleration of *G*. The second is related to the motion of the body relative to a centroidal frame of reference.

$$\Sigma \mathbf{M}_G = \dot{\mathbf{H}}_G$$

GENERAL EQUATIONS FOR PLANE MOTION

$$\Sigma \mathbf{F} = m\bar{\mathbf{a}}$$

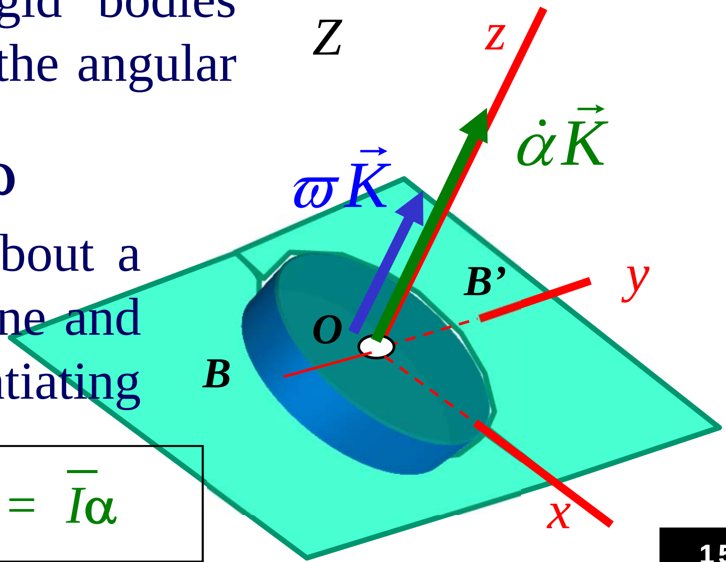
$$\Sigma \mathbf{M}_G = \dot{\mathbf{H}}_G$$

Where $\dot{\mathbf{H}}_G$ is the rate of change of the angular momentum $\mathbf{H}_G$ of the body about its mass centre G.

These equations express that ***<u>the system of the external forces is equipollent to the system consisting of the vector ma attached at G and the couple of moment $\dot{H}_G$</u>***.

For the **<u>plane motion</u>** of rigid slabs and rigid bodies symmetrical with respect to the reference plane, the angular momentum of the body is expressed as:

$$\mathbf{H}_G = \bar{I}\omega$$

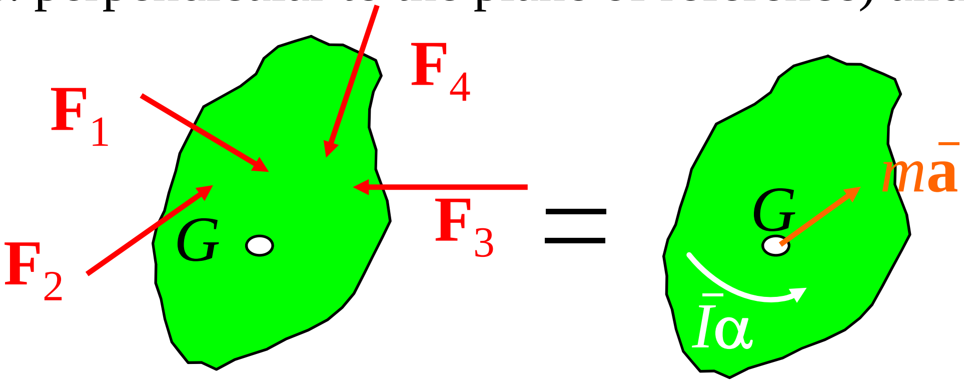


where $\bar{I}$ is the moment of inertia of the body about a centroidal axis perpendicular to the reference plane and **w** is the angular velocity of the body. Differentiating both members of this equation

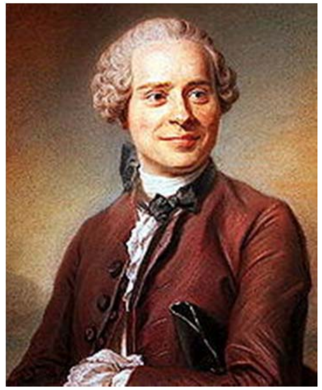
$$\dot{\mathbf{H}}_G = \bar{I}\dot{\omega} = \bar{I}\alpha$$

GENERAL EQUATIONS FOR PLANE MOTION

For the restricted case considered here, the rate of change for the angular momentum of the rigid body can be represented by a vector of the same direction as α (i.e. perpendicular to the plane of reference) and of magnitude $I\alpha$.



$$\sum M_G = \bar{I}\alpha \qquad \sum F_x = m\bar{a}_x \quad \sum F_y = m\bar{a}_y$$



Jean le Rond d'Alembert (November 16, 1717 – October 29, 1783) was a French mathematician, physicist a mechanician, and philosopher. He was also co-editor with Denis Diderot of the *Encyclopedia.*

The external forces acting on a rigid body are actually **equivalent** to the effective forces of the various particles forming the body. This statement is known as **d'Alembert's principle.** D'Alembert showed that one can transform an accelerating rigid body into an **equivalent static system** by adding the so-called "**inertial force**" and "**inertial torque**" or moment. The system can then be analyzed exactly as a static system subjected to this "inertial force and moment" and the external forces.

$$\sum M_G - \bar{I}\alpha = 0 \qquad \sum F_x - m\bar{a}_x = 0 \quad \sum F_y - m\bar{a}_y = 0$$

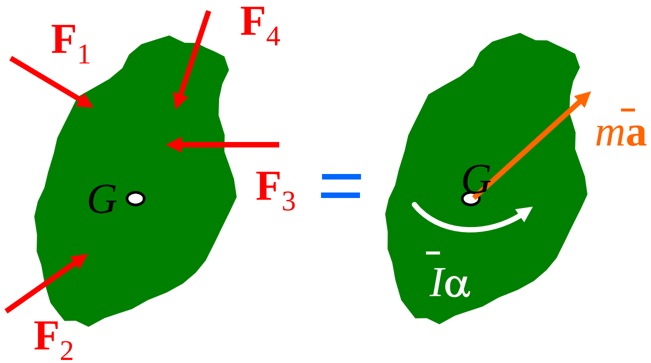
ALEMBERT'S PRINCIPLE

Alembert's principle can be expressed in the form of a vector diagram, where the effective forces are represented by a vector $m\mathbf{a}_G$ attached at G and a couple $I_G\alpha$.

•In the case of a slab in ***translation***, the effective forces reduce to a single vector $m\mathbf{a}_G$;

•while in the particular case of a slab in ***centroidal rotation***, they reduce to the single couple $I_G\alpha$;

•in any other case of plane motion, both the vector $m\mathbf{a}_G$ and $I_G\alpha$ should be included.

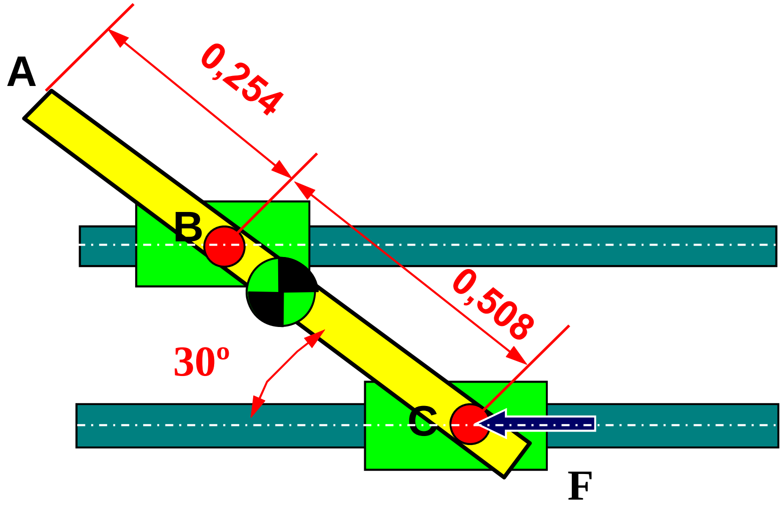


$$\sum \vec{F} - m\vec{a}_G = \vec{0}$$

$$\sum \vec{F} + \vec{F}_{inertia} = \vec{0}$$

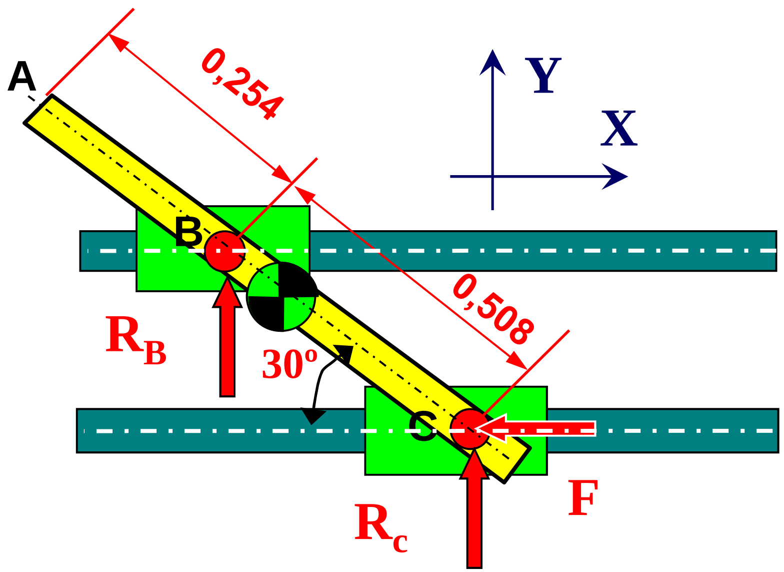
This method can be used to solve problems involving the plane motion of several connected rigid bodies. Some problems, such as ***noncentroidal rotation*** of rods and plates, the ***rolling motion*** of spheres and wheels, and the plane motion of ***various types of linkages***, which move under constraints, must be supplemented by ***kinematic analysis.***

THEMATIC EXERCISE



A homogenous bar with 71.2 [N] of weight is pinned connected to two distinct frictionless translation joints, located in a vertical plane. Calculate the force F necessary to promote a normal reaction of 35.6 [N] (vertical) in to point B and the corresponding bar acceleration.

$$\sum \vec{F} = m\vec{a}_G \Leftrightarrow \begin{cases} -F = ma_{Gx} \\ R_C + R_B - 71{,}2 = 0 \\ 0 = 0 \end{cases}$$

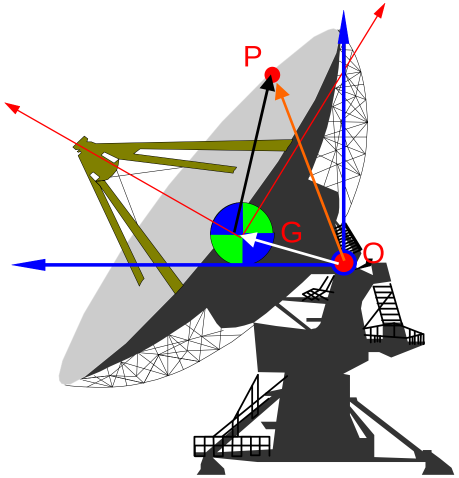


$$\sum \vec{M}_G = \dot{\vec{H}}_G = \vec{0} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} 0 = 0 \\ 0 = 0 \\ 0{,}33R_C - 0{,}109 \times 35{,}6 - F \times 0{,}1905 = 0 \end{cases}$$

Solution: **F=41,3 (N), a_{GX}=5,68 (m/s^2)**

ANGULAR MOMENTUM



By definition:

$$\vec{H}_G = \int_M G\vec{P} \times \vec{V}_P dm$$

$$\vec{H}_O = \int_M O\vec{P} \times \vec{V}_P dm$$

Relation between: (1° Koenig theorem)

$$\vec{H}_O = \int_M \left(O\vec{G} + G\vec{P}\right) \times \left(\vec{V}_G + W_{body} G\vec{P}\right) dm$$

$$= \int_M O\vec{G} \times \left(\vec{V}_G + W_{body} \times G\vec{P}\right) dm + \int_M G\vec{P} \times \left(\vec{V}_G + W_{body} \times G\vec{P}\right) dm$$

$$= O\vec{G} \times \vec{V}_G M + 0 + [I_G]\{W_{body}\}$$

$$= O\vec{G} \times \vec{V}_G M + \vec{H}_G$$

Samuel Koenig, German physicist , (1712-1757)

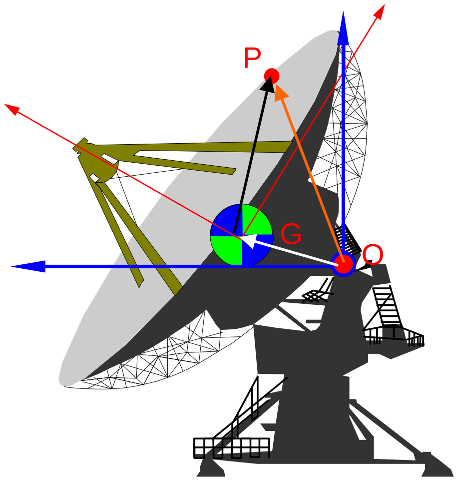
KINETIC MOMENTUM - SPECIAL CASES

1- Body in translation

$$\vec{H}_G = \vec{0}$$

Because W=0 !!!

$$\vec{H}_O = \vec{OG} \times M\vec{V}_G$$



2- Fix point rotation about point O

$$H_O = \int_M \vec{OP} \times (\vec{W} \times \vec{OP}) dm$$

$$= [I_O]\{W\}$$

3- Three-dimensional general movement

$$\vec{H}_G = \int_M \vec{GP} \times \left(\vec{V}_G + \vec{W} \times \vec{GP}\right) dm$$

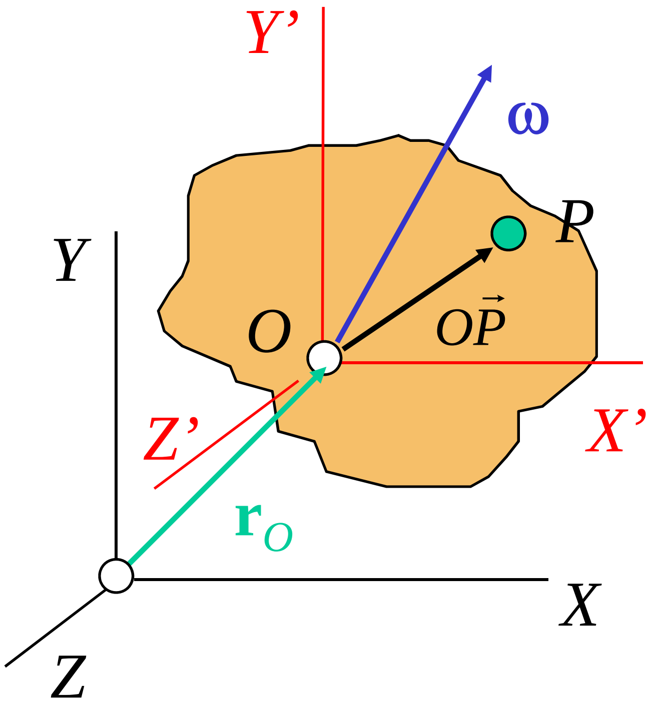
$$= [I_G]\{W\}$$

$$\vec{H}_O = \vec{H}_G + \vec{OG} \times M\vec{V}_G$$

(1º Koenig theorem)

DYNAMIC MOMENTUM

Time differentiating the angular momentum, normally calculated into a non fixed point, we obtain:



$$\dot{\vec{H}}_O = \frac{d}{dt}\int_M \vec{OP}\times\vec{V}_P dm$$

$$= \int_M \frac{d\left(\vec{OP}\right)}{dt}\times\vec{V}_P dm + \int_M \vec{OP}\times\frac{d\vec{V}_P}{dt}dm$$

$$= \int_M \left(\vec{V}_P - \vec{V}_O\right)\times\vec{V}_P dm + \int_M \vec{OP}\times\vec{a}_P dm$$

$$= -\vec{V}_O \times M\vec{V}_G + \vec{K}_O$$

Being "O" a fixed point → $\vec{K}_O = \dot{\vec{H}}_O$

Being "O" coincident to point "G" → $\vec{K}_G = \dot{\vec{H}}_G$

Other cases: → Use of general expression

CONCLUSION

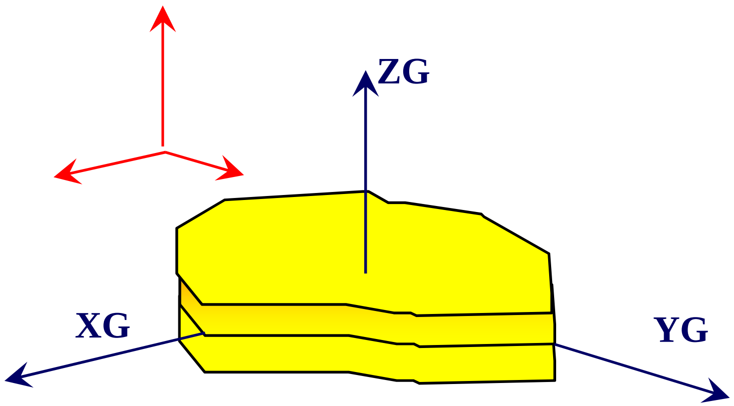
1- $\vec{H}_G = [I_G]\{W\}$ **ALWAYS !!!!**

2- $\vec{K}_G = \dot{\vec{H}}_G = [\dot{I}_G]\{W\} + [I_G]\{\dot{W}\} + \Omega \times \{[I_G]\{W\}\}$

Special case: general plane motion with symmetric body to plane OGY

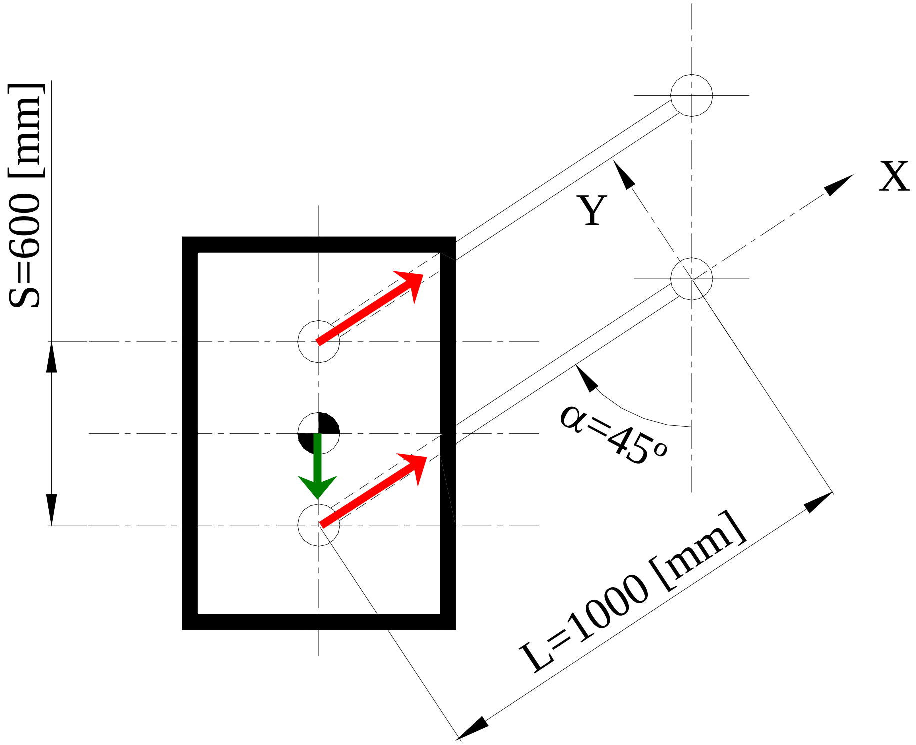
$$\vec{W} = \begin{Bmatrix} 0 \\ 0 \\ w \end{Bmatrix}$$

$$[I_G] = \begin{bmatrix} I_{X_G X_G} & -P_{X_G Y_G} & 0 \\ -P_{X_G Y_G} & I_{Y_G Y_G} & 0 \\ 0 & 0 & I_{Z_G Z_G} \end{bmatrix}$$



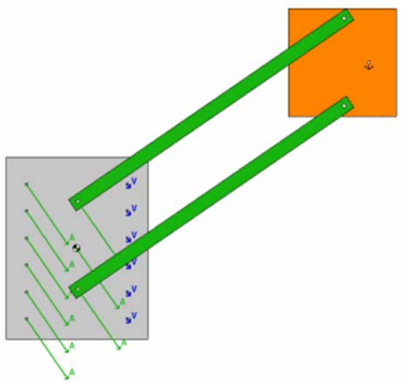
$$\vec{H}_G = \begin{Bmatrix} 0 \\ 0 \\ I_{Z_G Z_G} W \end{Bmatrix} \longrightarrow \vec{K}_G = \begin{Bmatrix} 0 \\ 0 \\ I_{Z_G Z_G} \dot{W} \end{Bmatrix} + \begin{Bmatrix} 0 \\ 0 \\ W \end{Bmatrix} \times \begin{Bmatrix} 0 \\ 0 \\ I_{Z_G Z_G} W \end{Bmatrix}$$

EXERCISE: RIGID BODY IN ROT. TRANSLATION



For the rigid plane body, moving in rotational translation, with mass M, connected with two straight massless bars to the same number of fix pin joints.

Knowing that the rigid plane body is moving in to a vertical plane, determine the connecting forces between the bars and the body, for the angle 45º.

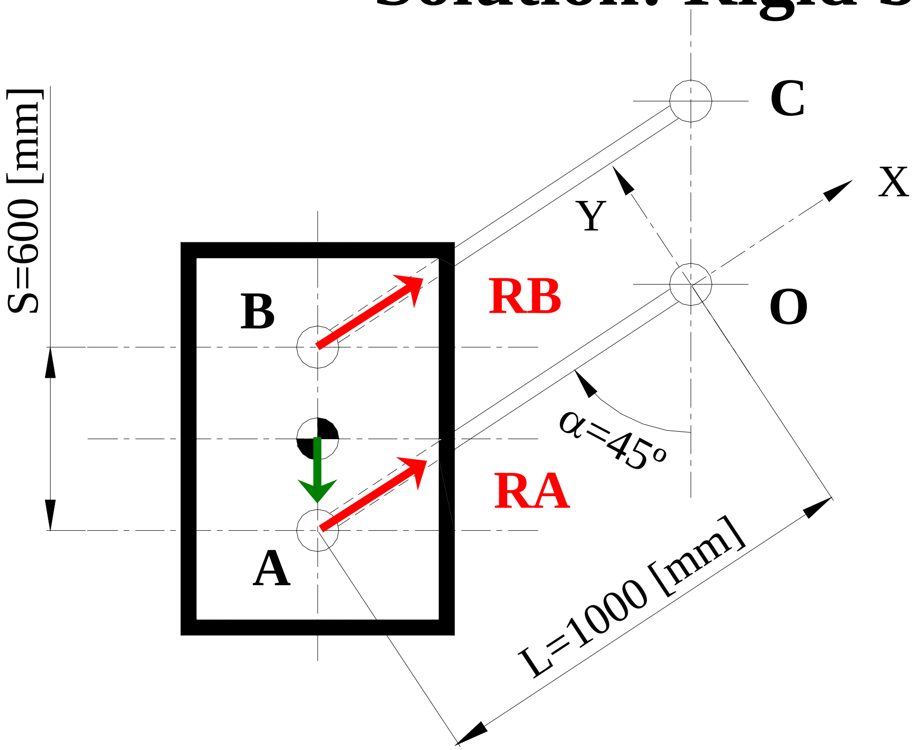


Dynamic Solution:

$$\sum \vec{F} = M\ \vec{a}_G$$

$$\sum \vec{M}_G = \vec{K}_G = \dot{\vec{H}}_G$$

Solution: Rigid body in translation



Cinematic solution: ***Determine the mass center acceleration.***

$$\vec{a}_G = \vec{a}_B + \dot{\vec{W}} \times B\vec{G} + \vec{W} \times \left(\vec{W} \times B\vec{G}\right)$$

$$\vec{a}_B = \vec{a}_C + \dot{\vec{\Omega}} \times C\vec{B} + \vec{\Omega} \times \left(\vec{\Omega} \times C\vec{B}\right)$$

$$= \vec{0} + \begin{Bmatrix} 0 \\ 0 \\ -\ddot{\alpha} \end{Bmatrix} \times \begin{Bmatrix} -L \\ 0 \\ 0 \end{Bmatrix} + \begin{Bmatrix} 0 \\ 0 \\ -\dot{\alpha} \end{Bmatrix} \times \left[\begin{Bmatrix} 0 \\ 0 \\ -\dot{\alpha} \end{Bmatrix} \times \begin{Bmatrix} -L \\ 0 \\ 0 \end{Bmatrix} \right]$$

$$= \begin{Bmatrix} \dot{\alpha}^2 L \\ \ddot{\alpha} L \\ 0 \end{Bmatrix}$$

Determine the dynamic momentum (time derivative of the angular momentum)

$$\vec{H}_G = [I_G]\{W\} = \vec{0}$$

$$\dot{\vec{H}}_G = \dot{\vec{0}} + \vec{\Omega} \times \vec{0} = \vec{0}$$

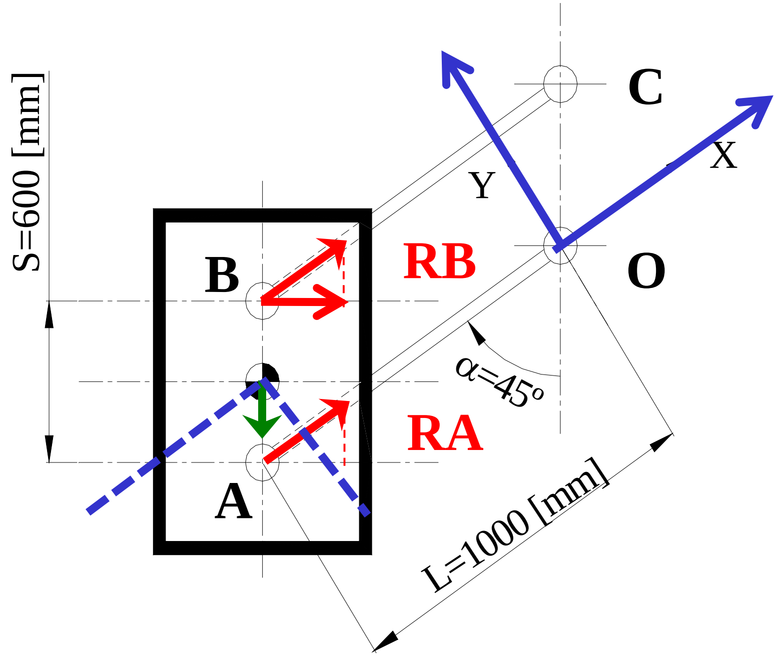
Solution: Rigid body in translation

Dynamic Solution system:

$$\sum \vec{F} = M\,\vec{a}_G \iff \begin{cases} RA + RB - Mg\cos(\alpha) = M\,(\dot{\alpha})^2 L \\ -Mg\sin(\alpha) = M\,\ddot{\alpha}\,L \end{cases}$$

$$\sum \vec{M}_G = \vec{K}_G = \dot{\vec{H}}_G \iff RA \times \frac{s}{2} \times \sin(\alpha) - RB \times \frac{s}{2} \times \sin(\alpha) = 0$$

$$\ddot{\alpha} = \frac{-g\,\sin(\alpha)}{L}$$



By direct substitution:

$$\ddot{\alpha} = \frac{-g\,\sin(\alpha)}{L} \iff \frac{d\dot{\alpha}}{d\alpha}\frac{d\alpha}{dt} = \frac{-g\,\sin(\alpha)}{L} \iff \frac{d\dot{\alpha}}{d\alpha}\dot{\alpha} = \frac{-g\,\sin(\alpha)}{L} \iff \dot{\alpha}\,d\dot{\alpha} = -\frac{g}{L}\sin(\alpha)\,d\alpha$$

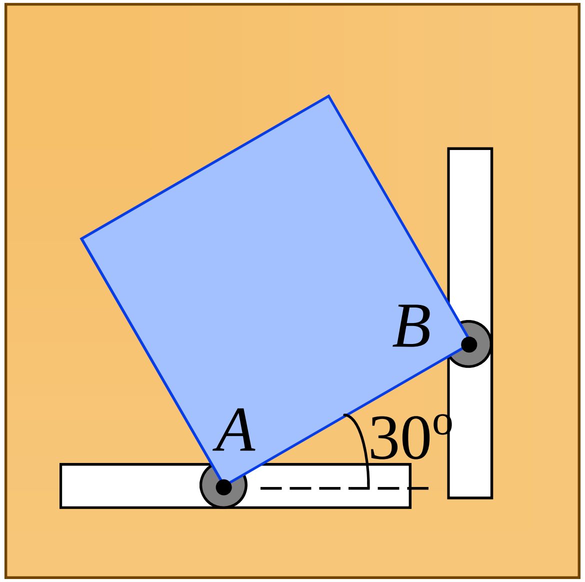
By direct integration:

$$\int \dot{\alpha}\,d\dot{\alpha} = \int -\frac{g}{L}\sin(\alpha)\,d\alpha \iff \frac{\dot{\alpha}^2}{2} = \frac{g}{L}\cos(\alpha)$$

By direct substitution:

$$RA = RB = \frac{3}{2}Mg\,\cos(\alpha)$$

EP 16.163 – DYNAMIC R. – Thematic exercise 13

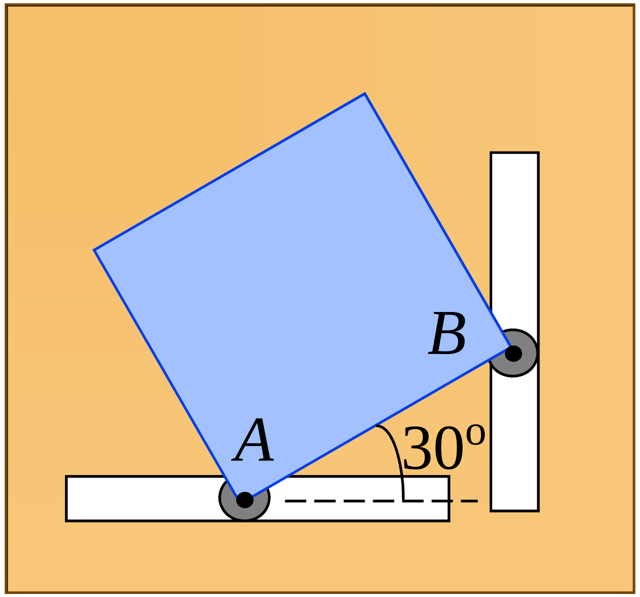


The motion of a square plate of side 150 mm and mass 2.5 kg is guided by pins at corners A and B that slide in slots cut in a vertical wall. Immediately after the plate is released from rest in the position shown, determine:

(a) the angular acceleration of the plate;

(b) the reaction at corner A.

EP 16.163 – DYNAMIC REACTIONS



1. ***Kinematics:*** Express the acceleration of the center of mass of the body, and the angular acceleration.
2. ***Kinetics:*** Draw a free body diagram showing the applied forces and the inertial components.
3. ***Write three equations of motion:*** Three equations of motion can be obtained by expressing the *x* components, *y* components, and moments about an arbitrary point.

1. *Kinematics:*

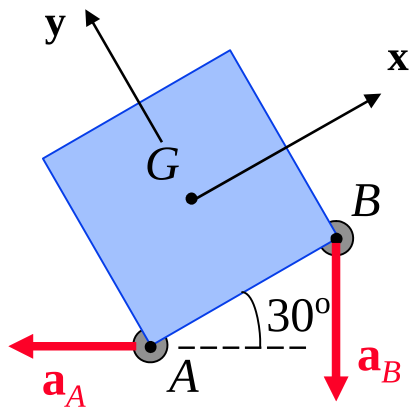
$$\vec{a}_B = \vec{a}_A + \dot{\vec{w}} \times \vec{AB} + \vec{w} \times \left(\vec{w} \times \vec{AB}\right)$$

For this instant, $\vec{w} = \vec{0}$.

$$\begin{Bmatrix} -a_B \sin(30) \\ -a_B \cos(30) \\ 0 \end{Bmatrix} = \begin{Bmatrix} -a_A \sqrt{3}/2 \\ a_A 1/2 \\ 0 \end{Bmatrix} + \begin{Bmatrix} 0 \\ 0 \\ -\alpha \end{Bmatrix} \times \begin{Bmatrix} 0.150 \\ 0 \\ 0 \end{Bmatrix}$$

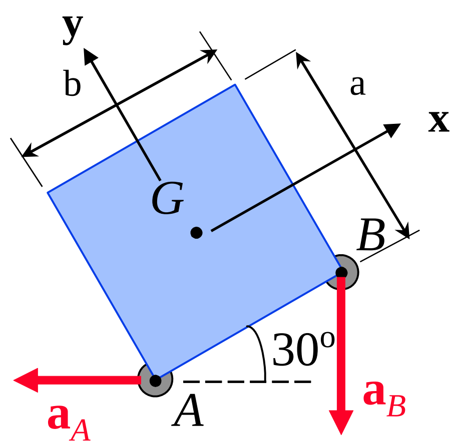
$$\begin{Bmatrix} -a_B \sin(30) \\ -a_B \cos(30) \\ 0 \end{Bmatrix} = \begin{Bmatrix} -a_A \sqrt{3}/2 \\ a_A 1/2 - 0.150\alpha \\ 0 \end{Bmatrix}$$

$$\begin{Bmatrix} a_B \\ a_A \\ 0 \end{Bmatrix} = \begin{Bmatrix} a_A \sqrt{3} \\ \alpha \times 0.075 \\ 0 \end{Bmatrix}$$



EP 16.163 – DYNAMIC REACTIONS

1. Kinematics: (cont.)



$$\vec{a}_G = \vec{a}_A + \dot{\vec{w}} \times \vec{AG} + \vec{w} \times \left(\vec{w} \times \vec{AG}\right)$$

$$= \begin{Bmatrix} -a_A \cos(30) \\ a_A \sin(30) \\ 0 \end{Bmatrix} + \begin{Bmatrix} 0 \\ 0 \\ -\alpha \end{Bmatrix} \times \begin{Bmatrix} 0.075 \\ 0.075 \\ 0 \end{Bmatrix}$$

$$= \begin{Bmatrix} -a_A \sqrt{3}/2 + 0.075\alpha \\ a_A 1/2 - 0.075\alpha \\ 0 \end{Bmatrix}$$

$$= \begin{Bmatrix} 0.0101\,\alpha \\ -0.0375\,\alpha \\ 0 \end{Bmatrix}$$

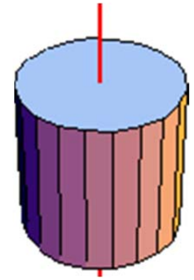
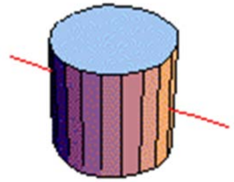
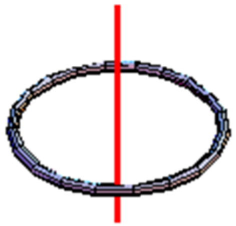
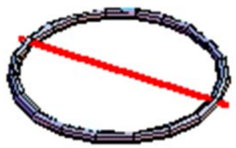
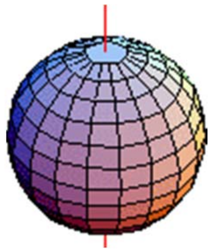
2. Kinetics: Angular and dynamic momentum

$$\vec{H}_G = [I_G]\{w\} = \begin{bmatrix} I_{xx} & 0 & 0 \\ 0 & I_{yy} & 0 \\ 0 & 0 & I_{zz} \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ -w \end{Bmatrix} \Leftrightarrow \vec{H}_G = \begin{Bmatrix} 0 \\ 0 \\ -I_{zz} w \end{Bmatrix} \Rightarrow \dot{\vec{H}}_G = \begin{Bmatrix} 0 \\ 0 \\ -I_{zz}\dot{w} \end{Bmatrix} + \vec{\Omega} \times \begin{Bmatrix} 0 \\ 0 \\ -I_{zz} w \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ -I_{zz}\dot{w} \end{Bmatrix} + \begin{Bmatrix} 0 \\ 0 \\ -w \end{Bmatrix} \times \begin{Bmatrix} 0 \\ 0 \\ -I_{zz} w \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ -I_{zz}\alpha \end{Bmatrix}$$

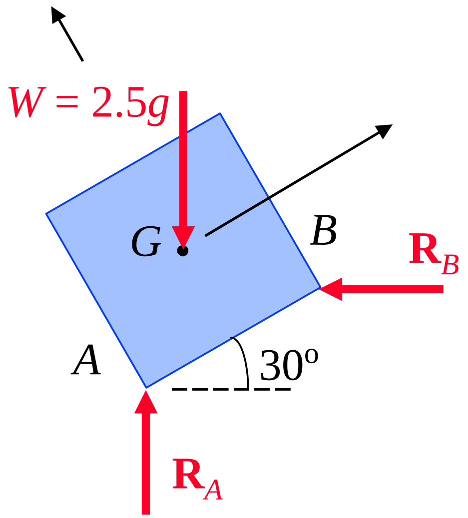
$$I_{zz} = I_{xx} + I_{yy} = \frac{1}{12} m a^2 + \frac{1}{12} m b^2 = \frac{1}{12} m\left(a^2 + b^2\right) = 0.009375\,\left[kg.m^2\right]$$

MOMENTS OF INERTIA FOR COMMON SOLIDS AROUND SOME OF THEIR PRINCIPAL AXES

SOLID/AXIS	MOMENT OF INERTIA
cylinder about symmetry axis	$\frac{1}{2}MR^2$
cylinder about central diameter	$\frac{1}{12}Mh^2 + \frac{1}{4}MR^2$
ellipsoid about principal axis	$\frac{1}{5}M(b^2 + c^2)$
elliptical slab about major axis	$\frac{1}{6}M(3b^2 + 4h^2)$
elliptical slab about vertical	$\frac{1}{2}M(a^2 + b^2)$
rectangular parallelepiped about major axis	$\frac{1}{3}(b^2 + c^2)M$
ring about perpendicular axis	MR^2
ring about diameter	$\frac{1}{2}MR^2$
rod about end	$\frac{1}{3}Mh^2$
rod about center	$\frac{1}{12}Mh^2$
sphere about diameter	$\frac{2}{5}MR^2$
spherical shell	$\frac{2}{3}MR^2$
torus about diameter	$\frac{1}{8}(5a^2 + 4c^2)M$
torus about symmetry axis	$(\frac{3}{4}a^2 + c^2)M$

EP 16.163 – DYNAMIC REACTIONS



$$\Sigma \mathbf{F} = m\bar{\mathbf{a}}$$

$$\Sigma \mathbf{M}_G = \dot{\mathbf{H}}_G$$

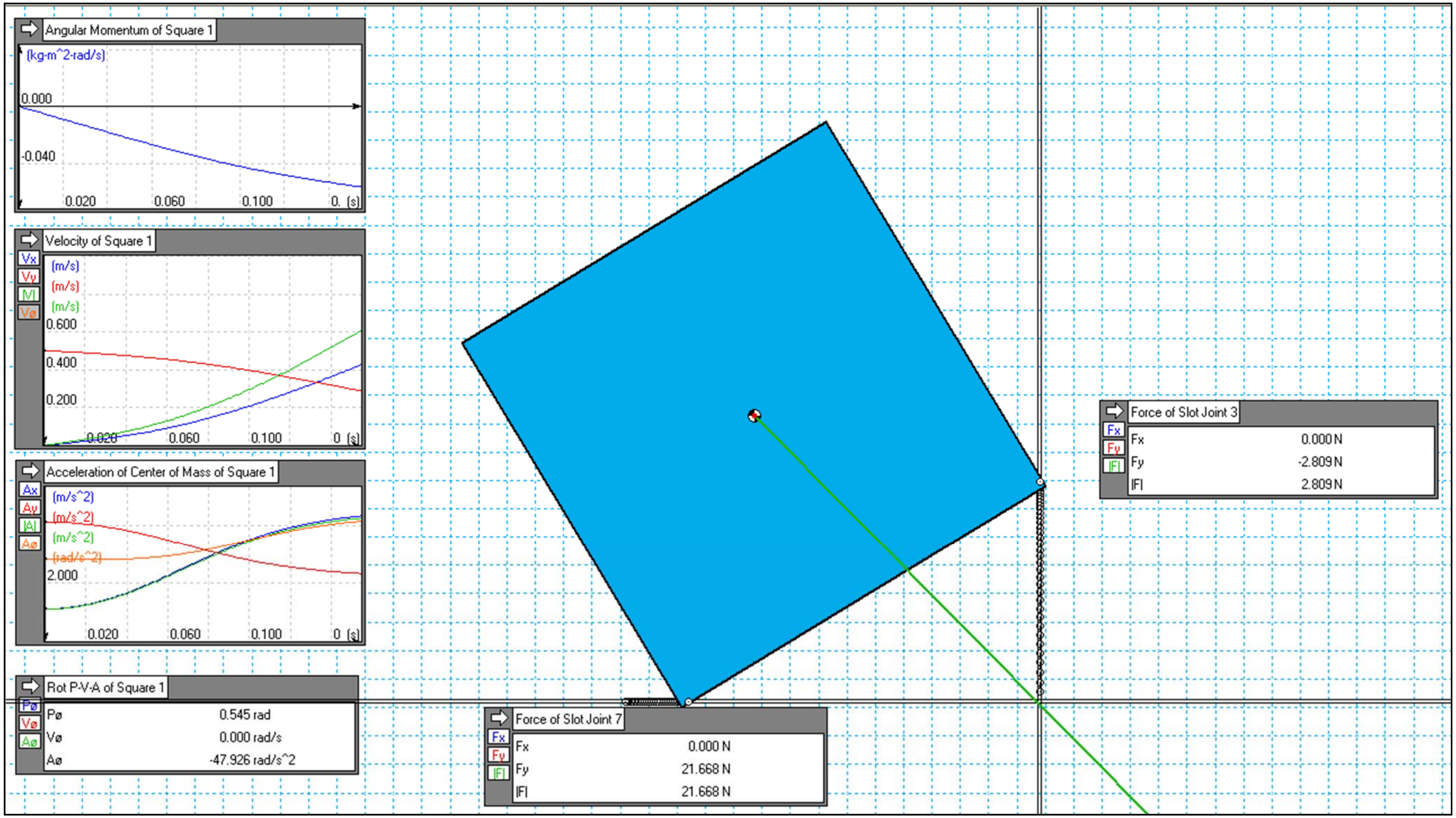
$$\Leftrightarrow \begin{cases} -R_B\cos(30) + R_A\sin(30) - 2.5g\sin(30) = m\,0.0101\alpha \\ R_A\cos(30) + R_B\sin(30) - 2.5g\cos(30) = m\,(-0.0375)\alpha \\ 0.075R_B\,1/2 - 0.075R_B\sqrt{3}/2 - 0.075R_A\sqrt{3}/2 + 0.075R_A\,1/2 = -0.009375\alpha \end{cases}$$

$$\Leftrightarrow \begin{bmatrix} 0.5 & -0.866 & -0.02525 \\ 0.866 & 0.5 & 0.09375 \\ -0.02745 & -0.01745 & 0.009375 \end{bmatrix} \begin{Bmatrix} R_A \\ R_B \\ \alpha \end{Bmatrix} = \begin{Bmatrix} 12.2625 \\ 21.2393 \\ 0 \end{Bmatrix}$$

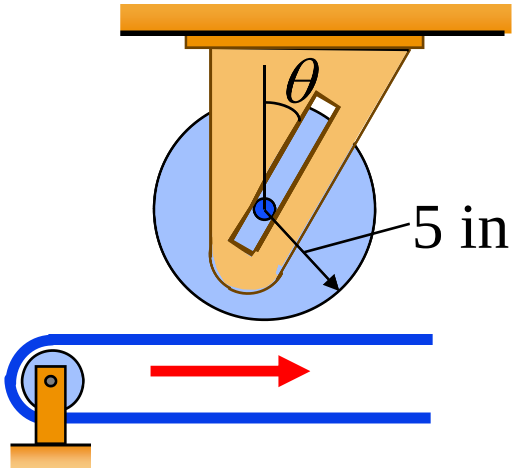
$$\Leftrightarrow \begin{Bmatrix} R_A \\ R_B \\ \alpha \end{Bmatrix} = \begin{Bmatrix} 21 \\ -3.52 \\ 51.2 \end{Bmatrix}$$

COMPUTATIONAL SOLUTION

Interactive physics: Angular momentum, velocity, acceleration, contact force.

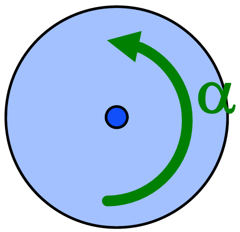


PROBLEM 16.153



The axis of a 5-in.-radius disk is fitted into a slot that forms an angle $\theta = 30^{o}$ with the vertical. The disk is at rest when it is placed in contact with a conveyor belt moving at constant speed. Knowing that the coefficient of kinetic friction between the disk and the belt is 0.2 and neglecting bearing friction, determine the angular acceleration of the disk while slipping occurs.

1. Kinematics: Express the acceleration of the center of mass of the body, and the angular acceleration.

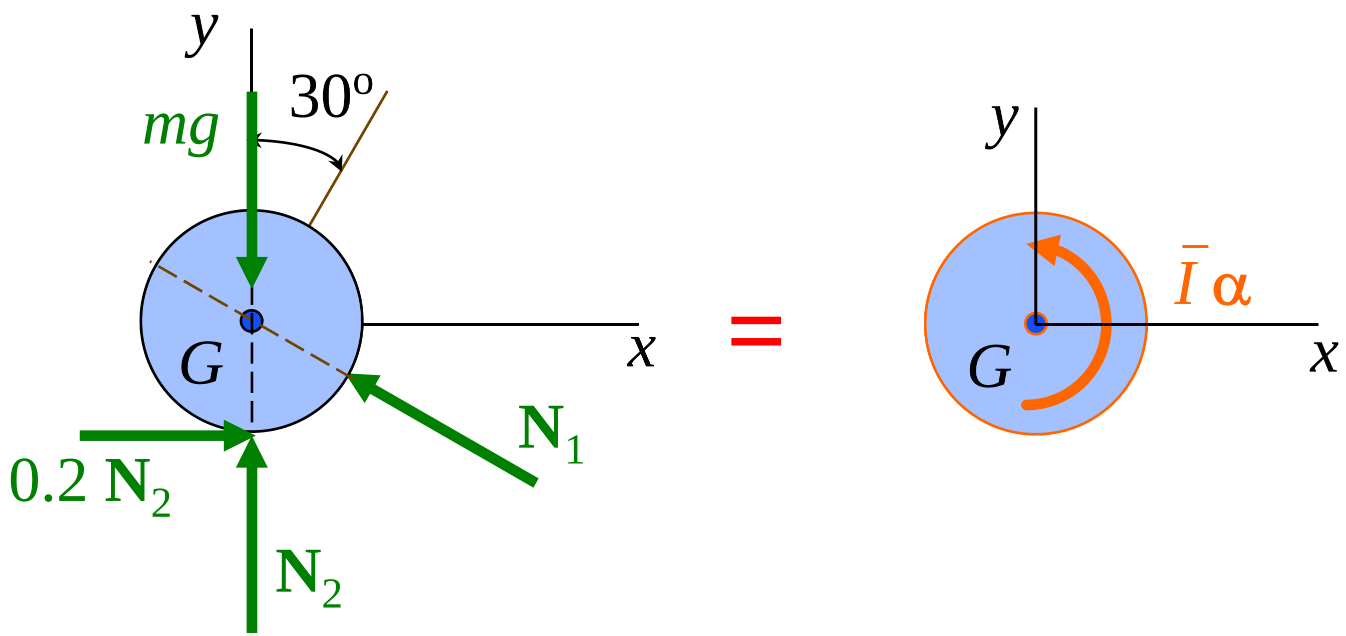


Once in contact with the belt the disk rotates about a fixed point (its center). The acceleration of the mass center is zero, and the angular acceleration is α.

2. Kinetics: Draw a free body diagram showing the applied forces and an equivalent force diagram showing the vector $m\bar{\mathbf{a}}$ or its components and the couple $\bar{I}\mathbf{a}$.

PROBLEM 16.153 - SOLUTION

Kinetics; draw a free body diagram.



Write three equations of motion.

$\Sigma F_x = m\,a_x$: $\quad 0.2\,N_2 - N_1 \cos 30^o = 0$

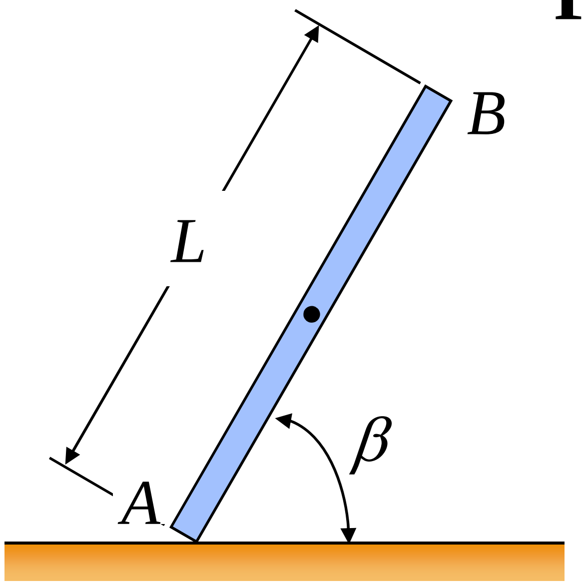
$\Sigma F_y = m\,a_y$: $\quad N_2 + N_1 \sin 30^o - mg = 0$

$\Sigma M_G = \bar{I}\alpha$: $\quad 0.2\,N_2\left(\frac{5}{12}\right) = \frac{1}{2}\mathrm{m}\left(\frac{5}{12}\right)^2 \alpha$

$N_2 = 0.896\,mg$

$\alpha = 27.7\ \text{rad/s}^2$

PROBLEM 16.158



The uniform rod AB of weight W is released from rest when $\beta = 70^o$. Assuming that the friction force is zero between end A and the surface, determine immediately after release (a) the angular acceleration of the rod, (b) the acceleration of the mass center of the rod, (c) the reaction at A.

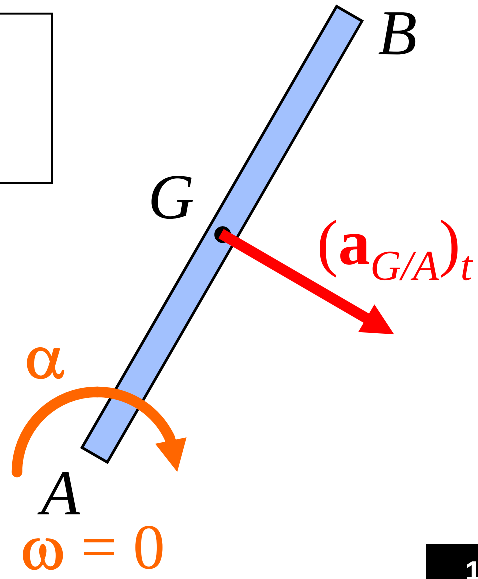
1. Kinematics: Express the acceleration of the center of mass of the body, and the angular acceleration.

$$(\mathbf{a}_{G/A})_t = \alpha\, r_{G/A} = \alpha \frac{L}{2}$$

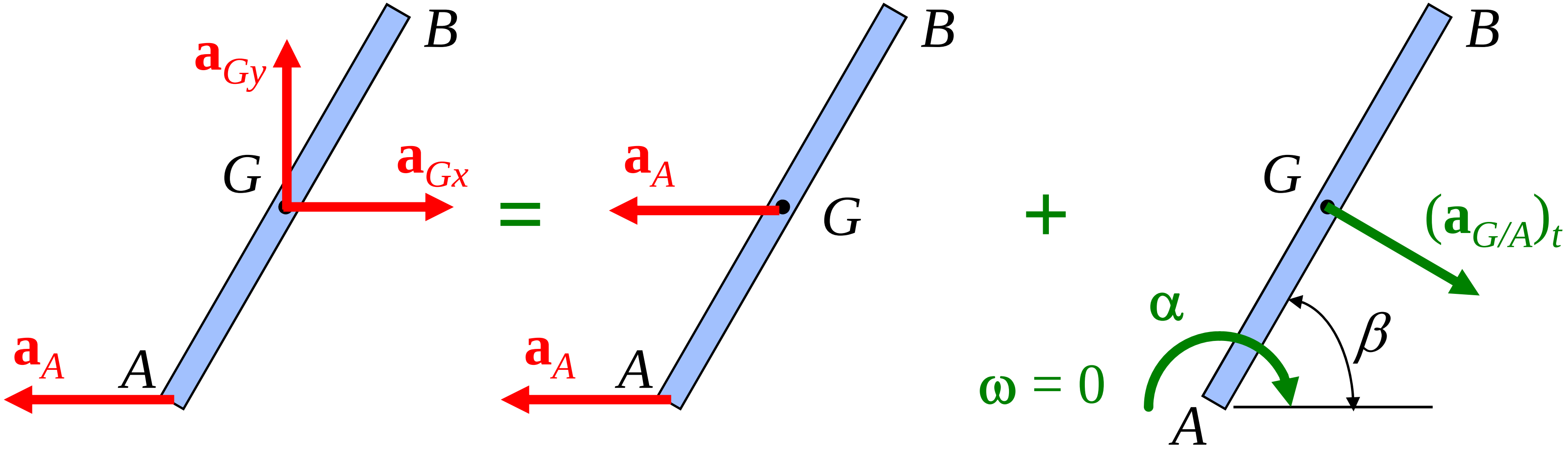
$$\mathbf{a}_G = \mathbf{a}_A + \mathbf{a}_{G/A}$$

$$\mathbf{a}_G = -a_A\,\mathbf{i} + \alpha \frac{L}{2} \sin 70^o\,\mathbf{i} - \alpha \frac{L}{2} \cos 70^o\,\mathbf{j}$$

$$\mathbf{a}_G = (-a_A + \alpha \frac{L}{2} \sin 70^o\,)\,\mathbf{i} - \alpha \frac{L}{2} \cos 70^o\,\mathbf{j}$$



PROBLEM 16.158 - SOLUTION

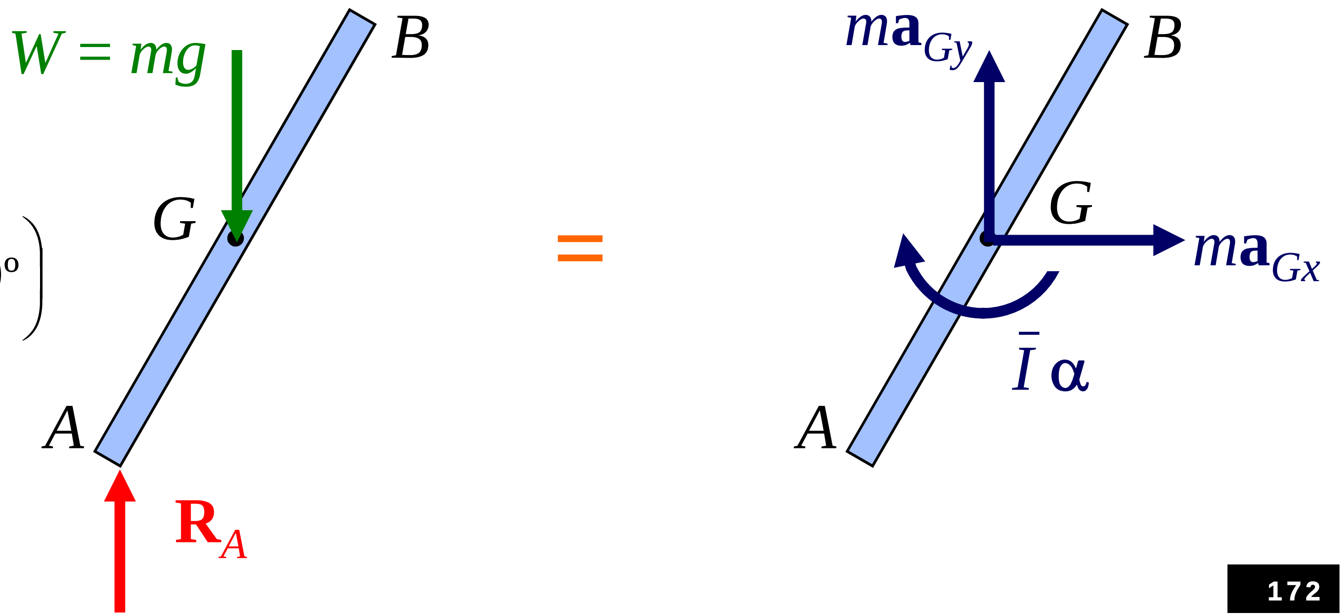


2. Kinetics: Draw a free body diagram showing the applied forces and an effective force diagram showing the vector $m\mathbf{a}$ or its components and the couple $\bar{I}\alpha$.

$$\bar{I}_{\alpha} = \frac{1}{12} m L^2 \alpha$$

$$ma_{Gx} = m\left(-a_A + \alpha \frac{L}{2} Sin70^{o}\right)$$

$$ma_{Gy} = -m\left(\alpha \frac{L}{2} Cos70^{o}\right)$$



PROBLEM 16.158 - SOLUTION

3. Write three equations of motion: Three equations of motion can be obtained by equating the *x* components, *y* components, and moments about an arbitrary point.

(a) The angular acceleration of the rod: Moments about point *P*

$$mg\left(\frac{L}{2}\cos 70^o\right) = m\alpha\frac{L}{2}\cos 70^o\left(\frac{L}{2}\cos 70^o\right) + \frac{1}{12}mL^2\alpha$$

$$\alpha = \frac{6\,g\cos 70^o}{L\,[\,1 + 3\,(\cos 70^o)^2\,]}$$

$$\boldsymbol{\alpha} = 1.519\,(\,g/L)$$

(b) The acceleration of the mass center:

$$\Sigma F_x = m\,a_x: \quad 0 = m\left(-a_A + \alpha\frac{L}{2}\sin 70^o\right)$$

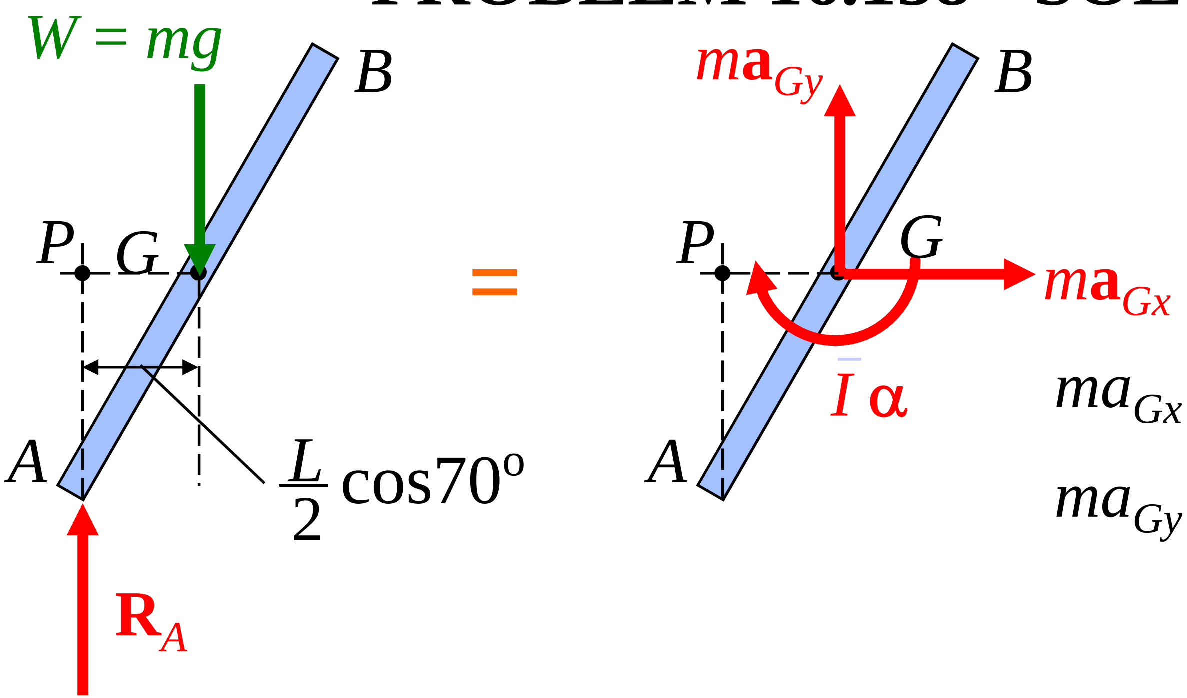
$$a_A = \alpha\frac{L}{2}\sin 70^o = 1.519\frac{L}{2}\frac{g}{L}\sin 70^o = 0.760\,g$$

$$\mathbf{a}_G = \left(-a_A + \alpha\frac{L}{2}\sin 70^o\right)\mathbf{i} - \alpha\frac{L}{2}\cos 70^o\,\mathbf{j}$$

Substitute for a_A and α :

$$\mathbf{a}_G = 0\,\mathbf{i} - 0.260\,g\,\mathbf{j}$$

PROBLEM 16.158 - SOLUTION



$$\alpha = 1.519\ (g/L)$$

$$m a_{Gx} = m\,(-a_A + \alpha \frac{L}{2} \sin 70^{\circ})$$

$$m a_{Gy} = -\,m\,\alpha \frac{L}{2} \cos 70^{\circ}$$

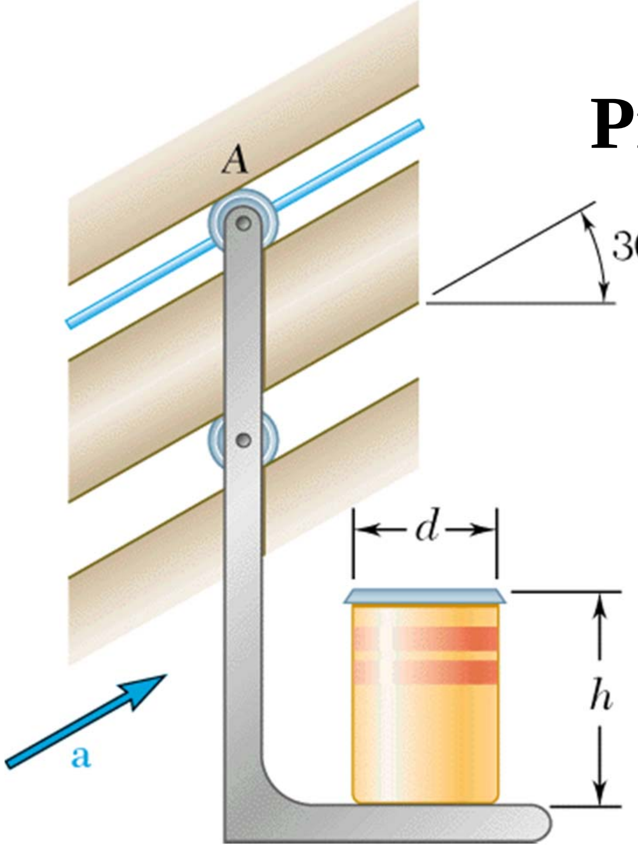
(c) The reaction at *A*:

$$\Sigma F_y = m\,a_y: \quad R_A - mg = -\,m\,\alpha \frac{L}{2} \cos 70^{\circ}$$

$$R_A = mg - m\,\alpha \frac{L}{2} \cos 70^{\circ}$$

Substitute for α: $\quad R_A = 0.740\ mg \qquad \mathbf{R}_A = 0.740\ mg$

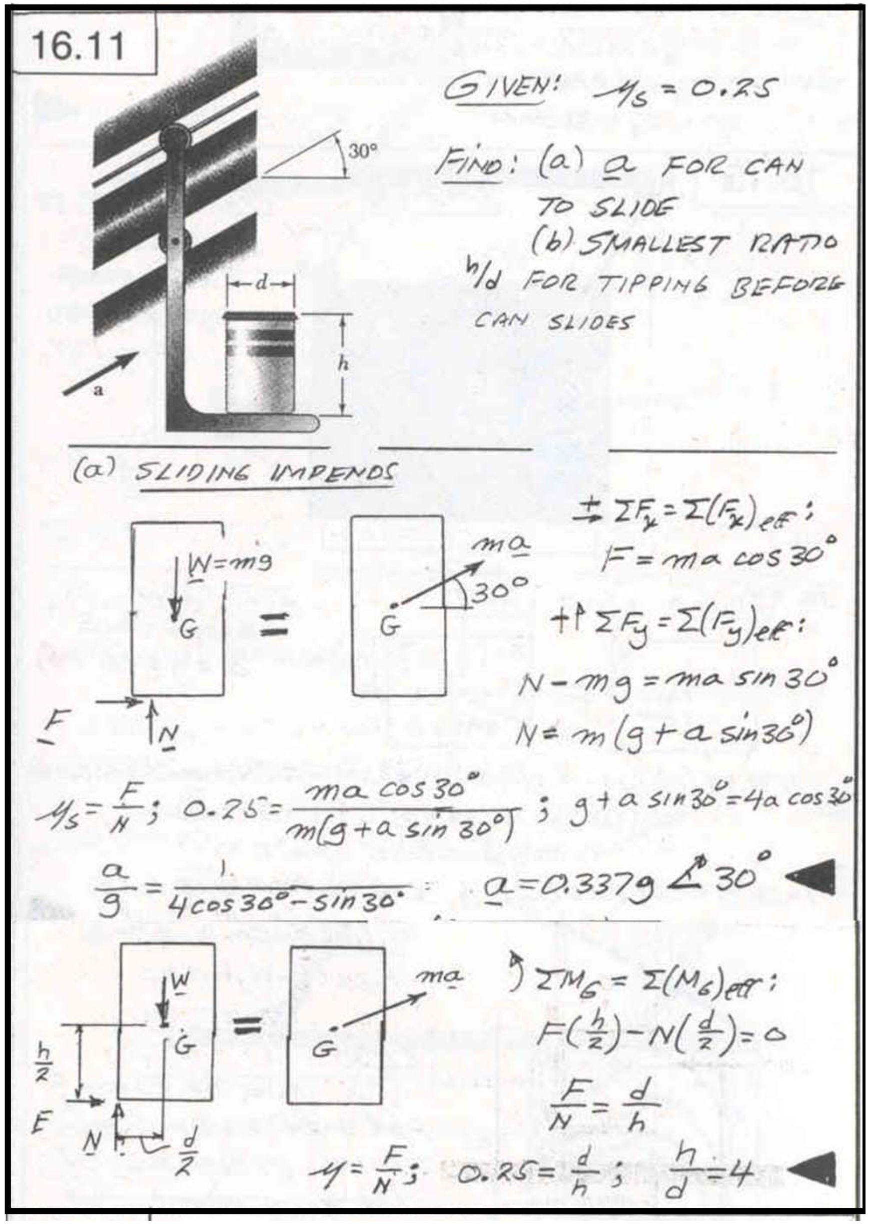
Problem 16.11



The support structure shown in the figure is used to move up cylindrical objects from one level to another. Knowing that the static friction coefficient is equal to 0.25 between the support and the object, determine:

a)The acceleration “a” that tends object to slip.

b)The smaller ratio between h/d that tends object to rotate down, before slipping.



16.11

GIVEN: $\mu_s = 0.25$

FIND: (a) a FOR CAN TO SLIDE

(b) SMALLEST RATIO h/d FOR TIPPING BEFORE CAN SLIDES

(a) SLIDING IMPENDS

$\pm \Sigma F_x = \Sigma (F_x)_{eff}$: $F = ma\cos 30°$

$+\uparrow \Sigma F_y = \Sigma (F_y)_{eff}$: $N - mg = ma\sin 30°$

$N = m(g + a\sin 30°)$

$\mu_s = \frac{F}{N}$; $0.25 = \frac{ma\cos 30°}{m(g + a\sin 30°)}$; $g + a\sin 30° = 4a\cos 30°$

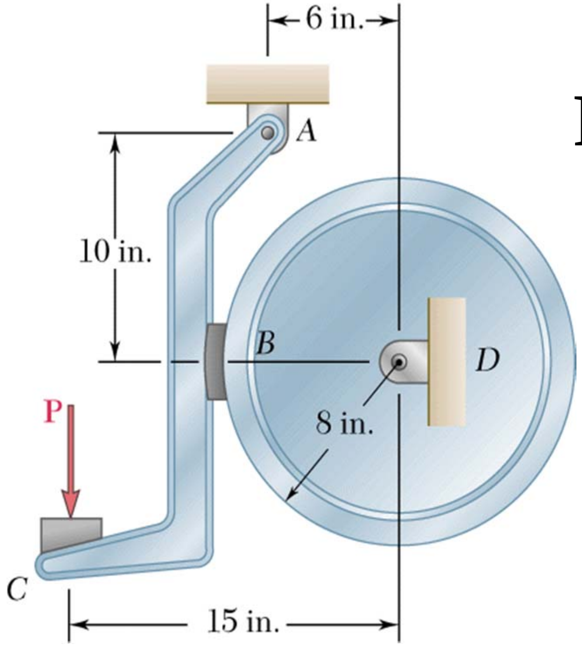
$\frac{a}{g} = \frac{1}{4\cos 30° - \sin 30°}$; $a = 0.337g \angle 30°$ ◀

$\Sigma M_G = \Sigma (M_G)_{eff}$:

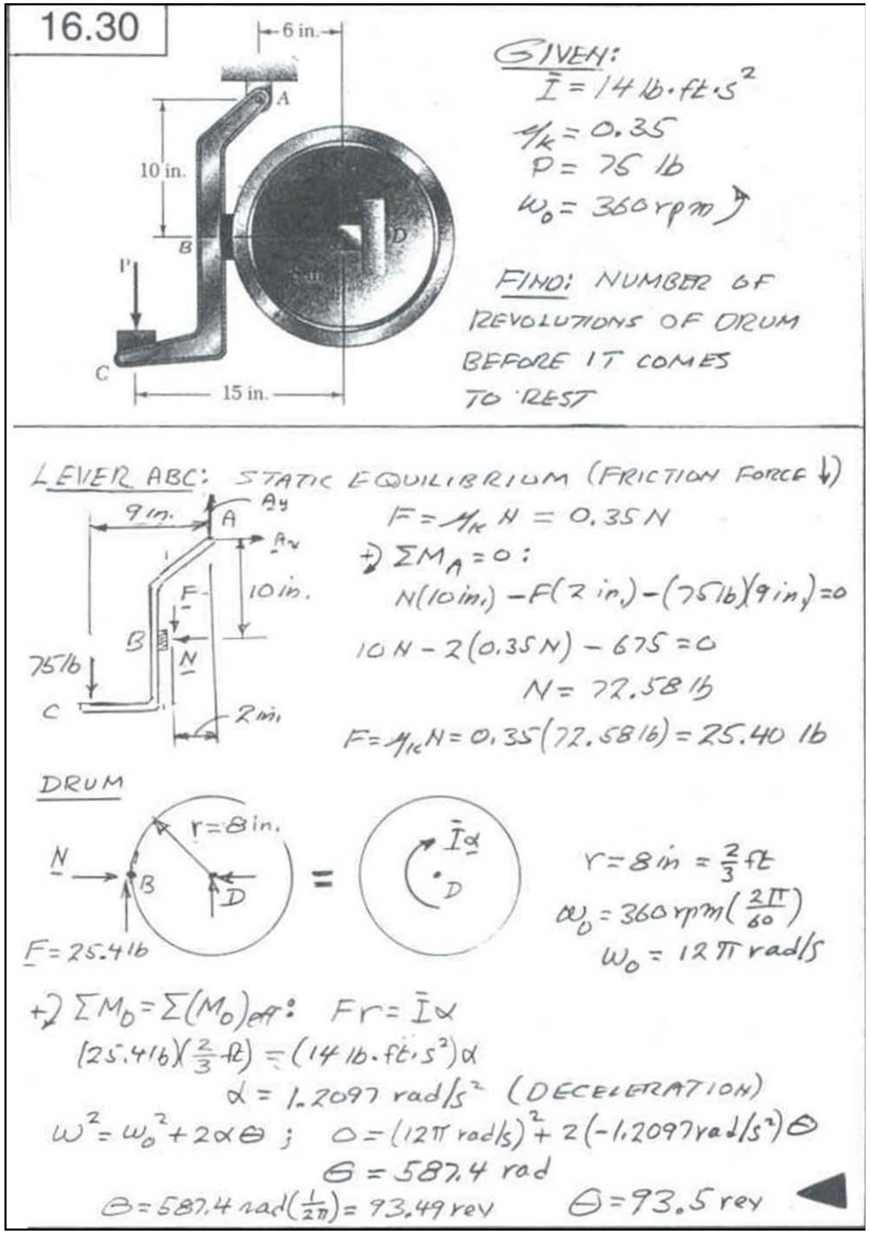
$F\left(\frac{h}{2}\right) - N\left(\frac{d}{2}\right) = 0$

$\frac{F}{N} = \frac{d}{h}$

$\mu = \frac{F}{N}$; $0.25 = \frac{d}{h}$; $\frac{h}{d} = 4$ ◀



Problem 16.30


16.30

GIVEN:
$\bar{I} = 14 \text{ lb}\cdot\text{ft}\cdot\text{s}^2$
$\mu_k = 0.35$
$P = 75 \text{ lb}$
$\omega_0 = 360 \text{ rpm}$

FIND: NUMBER OF REVOLUTIONS OF DRUM BEFORE IT COMES TO REST

LEVER ABC: STATIC EQUILIBRIUM (FRICTION FORCE ↓)

$F = \mu_k N = 0.35 N$

$\Sigma M_A = 0$:

$N(10 \text{ in.}) - F(2 \text{ in.}) - (75 \text{ lb})(9 \text{ in.}) = 0$

$10N - 2(0.35N) - 675 = 0$

$N = 72.58 \text{ lb}$

$F = \mu_k N = 0.35(72.58 \text{ lb}) = 25.40 \text{ lb}$

DRUM

$r = 8 \text{ in} = \frac{2}{3} \text{ ft}$

$\omega_0 = 360 \text{ rpm}\left(\frac{2\pi}{60}\right)$

$\omega_0 = 12\pi \text{ rad/s}$

$\Sigma M_D = \Sigma(M_D)_{eff}$: $Fr = \bar{I}\alpha$

$(25.4 \text{ lb})(\frac{2}{3} \text{ ft}) = (14 \text{ lb}\cdot\text{ft}\cdot\text{s}^2)\alpha$

$\alpha = 1.2097 \text{ rad/s}^2$ (DECELERATION)

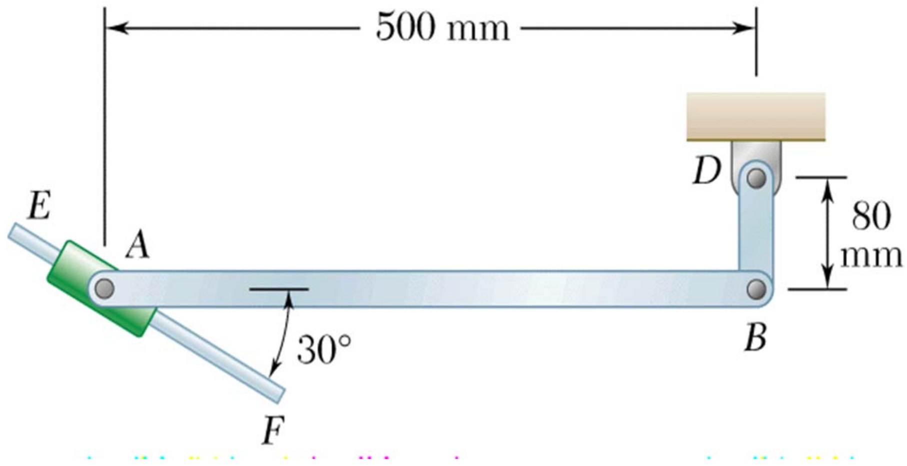
$\omega^2 = \omega_0^2 + 2\alpha\theta$; $0 = (12\pi \text{ rad/s})^2 + 2(-1.2097 \text{ rad/s}^2)\theta$

$\theta = 587.4 \text{ rad}$

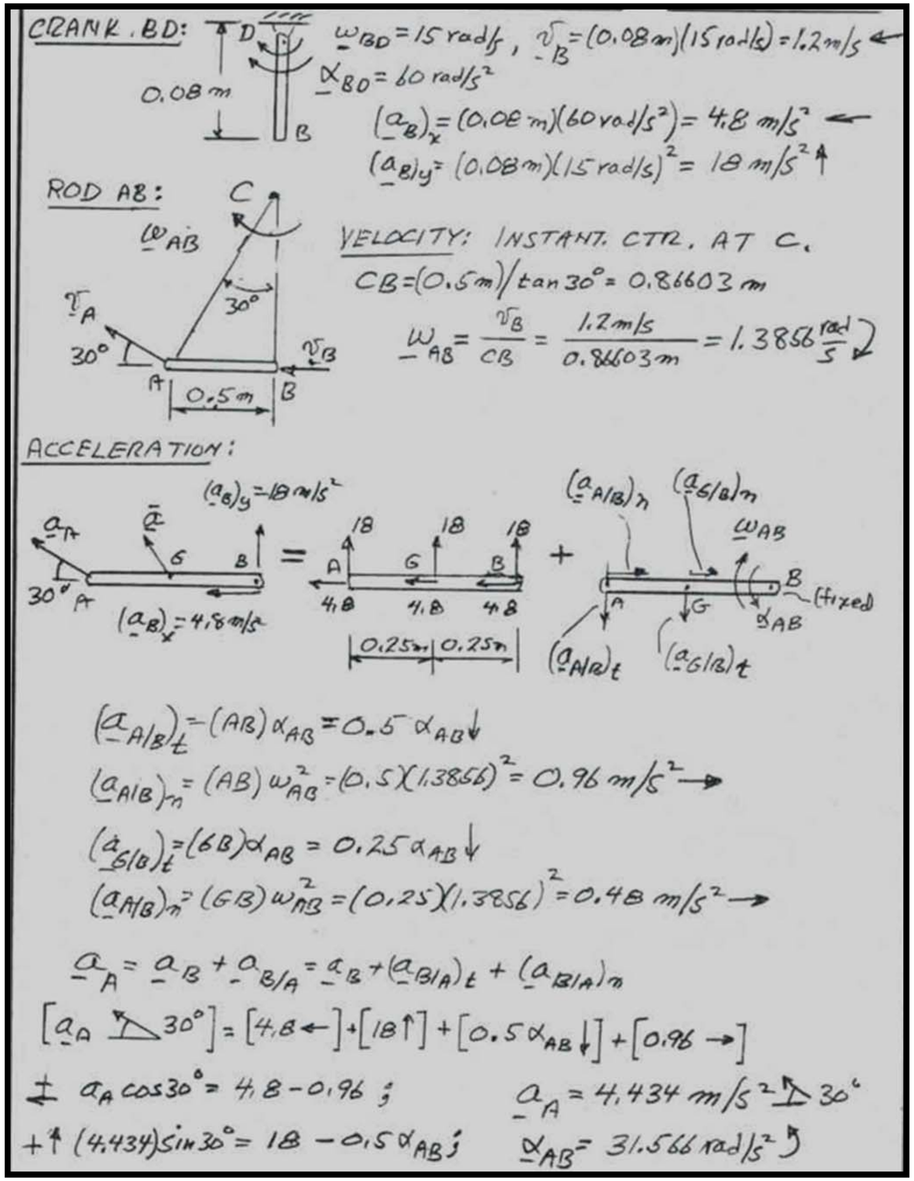
$\theta = 587.4 \text{ rad}\left(\frac{1}{2\pi}\right) = 93.49 \text{ rev}$ $\quad \theta = 93.5 \text{ rev}$

A disk with 203 [mm] radius is part of a breaking system which is connected to a flying wheel (not represented). The inertial moment of both components is 18.98 [kgm^2]. The movement is controlled by means of a brace, being the kinetic friction coefficient equal to 0.35. Knowing that initial angular velocity equals 360 [rpm] anticlockwise, when a 333.6 [N] force is applied under the pedal, determine the number of revolutions necessary to stop the disc.

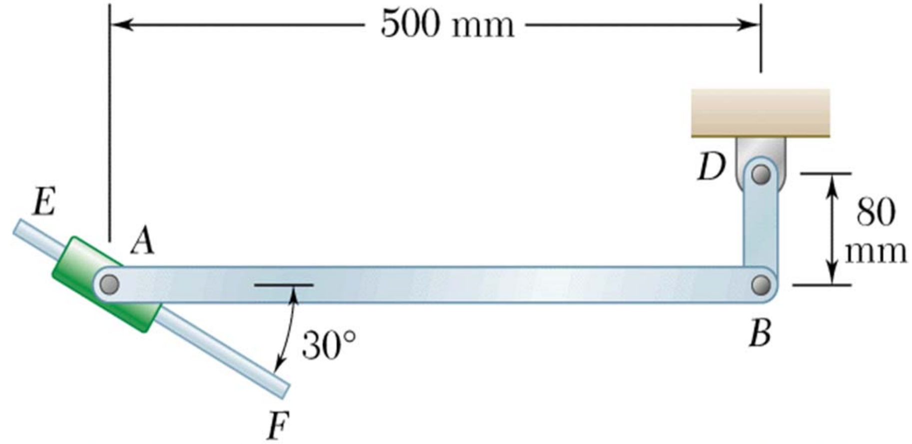
Problem 16.129



A uniform bar AB with 3[kg] mass is connected to the winch BD and to the mass less cursor, which may slip over EF. Knowing that for the represented position BD rotates with 15 (rad/s) of angular velocity and with 60 [rad/s²] of angular acceleration, both clockwise, determine the reaction at A.



Problem 16.129



$$\underline{a}_A = \underline{a}_B + \underline{a}_{B/A} = \underline{a}_B + (\underline{a}_{B/A})_t + (\underline{a}_{B/A})_n$$

$$[a_A \measuredangle 30°] = [4.8 \leftarrow] + [18 \uparrow] + [0.5\alpha_{AB} \downarrow] + [0.96 \rightarrow]$$

$$\overset{+}{\leftarrow} \; a_A \cos 30° = 4.8 - 0.96 \; ; \qquad \underline{a}_A = 4.434 \text{ m/s}^2 \measuredangle 30°$$

$$+\uparrow \; (4.434)\sin 30° = 18 - 0.5\alpha_{AB} \; ; \qquad \alpha_{AB} = 31.566 \text{ rad/s}^2 \circlearrowright$$

$$\bar{a} = a_B + a_{G/B} = a_B + (a_{G/B})_t + (a_{G/B})_n$$

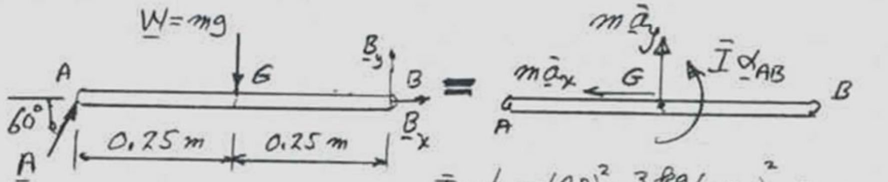
$$\bar{\underline{a}} = [4.8 \leftarrow] + [18 \uparrow] + [0.25(31.566) \downarrow] + [0.48 \rightarrow]$$

$$\overset{+}{\leftarrow} \; \bar{a}_x = 4.8 - 0.48 = 4.32 \; ; \qquad \bar{a}_x = 4.32 \text{ m/s}^2 \leftarrow$$

$$+\uparrow \; \bar{a}_y = 18 - 7.892 = 10.108 \; ; \qquad \bar{a}_y = 10.108 \text{ m/s}^2 \uparrow$$

KINETICS:

$$\bar{I} = \tfrac{1}{12} m (AB)^2 = \frac{3 \text{ kg}}{12}(0.5 \text{ m})^2 = 0.0625 \text{ kg}\cdot\text{m}^2$$



$$\bar{I} = \tfrac{1}{12} m (AB)^2 = \frac{3 \text{ kg}}{12}(0.5 \text{ m})^2 = 0.$$

$$+\circlearrowleft \Sigma M_B = \Sigma (M_B)_{eff} :$$

$$(A \sin 60°)(0.5 \text{ m}) - mg(0.25 \text{ m}) = -\bar{I}\alpha_{AB} + m\bar{a}_y (0.25 \text{ m})$$

$$0.433 A - (3 \text{ kg})(9.81 \text{ m/s}^2)(0.25 \text{ m}) = -(0.0625 \text{ kg}\cdot\text{m}^2)(31.566 \tfrac{\text{rad}}{\text{s}^2}) + (3 \text{ kg})(10.108 \text{ m/s}^2)(0.25 \text{ m})$$

$$0.433 A - 7.358 = -1.973 + 7.581$$

$$A = 29.94 \text{ N}$$

$$\underline{A} = 29.9 \text{ N} \measuredangle 60°$$

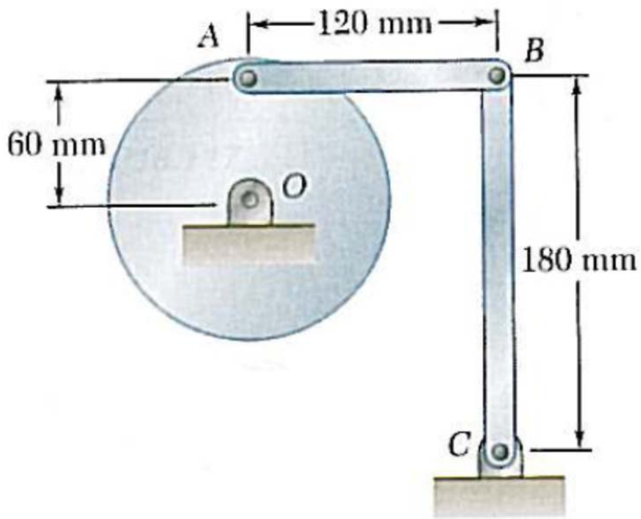
Test exercise - revisions

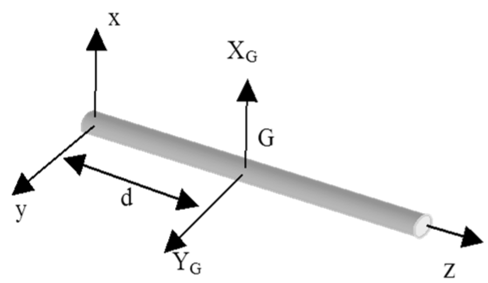
- Right figure represents a complex mechanism, built with two straight bars AB and BC, each with 2 and 3 [kg] mass. Bar AB is connect to a disc in vertical position. Disc is rotating clockwise at constant rate, with angular velocity of 6 [rad/s]. For the represented position, determine the dynamic reactions affecting bar AB.
- Formulae:
  – Tabulated data:

$$I_{Y_G} = I_{Xg} = \frac{1}{12} mL^2$$

  – Parallel axis theorem:

$$I = I_G + m.d^2$$

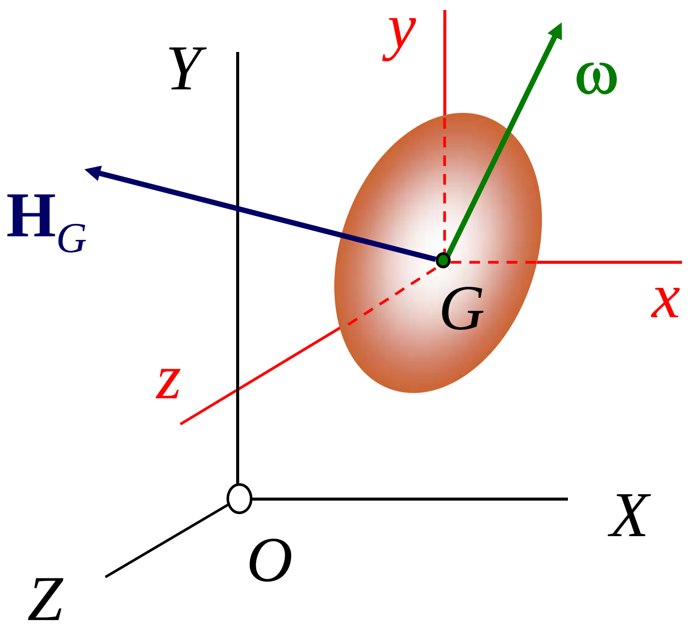




KINETICS OF RIGID BODIES IN THREE DIMENSIONS

The two fundamental equations for the motion of a system of particles

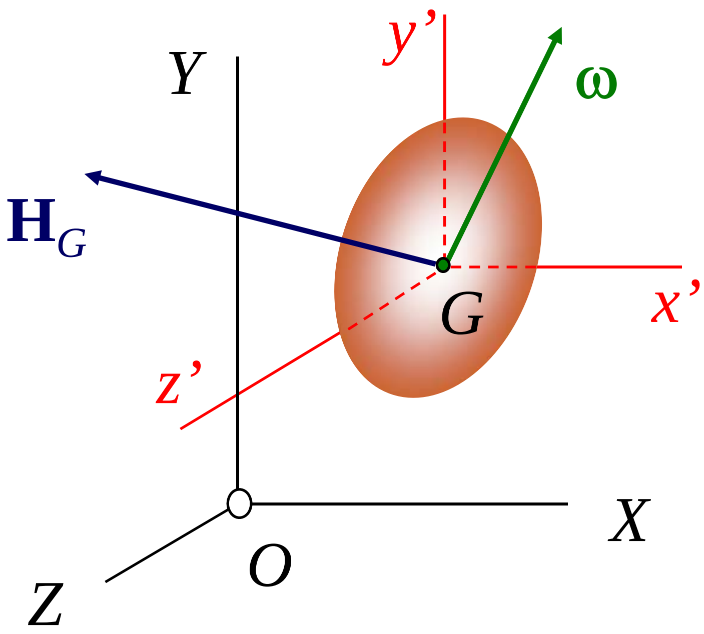
$$\Sigma \mathbf{F} = m\bar{\mathbf{a}} \qquad \Sigma \mathbf{M}_G = \dot{\mathbf{H}}_G$$



provide the foundation for three dimensional analysis, just as they do in the case of plane motion of rigid bodies. The computation of the angular momentum $\mathbf{H}_G$ and its derivative $\mathbf{k}_G$, however, are now considerably more involved.

KINETICS OF RIGID BODIES IN THREE DIMENSIONS (CONT.)

The rectangular components of the angular momentum $\mathbf{H}_G$ of a rigid body may be expressed in terms of the components of its angular velocity **w** and of its centroidal moments and products of inertia:



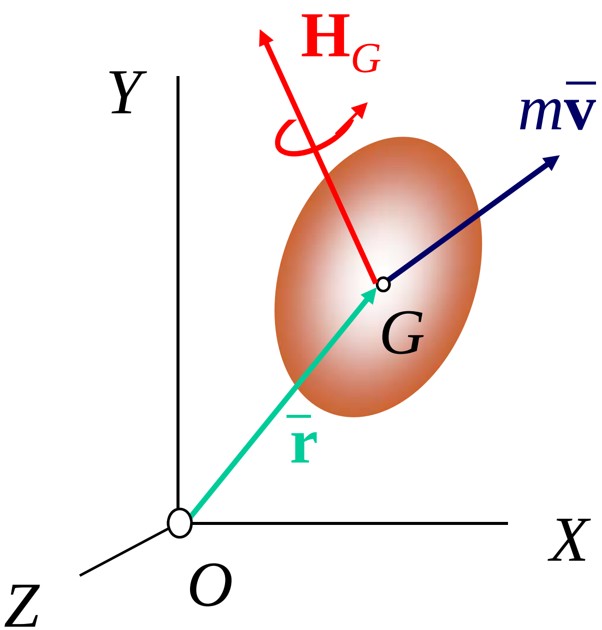
$$H_x = +\bar{I}_x\,\omega_x - \bar{I}_{xy}\omega_y - \bar{I}_{xz}\omega_z$$
$$H_y = -\bar{I}_{yx}\omega_x + \bar{I}_y\,\omega_y - \bar{I}_{yz}\omega_z$$
$$H_z = -\bar{I}_{zx}\omega_x - \bar{I}_{zy}\omega_y + \bar{I}_z\,\omega_z$$

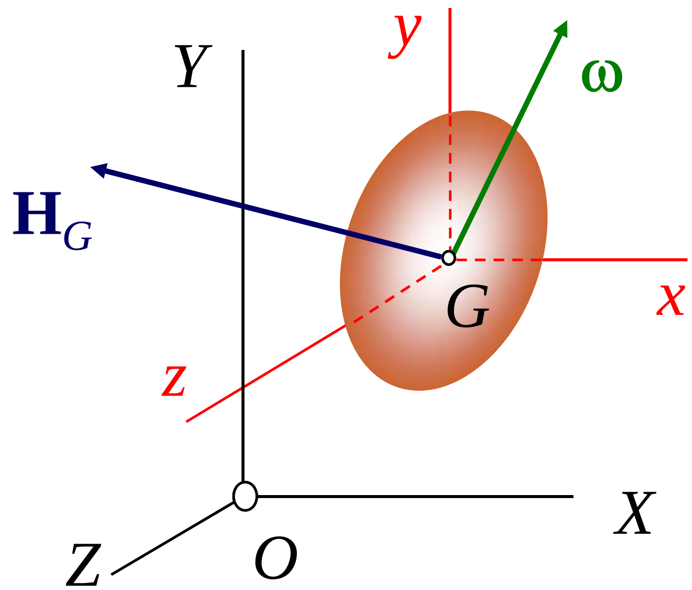
If principal axes of inertia *Gx'y'z'* are used, these relations reduce to

$$H_{x'} = \bar{I}_{x'}\omega_{x'}$$
$$H_{y'} = \bar{I}_{y'}\omega_{y'}$$
$$H_{z'} = \bar{I}_{z'}\omega_{z'}$$

KINETICS OF RIGID BODIES

In general, the ***angular momentum*** $\mathbf{H}_G$ and ***the angular velocity*** **w** *do not have the same direction*. They will, however, have the same direction if **w** is directed along one of the principal axes of inertia of the body.



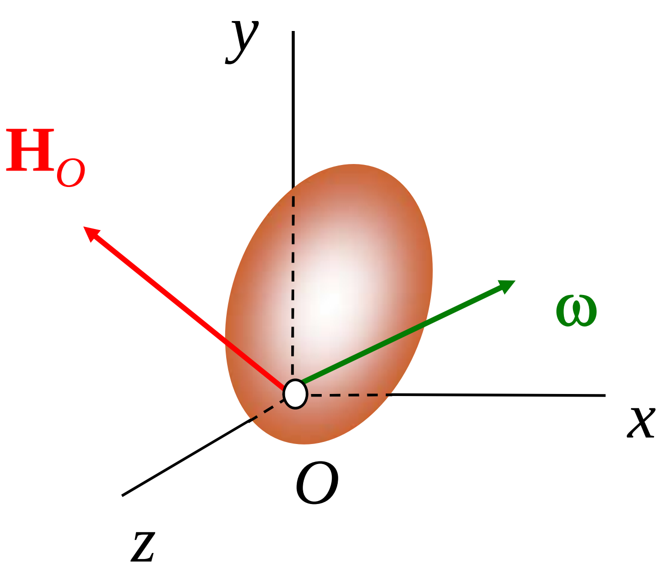


The system of the momenta of the particles forming a rigid body may be reduced to the vector $m\bar{\mathbf{v}}$ attached at G and the couple $\mathbf{H}_G$. Once these are determined, the angular momentum $\mathbf{H}_O$ of the body about any given point O may be obtained by writing

$$\mathbf{H}_O = \bar{\mathbf{r}} \times m\bar{\mathbf{v}} + \mathbf{H}_G$$

FIXED POINT ROTATION

In the particular case of a rigid body constrained to rotate about a fixed point *O*, the components of the angular momentum $\mathbf{H}_O$ of the body about *O* may be obtained directly from the components of its angular velocity and from its moments and products of inertia with respect to axes through *O*.



$$\mathbf{H}_O = \overline{\mathbf{r}} \times m\overline{\mathbf{v}} + \mathbf{H}_G$$
$$\mathbf{H}_O = [\mathrm{I}o].\mathbf{w}$$

since: $\vec{v}_G = \vec{w} \times \vec{r}$

$$I_o = I_G + Md^2$$

$$H_x = +I_x\,\omega_x - I_{xy}\omega_y - I_{xz}\omega_z$$
$$H_y = -I_{yx}\omega_x + I_y\,\omega_y - I_{yz}\omega_z$$
$$H_z = -I_{zx}\omega_x - I_{zy}\omega_y + I_z\,\omega_z$$

PRINCIPLE OF IMPULSE AND MOMENTUM

The ***principle of impulse and momentum*** for a rigid body in three- dimensional motion is expressed by the same fundamental formula used for a rigid body in plane motion.

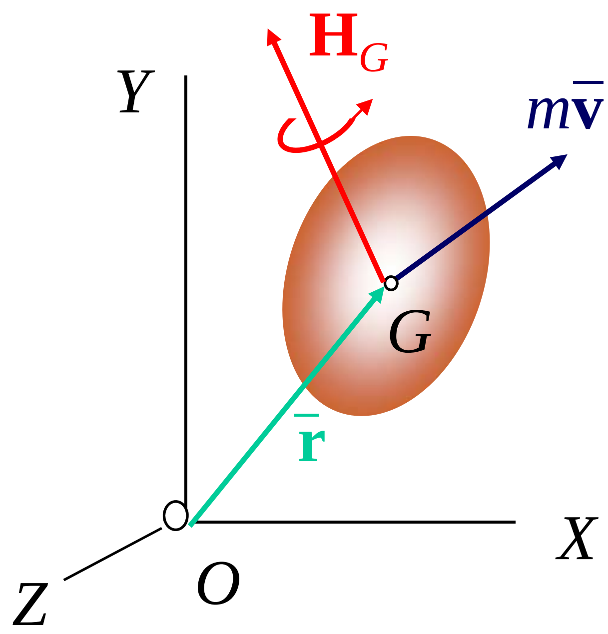
$$\textbf{Syst Momenta}_1 + \textbf{Syst Ext Imp}_{1\rightarrow 2} = \textbf{Syst Momenta}_2$$

The initial and final system momenta should be represented as shown in the figure and computed from

$$H_x = +\bar{I}_x \omega_x - \bar{I}_{xy}\omega_y - \bar{I}_{xz}\omega_z$$

$$H_y = -\bar{I}_{yx}\omega_x + \bar{I}_y \omega_y - \bar{I}_{yz}\omega_z$$

$$H_z = -\bar{I}_{zx}\omega_x - \bar{I}_{zy}\omega_y + \bar{I}_z \omega_z$$



Or, in the case of principal axes of inertia:

$$H_{x'} = \bar{I}_{x'}\omega_{x'} \qquad H_{y'} = \bar{I}_{y'}\omega_{y'} \qquad H_{z'} = \bar{I}_{z'}\omega_{z'}$$

KINETIC ENERGY OF A RIGID BODY

The kinetic energy of a rigid body in three-dimensional motion may be divided into two parts, one associated with the motion of its mass center *G*, and the other with its motion about *G*. Using principal axes *x'*, *y'*, *z'*, we write

$$T = \frac{1}{2} m\bar{v}^{\,2} + \frac{1}{2}\left(\bar{I}_{x'}\omega_{x'}^2 + \bar{I}_{y'}\omega_{y'}^2 + \bar{I}_{z'}\omega_{z'}^2\right)$$

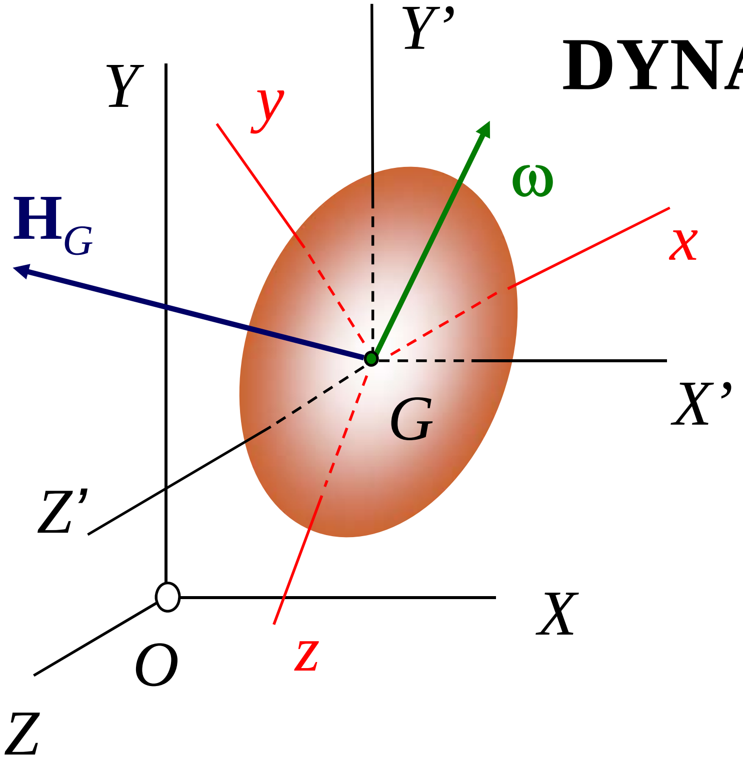
where $\bar{\mathbf{v}}$ = velocity of the mass center
$\mathbf{w}$ = angular velocity
m = mass of rigid body
$\bar{I}_{x'}, \bar{I}_{y'}, \bar{I}_{z'}$ = principal centroidal moments of inertia.

In the case of a rigid body ***constrained to rotate about a fixed point O***, the Kinetic energy may be expressed as

$$T = \frac{1}{2}\left(I_{x'}\omega_{x'}^2 + I_{y'}\omega_{y'}^2 + I_{z'}\omega_{z'}^2\right)$$

The equations for kinetic energy make it possible to extend to the three-dimensional motion of a rigid body the application of the *principle of work and energy* and of the *principle of conservation of energy*.

DYNAMIC ANALYSIS



$$\Sigma \mathbf{F} = m\bar{\mathbf{a}} \qquad \Sigma \mathbf{M}_G = \dot{\mathbf{H}}_G$$

The fundamental equations can be applied to the motion of a rigid body in three dimensions. We first recall that $\mathbf{H}_G$ represents the angular momentum of the body relative to a centroidal frame *GX'Y'Z'* of fixed orientation and that $\dot{\mathbf{H}}_G$ represents the rate of change of $\mathbf{H}_G$ with respect to that frame. As the body rotates, its moments and products of inertia with respect to *GX'Y'Z'* change continually. It is therefore more convenient to use a frame *Gxyz* rotating with the body to resolve **w** into components and to compute the moments and products of inertia which are used to determine $\mathbf{H}_G$.

$\dot{\mathbf{H}}_G$ represents the rate of change of $\mathbf{H}_G$ with respect to the frame *GX'Y'Z'* of fixed orientation, and $\boldsymbol{\Omega}$ equals the angular velocity of the rotating frame *Gxyz*.

$$\dot{\mathbf{H}}_G = (\dot{\mathbf{H}}_G)_{Gxyz} + \boldsymbol{\Omega} \times \mathbf{H}_G$$

EULER'S EQUATIONS OF MOTON

If the rotating frame is attached to the body, its angular velocity $\mathbf{\Omega}$ is identical to the angular velocity $\boldsymbol{\omega}$ of the body.

Setting $\mathbf{\Omega} = \boldsymbol{\omega}$, using principal axes, and writing this equation in scalar form, we obtain ***Euler's equations of motion***.

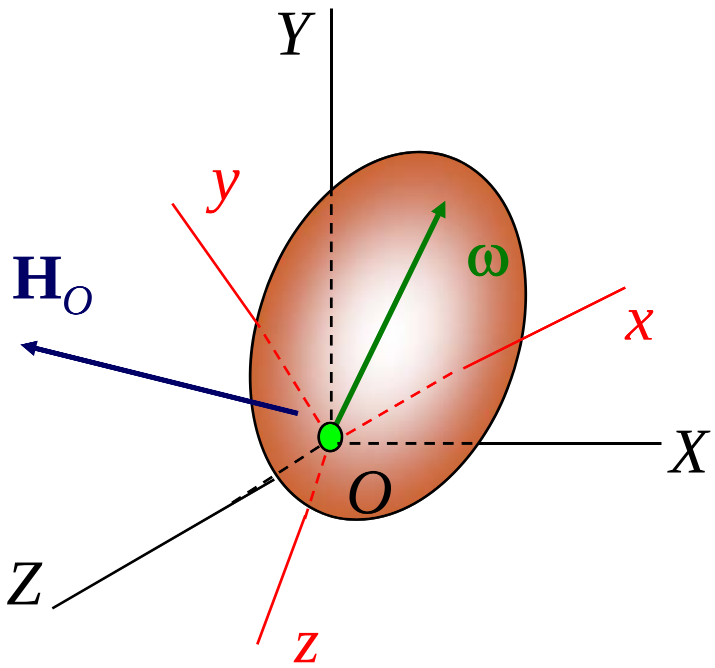
$$\Sigma \mathbf{F} = m\bar{\mathbf{a}} \qquad \Sigma \mathbf{M}_G = \dot{\mathbf{H}}_G$$

$$\dot{\mathbf{H}}_G = (\dot{\mathbf{H}}_G)_{Gxyz} + \mathbf{\Omega} \times \mathbf{H}_G$$

Substituting $\dot{\mathbf{H}}_G$ above into $\Sigma\, \mathbf{M}_G$,

$$\Sigma \mathbf{M}_G = (\dot{\mathbf{H}}_G)_{Gxyz} + \mathbf{\Omega} \times \mathbf{H}_G$$

FIXED POINT ROTATION MOTION



In the case of a rigid body *constrained to rotate about a fixed point O*, an alternative method of solution may be used, involving moments of the forces and the rate of change of the angular momentum about point *O*.

$$\Sigma \mathbf{M}_O = (\dot{\mathbf{H}}_O)_{Oxyz} + \mathbf{\Omega} \times \mathbf{H}_O$$

Where: $\Sigma \mathbf{M}_O$ = sum of the moments about *O* of the forces applied to the rigid body

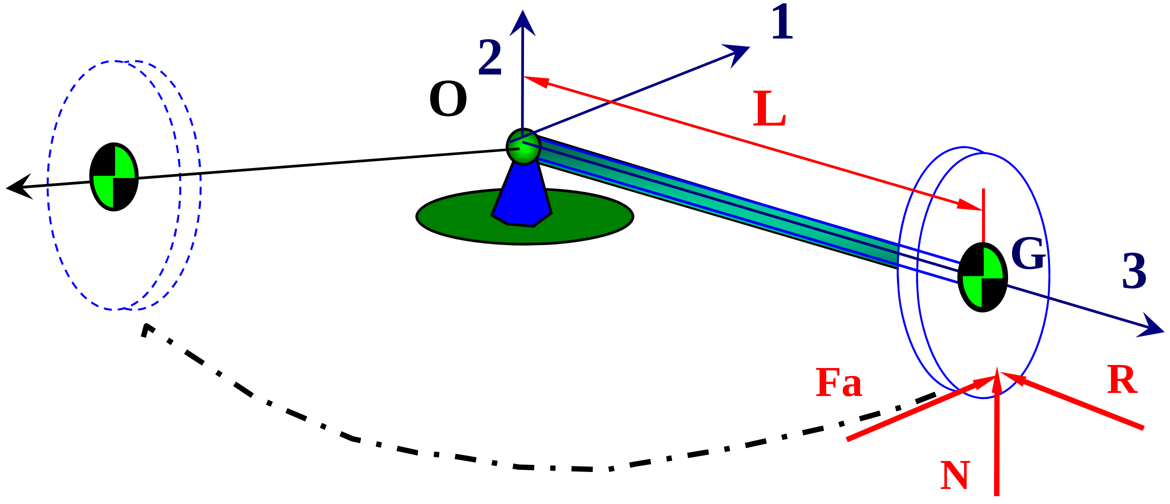
$\mathbf{H}_O$ = angular momentum of the body with respect to the frame *OXYZ*

$(\dot{\mathbf{H}}_O)_{Oxyz}$ = rate of change of $\mathbf{H}_O$ with respect to the rotating frame *Oxyz*

$\mathbf{\Omega}$ = angular velocity of the rotating frame *Oxyz*

Thematic exercise 14

A disc of radius “r” and mass “M” rotates without slipping in the plane ground. The disc axis OG rotates in a socket ball fixed rotational joint, point O, with a constant linear velocity “V” at point “G”, being always in vertical position.
Calculate the contact reactions.



1. Kinematics: Express the acceleration of the body mass center, and the angular acceleration.

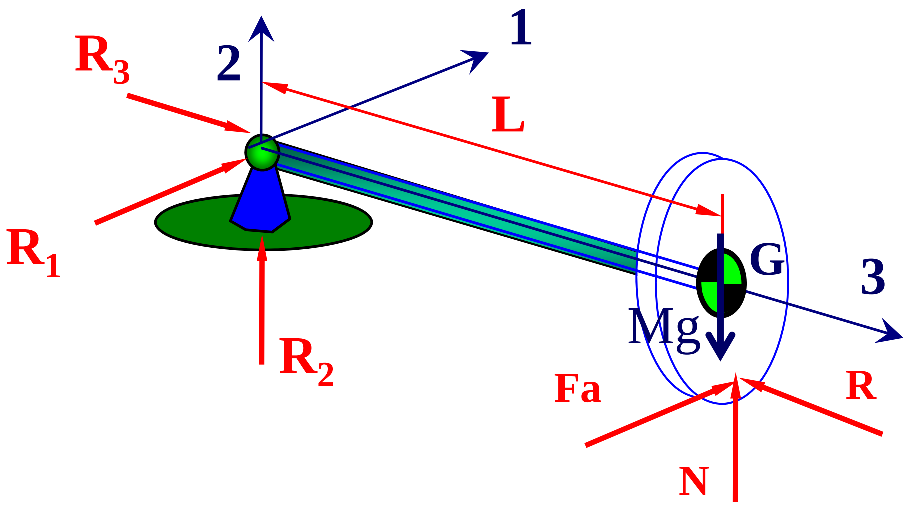
$$\vec{a}_G = \vec{a}_O + \dot{\vec{W}}_{arm} \times \vec{OG} + \vec{W}_{arm} \times (\vec{W}_{arm} \times \vec{OG})$$

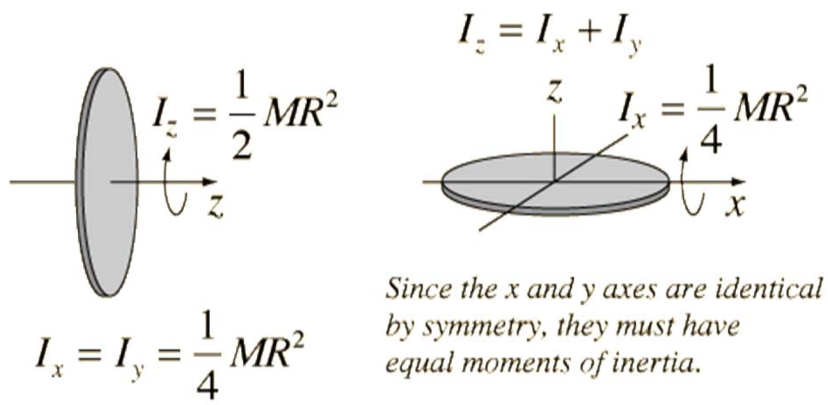
2. Kinetics: Draw a free body diagram showing the applied forces and an equivalent force diagram showing the vector $m\bar{\mathbf{a}}$ or its components and the couple $\bar{I}\alpha$.

3. Mass properties: Recall geometric mass properties for a disc.

THEMATIC PROBLEM

2. Kinetics: Draw a free body diagram showing the applied forces and an equivalent force diagram showing the vector $m\bar{\mathbf{a}}$ or its components and the couple $\bar{I}\alpha$.





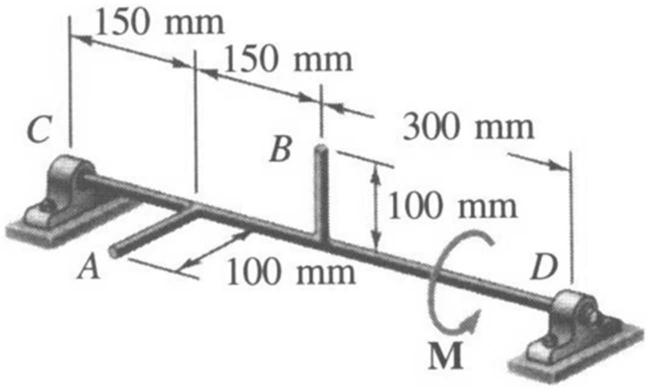
$$\sum \vec{F} = m \times \vec{a}_G \quad \Leftrightarrow \begin{cases} R1 + Fa = 0 \\ -Mg + N + R2 = 0 \\ R3 = -M\dot{\theta}^2 L \end{cases}$$

$$\sum \vec{M}_G = \dot{\vec{H}}_G \quad \Leftrightarrow \begin{cases} L.R2 = -\frac{1}{4} Mr^2 \dot{\beta}\dot{\theta} \\ -L.R1 = 0 \\ r.Fa = 0 \end{cases}$$

Solution: *R1=0; Fa=0; R3=-Mv²/L; R2=-1/4Mv²r/L²; N=Mg+1/2Mv²r/L²*

Thematic exercise 15

Un unbalance axis may be approximated by the sketch represented in the figure. Knowing that the mass of each bar is equal to 0.3 [kg] and the principal rod rotates at a instant speed of 1200 [r.p.m], when a couple M of 6 [Nm] is applied, determine the dynamic reactions at the supports, neglecting the inertia of the principal rod CD.



Equations to be solved:

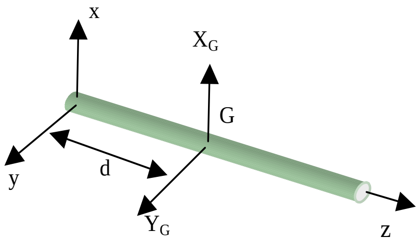
$$\sum \vec{F} = m\, \vec{a}_G$$

$$\sum \vec{M}_C = \dot{\vec{H}}_C$$

Angular momentum H_C:

$$I = I_G + m.d^2$$

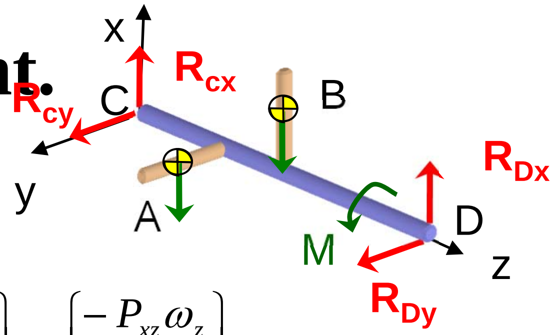
$$I_{y_G} = \frac{1}{12} mL^2$$



Mass centre:

$$\bar{x} = \frac{\sum_{i=1}^{2} x_i M_i}{\sum_{i=1}^{2} M_i} = 25\,[mm] \quad \bar{y} = \frac{\sum_{i=1}^{2} y_i M_i}{\sum_{i=1}^{2} M_i} = 25\,[mm] \quad \bar{z} = \frac{\sum_{i=1}^{2} y_i M_i}{\sum_{i=1}^{2} M_i} = 225\,[mm]$$

TEST EXERCISE – cont.



Angular momentum H_C:

$$\vec{H}_C = I_C\vec{\omega} = \begin{bmatrix} I_x & -P_{xy} & -P_{xz} \\ -P_{yx} & I_y & -P_{yz} \\ -P_{zx} & -P_{zy} & I_z \end{bmatrix} \bullet \begin{Bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{Bmatrix} = \begin{bmatrix} I_x & -P_{xy} & -P_{xz} \\ -P_{yx} & I_y & -P_{yz} \\ -P_{zx} & -P_{zy} & I_z \end{bmatrix} \bullet \begin{Bmatrix} 0 \\ 0 \\ \omega_z \end{Bmatrix} = \begin{Bmatrix} -P_{xz}\omega_z \\ -P_{yz}\omega_z \\ I_z\omega_z \end{Bmatrix}$$

$$I_{zz} = \sum_{i=1}^{2} I_{Gz} + m_i * z_i^2 = 2\left(\frac{1}{12} * 0.3 * 0.1^2 + 0.3 * 0.05^2\right) = 0.002\ kgm^2$$

$$P_{xz} = \sum_{i=1}^{2} m_i * x_i * z_i = 0.3 * 0 * 0.15 + 0.3 * 0.05 * 0.3 = 0.0045\ kgm^2$$

$$P_{yz} = \sum_{i=1}^{2} m_i * y_i * z_i = 0.3 * 0.05 * 0.15 + 0.3 * 0 * 0.3 = 0.00225\ kgm^2$$

Time derivative of H_C:

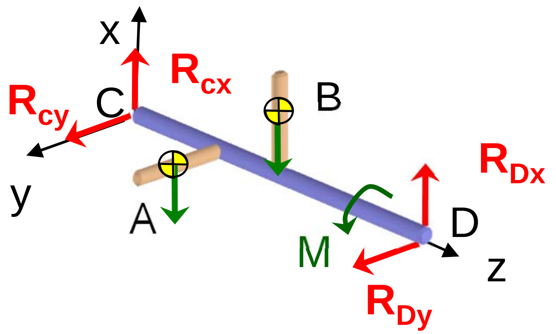
$$\dot{\vec{H}}_C = \begin{Bmatrix} -P_{xz}\dot{\omega}_z \\ -P_{yz}\dot{\omega}_z \\ I_z\dot{\omega}_z \end{Bmatrix} + \vec{\Omega} \times \begin{Bmatrix} -P_{xz}\omega_z \\ -P_{yz}\omega_z \\ I_z\omega_z \end{Bmatrix} = \begin{Bmatrix} -P_{xz}\dot{\omega}_z \\ -P_{yz}\dot{\omega}_z \\ I_z\dot{\omega}_z \end{Bmatrix} + \begin{Bmatrix} 0 \\ 0 \\ \omega_z \end{Bmatrix} \times \begin{Bmatrix} -P_{xz}\omega_z \\ -P_{yz}\omega_z \\ I_z\omega_z \end{Bmatrix} = \begin{Bmatrix} -P_{xz}\dot{\omega}_z + P_{yz}\omega_z^{\ 2} \\ -P_{yz}\dot{\omega}_z - P_{xz}\omega_z^{\ 2} \\ I_z\dot{\omega}_z \end{Bmatrix}$$

TEST EXERCISE – cont.

$$\sum \vec{M}_C = \dot{\vec{H}}_C \Leftrightarrow \begin{Bmatrix} 0 \\ 0.05 \\ 0.15 \end{Bmatrix} \times \begin{Bmatrix} -mg \\ 0 \\ 0 \end{Bmatrix} + \begin{Bmatrix} 0.05 \\ 0 \\ 0.30 \end{Bmatrix} \times \begin{Bmatrix} -mg \\ 0 \\ 0 \end{Bmatrix} + \begin{Bmatrix} 0 \\ 0 \\ 0.6 \end{Bmatrix} \times \begin{Bmatrix} R_{Dx} \\ R_{Dy} \\ 0 \end{Bmatrix} + \begin{Bmatrix} 0 \\ 0 \\ M \end{Bmatrix} = \begin{Bmatrix} -P_{xz}\alpha_z + P_{yz}\omega_z^2 \\ -P_{yz}\alpha_z + P_{xz}\omega_z^2 \\ I_z\alpha_z \end{Bmatrix}$$

$$\begin{Bmatrix} -0.6R_{Dy} \\ -0.15mg - 0.3mg + 0.6R_{Dx} \\ 0.05mg + 6 \end{Bmatrix} = \begin{Bmatrix} -0.0045\alpha_z + 0.00225 * (125.66)^2 \\ -0.00225\alpha_z + 0.0045 * (125.66)^2 \\ 0.002\alpha_z \end{Bmatrix} \Leftrightarrow \begin{cases} R_{Dx} = 109.15\,[N] \\ R_{Dy} = -36.1\,[N] \\ \alpha_z = 3075\,[rad/s^2] \end{cases}$$

Kinematics of mass centre:



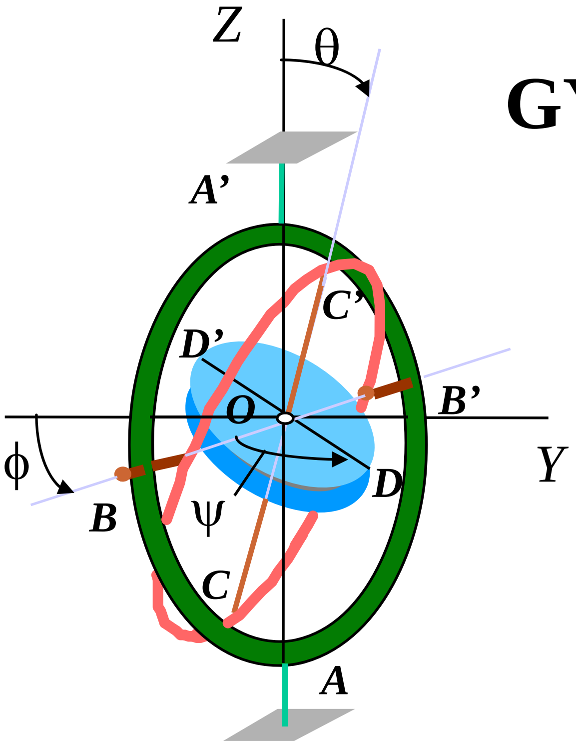
$$\vec{a}_G = \vec{a}_C + \vec{\alpha} \times \vec{CG} + \vec{w} \times \left(\vec{w} \times \vec{CG}\right) =$$

$$\begin{Bmatrix} a_{Gx} \\ a_{Gy} \\ 0 \end{Bmatrix} = \vec{0} + \begin{Bmatrix} 0 \\ 0 \\ \alpha_z \end{Bmatrix} \times \begin{Bmatrix} 0.025 \\ 0.025 \\ 0.225 \end{Bmatrix} + \begin{Bmatrix} 0 \\ 0 \\ 125.66 \end{Bmatrix} \times \left[\begin{Bmatrix} 0 \\ 0 \\ 125.66 \end{Bmatrix} \times \begin{Bmatrix} 0.025 \\ 0.025 \\ 0.225 \end{Bmatrix} \right]$$

Kinetics of mass centre:

After mass centre determination, acceleration calculation and establishing force equilibrium:

$$\sum \vec{F} = m\,\vec{a}_G \Leftrightarrow \begin{Bmatrix} R_{Cx} \\ R_{Cy} \\ 0 \end{Bmatrix} + \begin{Bmatrix} R_{Dx} \\ R_{Dy} \\ 0 \end{Bmatrix} - \begin{Bmatrix} 2mg \\ 0 \\ 0 \end{Bmatrix} = 2m \begin{Bmatrix} a_{Gx} \\ a_{Gy} \\ 0 \end{Bmatrix} \Leftrightarrow \begin{cases} R_{Cx} = -384.5\,[N] \\ R_{Cy} = -155.3\,[N] \\ 0 = 0 \end{cases}$$



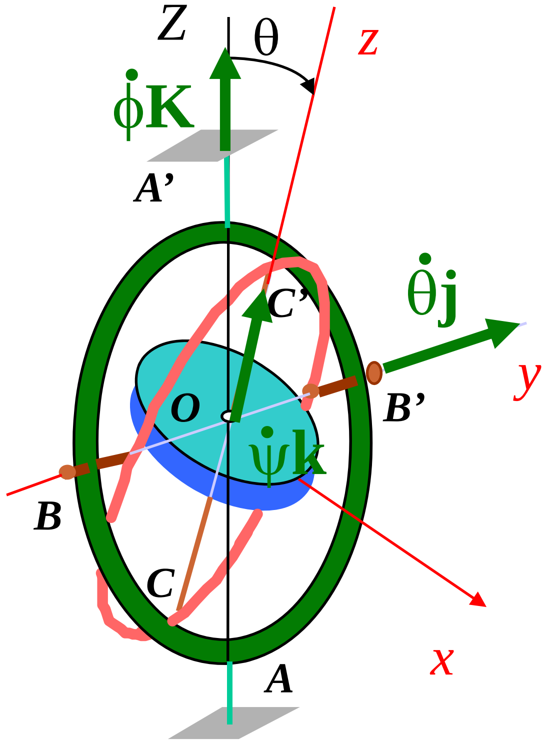
GYROSCOPE'S MOTION

When the motion of *gyroscopes* and other *axisymmetrical bodies* are considered, the *Eulerian angles* ϕ, θ, and ψ are introduced to define the position of a gyroscope. The time derivatives of these angles represent, respectively, the rates of *precession*, *nutation*, and *spin* of the gyroscope. The angular velocity **ω** is expressed in terms of these derivatives as

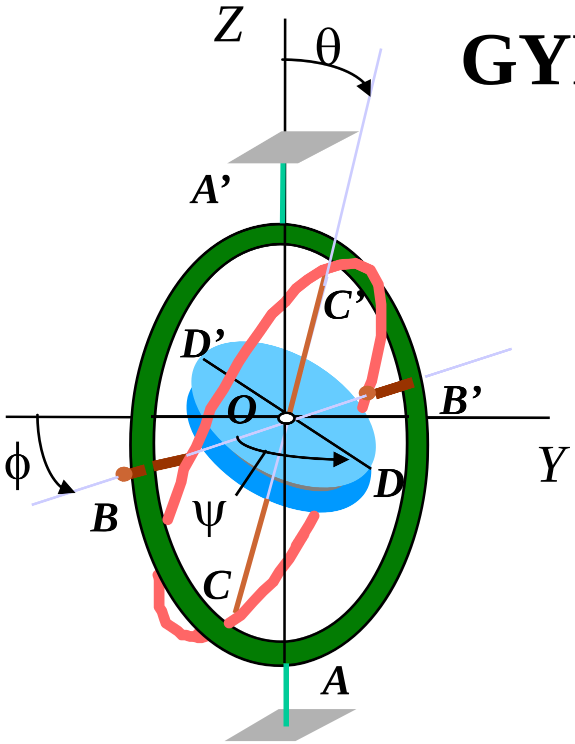
$$\boldsymbol{\omega} = -\dot{\phi}\sin\theta\,\mathbf{i} + \dot{\theta}\,\mathbf{j} + (\dot{\psi} + \dot{\phi}\cos\theta)\mathbf{k}$$

The unit vectors are associated with the frame *Oxyz* attached to the inner gimbal of the gyroscope (figure to the right) and rotate, therefore, with the angular velocity

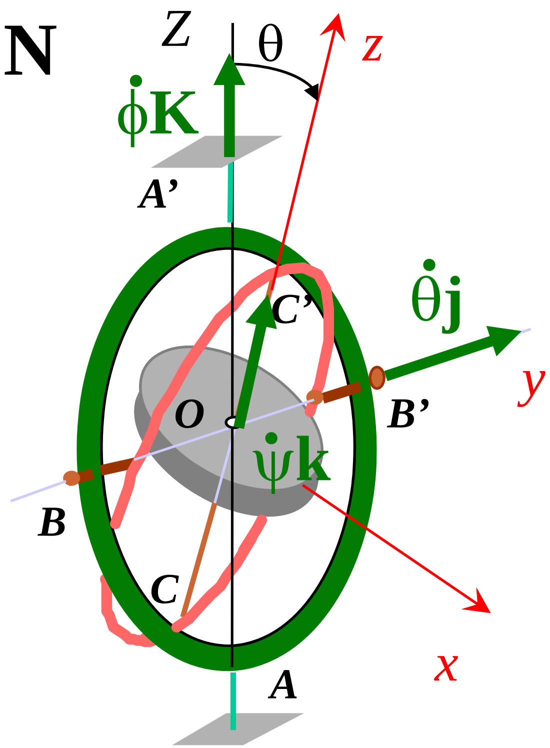
$$\boldsymbol{\Omega} = -\dot{\phi}\sin\theta\,\mathbf{i} + \dot{\theta}\,\mathbf{j} + \dot{\phi}\cos\theta\,\mathbf{k}$$



GYROSCOPE'S MOTION



Denoting by I the moment of inertia of the gyroscope with respect to its spin axis z and by I' its moment of inertia with respect to a transverse axis through O, we write:



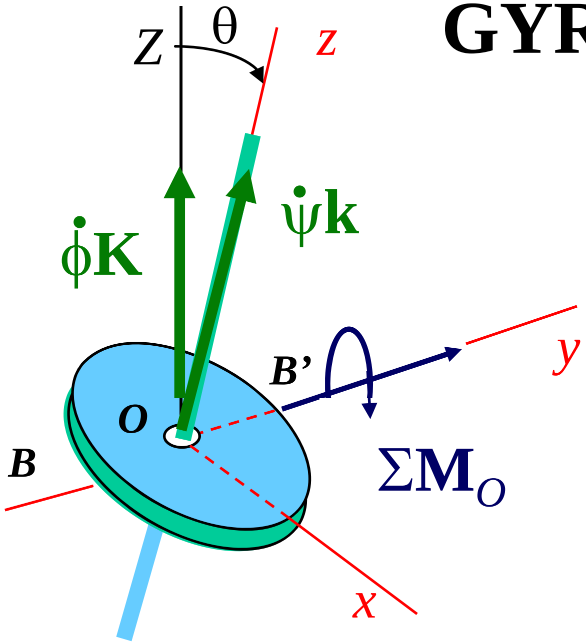
$$\mathbf{H}_O = -I'\dot{\phi}\sin\theta\,\mathbf{i} + I'\dot{\theta}\,\mathbf{j} + I(\dot{\psi} + \dot{\phi}\cos\theta)\mathbf{k}$$

Substituting for $\mathbf{H}_O$ and $\mathbf{\Omega} = -\dot{\phi}\sin\theta\,\mathbf{i} + \dot{\theta}\,\mathbf{j} + \dot{\phi}\cos\theta\,\mathbf{k}$ into

$$\Sigma\mathbf{M}_O = (\dot{\mathbf{H}}_O)_{Oxyz} + \mathbf{\Omega} \times \mathbf{H}_O$$

leads to the differential equations defining the motion of the gyroscope.

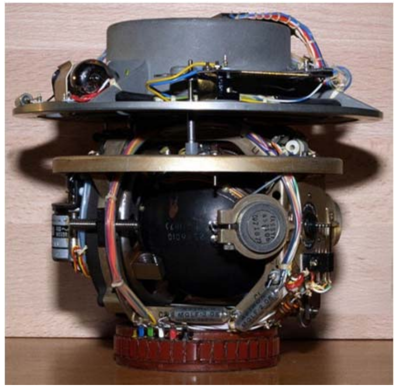
GYROSCOPE'S MOTION



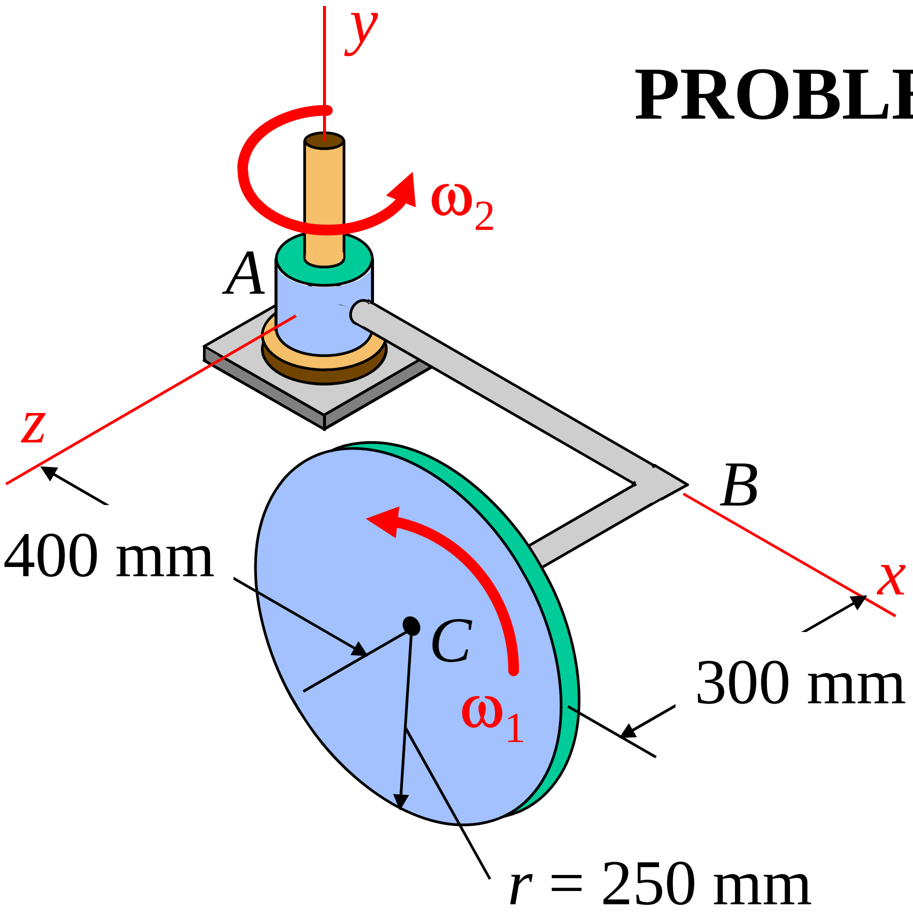
In the particular case of the *steady precession* of a gyroscope, the angle θ, the rate of precession $\dot{\phi}$, and the rate of spin $\dot{\psi}$ remain constant. Such motion is possible only if the moments of the external forces about *O* satisfy the relation

$$\Sigma \mathbf{M}_O = (I\omega_z - I'\dot{\phi}\cos\theta)\dot{\phi}\sin\theta\,\mathbf{j}$$

i.e., if the external forces reduce to a couple of moment equal to the right-hand member of the equation above and applied *about an axis perpendicular to the precession axis and to the spin axis.*



PROBLEM 18.147



A homogeneous disk of mass m = 5 kg rotates at a constant rate ω_1 = 8 rad/s with respect to the bent axle *ABC*, which itself rotates at the constant rate ω_2 = 3 rad/s about the y axis.

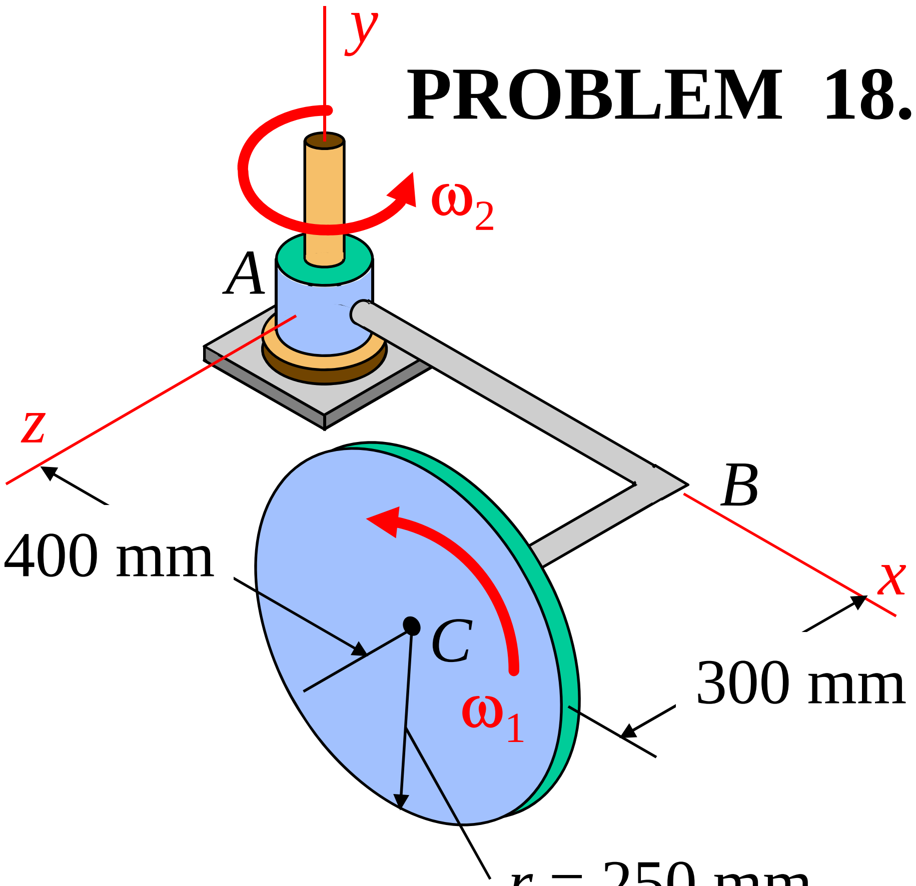
Determine the angular momentum $\mathbf{H}_C$ of the disk about its center *C*.

1. Determine the angular velocity w of the body: **w** is the angular velocity of the body with respect to a fixed frame of reference. The vector **w** may be resolved into components along the rotating axes. The angular velocity is often obtained by adding two components of angular velocities $\mathbf{w}_1$ and $\mathbf{w}_2$.

$$\omega = \omega_1 + \omega_2 \quad ; \qquad \omega_1 = 8\,\mathbf{k} \text{ rad/s}; \qquad \omega_2 = 3\,\mathbf{j} \text{ rad/s}$$

$$\omega = 3\,\mathbf{j} + 8\mathbf{k} \text{ rad/s}$$

PROBLEM 18.147 - SOLUTION



2. Determine the angular momentum of the body: If the principal axes of inertia x', y', z' of the body at G (mass center) are known, the components of the angular momentum $\mathbf{H}_G$ are given by:

$$(H_G)_{x'} = \bar{I}_{x'} \, \omega_{x'}$$
$$(H_G)_{y'} = \bar{I}_{y'} \, \omega_{y'}$$
$$(H_G)_{z'} = \bar{I}_{z'} \, \omega_{z'}$$

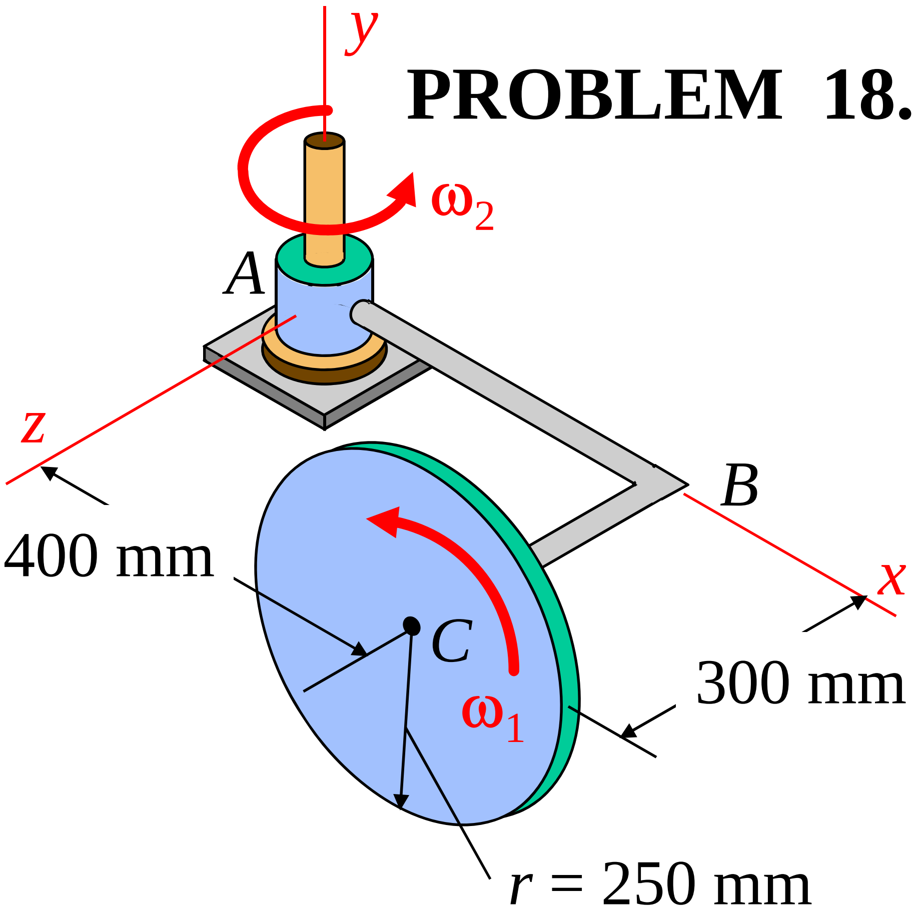
where $\bar{I}_{x'}$, $\bar{I}_{y'}$, and $\bar{I}_{z'}$, are the principal moments of inertia, and $\omega_{x'}$, $\omega_{y'}$, and $\omega_{z'}$, are the components of the angular velocity of the body.

Angular momentum about C :

$$(H_C)_{x'} = \bar{I}_{x'} \, \omega_{x'} \qquad (H_C)_{y'} = \bar{I}_{y'} \, \omega_{y'} \qquad (H_C)_{z'} = \bar{I}_{z'} \, \omega_{z'}$$

$$\bar{I}_{x'} = \bar{I}_{y'} = \frac{1}{4} \, m \, r^2 , \quad \bar{I}_{z'} = \frac{1}{2} \, m \, r^2$$

PROBLEM 18.147 - SOLUTION



$$\boldsymbol{\omega} = 3\,\mathbf{j} + 8\mathbf{k} \ \text{ rad/s}$$

$$\bar{I}_{x'} = \bar{I}_{y'} = \frac{1}{4}\, m\, r^2$$

$$\bar{I}_{z'} = \frac{1}{2}\, m\, r^2$$

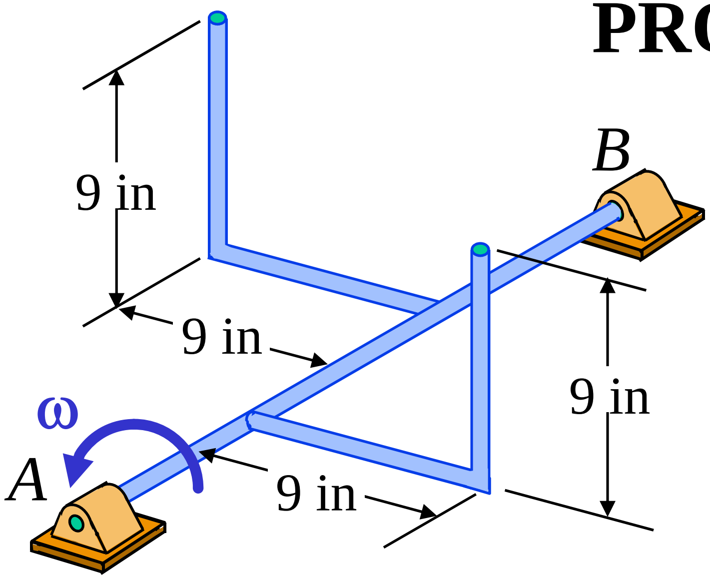
$$(H_C)_{x'} = \bar{I}_{x'}\, \omega_{x'} = 0$$

$$(H_C)_{y'} = \bar{I}_{y'}\, \omega_{y'} = \frac{1}{4}\, 5\, (0.25)^2\, 3 = 0.234 \ \text{ kg m}^2/\text{s}$$

$$(H_C)_{z'} = \bar{I}_{z'}\, \omega_{z'} = \frac{1}{2}\, 5\, (0.25)^2\, 8 = 1.25 \ \text{ kg m}^2/\text{s}$$

$$\mathbf{H}_C = 0.234\,\mathbf{j} + 1.25\,\mathbf{k} \quad \text{kg m}^2/\text{s}$$

PROBLEM 18.148



Two L-shaped arms, each weighing 5 lb, are welded to the one-third points of the 24-in. shaft *AB*. Knowing that shaft *AB* rotates at the constant rate *w* = 180 rpm, determine:

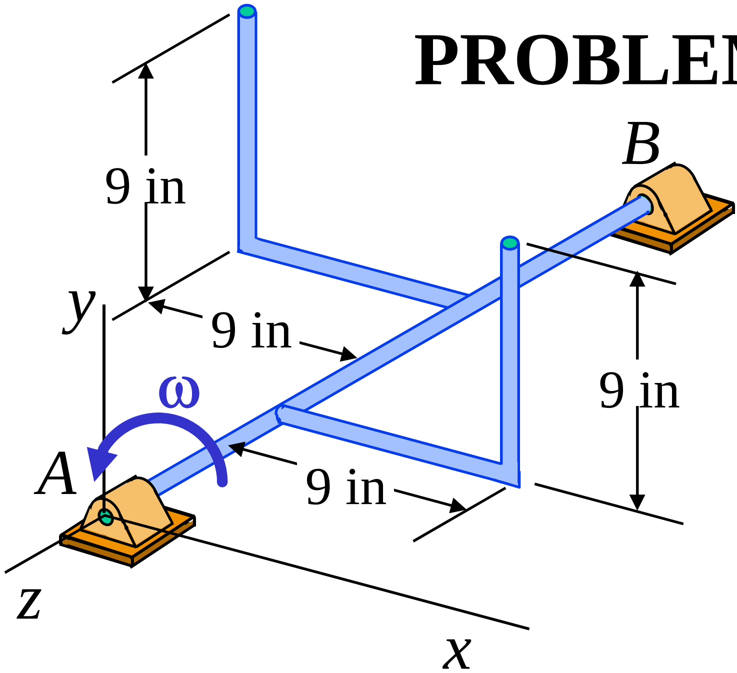
(a)the angular momentum $\mathbf{H}_A$ of the body about *A*;

(b) the angle that $\mathbf{H}_A$ forms with the shaft.

1. Determine the angular velocity w of the body : **w** is the angular velocity of the body with respect to a fixed frame of reference. The vector **w** may be resolved into components along the rotating axes.

$$\omega = 180 \text{ rpm} = 18.85 \text{ rad/s}$$

$$\boldsymbol{\omega} = 18.85\,\mathbf{k}\ \ \text{rad/s}$$



PROBLEM 18.148 – SOLUTION

2. *Determine the mass moments and products of inertia of the body:* For a three dimensional body these are the quantities I_x, I_y, I_z, I_{xy}, I_{xz}, and I_{yz}, where xyz is the rotating frame. If the rotating
frame is centered at G (mass center) and is in the direction of the principal axes of inertia ($G\bar{x}'\bar{y}'\bar{z}'$), then the products of inertia are zero and I_x, I_y, and I_z, are the principal centroidal moments of inertia.

Defining:

$$L = \frac{9}{12}\ \text{ft.} \qquad d = \frac{8}{12}\ \text{ft} \qquad m = \frac{2.5}{32.2}\ \text{slug}$$

$$I_z = 2\,[I_z \text{ of } ① + I_z \text{ of } ②]$$

$$I_z = 2\left\{\frac{1}{12} m L^2 + m\,[L^2 + (0.5 L)^2] + \frac{1}{12} m L^2 + m\,(0.5 L)^2\right\}$$

$$I = 0.1456\ \text{lb}\cdot\text{ft}\cdot\text{s}^2$$

PROBLEM 18.148 – SOLUTION

$$I_{xz} = [I_{xz} \text{ of } ① + I_{xz} \text{ of } ② + I_{xz} \text{ of } ③ + I_{xz} \text{ of } ④]$$

$$I_{xz} = [\, m\,(\text{-L})(\text{-}2d) + m\,(\text{-}0.5\text{L})(\text{-}2d) + m\,(\,0.5\,\text{L})(\text{-}d) + m\,(\,\text{L}\,)(\text{-}d)\,]$$

$$I_{xz} = 1.5\, m\, \text{L}\, d = 0.0582 \;\; \text{lb} \cdot \text{ft} \cdot \text{s}^2$$

$$I_{yz} = [I_{yz} \text{ of } ① + I_{yz} \text{ of } ④]$$

$$I_{yz} = [m(0.5\text{L})(\text{-}2d) + m(0.5\,\text{L})(\text{-}d)] = \text{-}1.5\, m\, \text{L}\, d = \text{-}\, 0.0582 \text{ lb} \cdot \text{ft} \cdot \text{s}^2$$

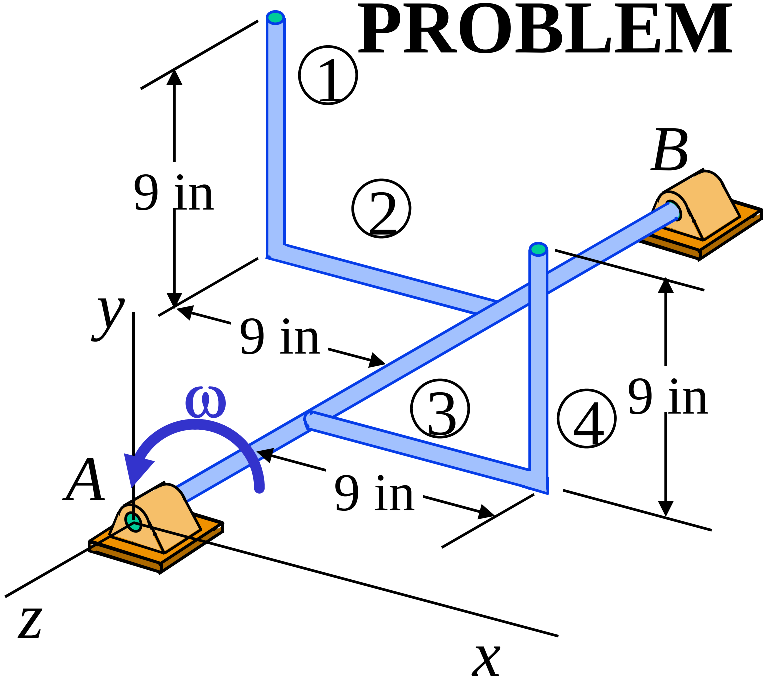
3. Determine the angular momentum of the body: The angular momentum $\mathbf{H}_A$ of a rigid body about point *A* can be expressed in terms of the components of its angular velocity **w** and its moments and products of inertia.

$$(\,H_A\,)_x = +\, I_x\, \omega_x - I_{xy}\, \omega_y - I_{xz}\, \omega_z$$

$$(\,H_A\,)_y = -\, I_{yx}\, \omega_x + I_y\, \omega_y - I_{yz}\, \omega_z$$

$$(\,H_A\,)_z = -\, I_{zx}\, \omega_x - I_{zy}\, \omega_y + I_z\, \omega_z$$

PROBLEM 18.148 – SOLUTION



$\omega_z = 18.85 \text{ rad/s}$

$I_z = 0.1456 \text{ lb} \cdot \text{ft} \cdot \text{s}^2$

$I_{xz} = 0.0582 \text{ lb} \cdot \text{ft} \cdot \text{s}^2$

$I_{yz} = -0.0582 \text{ lb} \cdot \text{ft} \cdot \text{s}^2$

(a) Angular momentum about A:

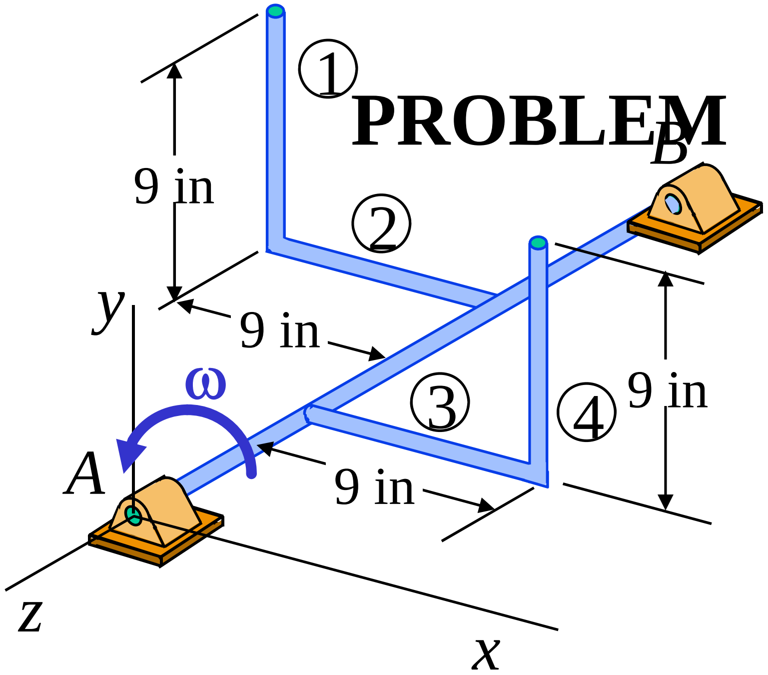
$$(H_A)_x = -I_{xz}\,\omega_z = -(0.0582)(18.85) = -1.098 \text{ lb} \cdot \text{ft} \cdot \text{s}$$

$$(H_A)_y = -I_{yz}\,\omega_z = -(-0.0582)(18.85) = 1.098 \text{ lb} \cdot \text{ft} \cdot \text{s}$$

$$(H_A)_z = +I_z\,\omega_z = +(0.1456)(18.85) = 2.744 \text{ lb} \cdot \text{ft} \cdot \text{s}$$

$$\mathbf{H}_A = -1.098\,\mathbf{i} + 1.098\,\mathbf{j} + 2.74\,\mathbf{k} \quad \text{lb} \cdot \text{ft} \cdot \text{s}$$

PROBLEM 18.148 – SOLUTION



(b) = The angle $\mathbf{H}_A$ forms with the shaft:

$$\mathbf{H}_A = -1.098\,\mathbf{i} + 1.098\,\mathbf{j} + 2.744\,\mathbf{k} \quad \text{lb} \cdot \text{ft} \cdot \text{s}$$

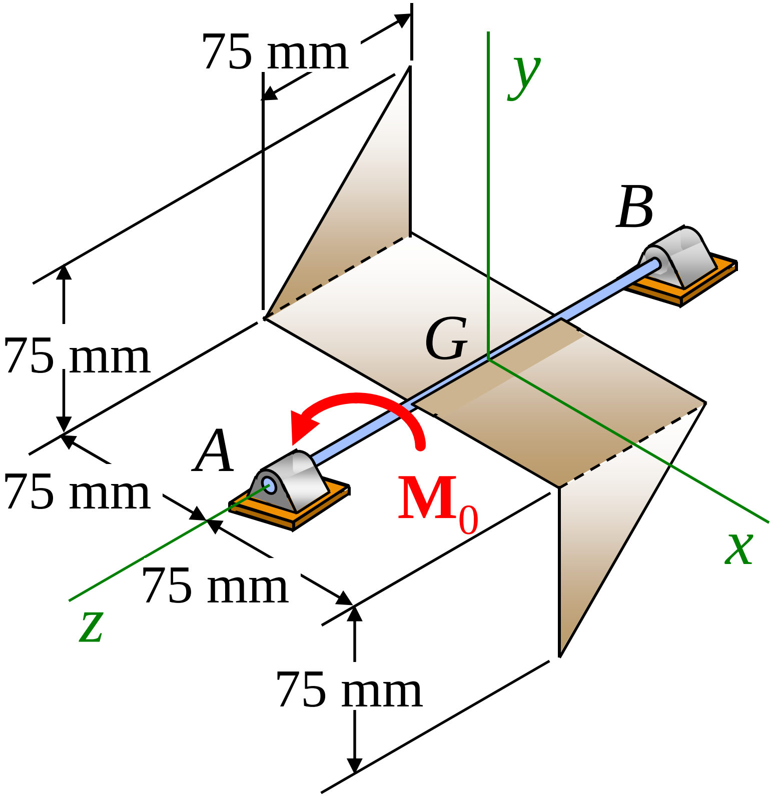
$$H_A = \sqrt{(-1.098)^2 + (1.098)^2 + (2.74)^2} = 3.153 \ \text{lb} \cdot \text{ft} \cdot \text{s}$$

$$\mathbf{H}_A \cdot \mathbf{k} = H_A \cos\theta$$

$$2.744 = 3.153 \cos\theta$$

$$\theta = 29.5^\circ$$

PROBLEM 18.153- Thematic exercise 16



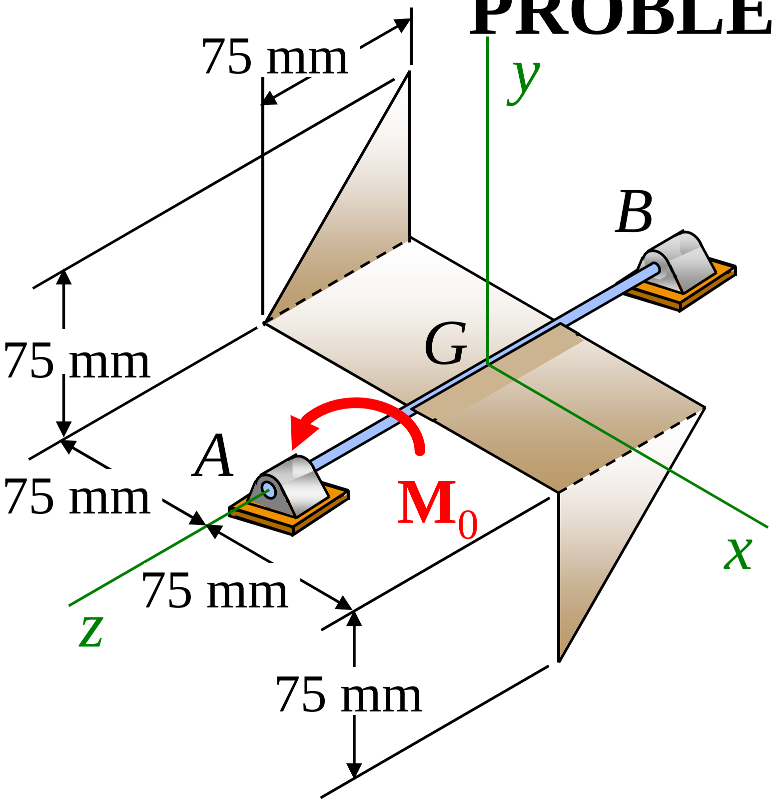
The sheet-metal component shown is of uniform thickness and has a mass of 600 g. It is attached to a light axle supported by bearings at *A* and *B* located 150 mm apart. The component is at rest when it is subjected to a couple $\mathbf{M}_0$ = (49.5 mN·m) **k**. Determine the dynamic reactions at *A* and *B:*

(a) immediately after the couple is applied;

(b) 0.6 [s] later.

1. Determine the mass moments and products of inertia of the body: For a three dimensional body these are the quantities I_x, I_y, I_z, I_{xy}, I_{xz}, and I_{yz}, where *xyz* is the rotating frame. If the rotating frame is centered at *G* (mass center) and is in the direction of the principal axes of inertia (*Gx'y'z'*), then the products of inertia are zero and $\bar{I}_x$, $\bar{I}_y$, and $\bar{I}_z$, are the principal centroidal moments of inertia.

PROBLEM 18.153 – solution



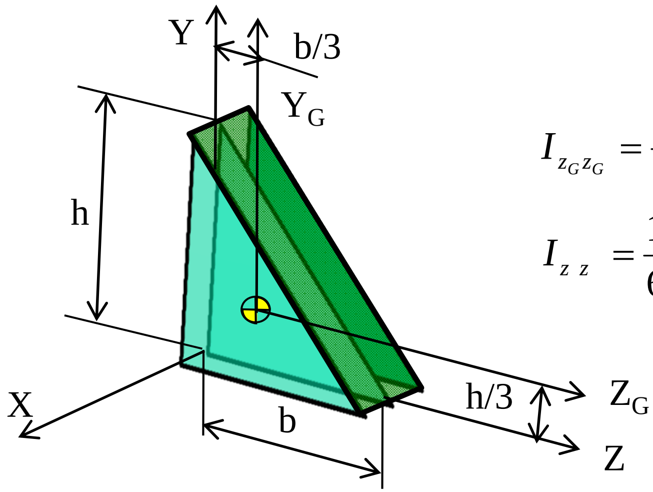
Moments and products of inertia :

Set : $b = 0.075$ m, $m = 0.6$ kg

By symmetry :

I_z , I_{xz} , I_{xz} of ① = I_z , I_{xz} , I_{xz} of ③

Recall:



$$I_{z_G z_G} = \frac{1}{18} M h^2$$

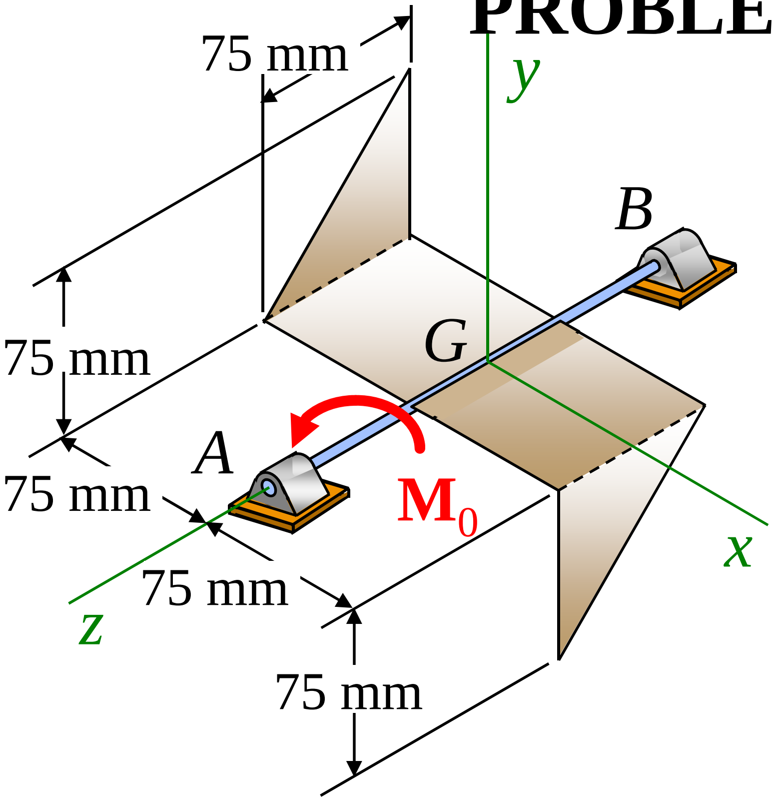
$$I_{z\,z} = \frac{1}{6} M h^2$$

$$I_z = 2\,[\,I_z \text{ of } ①\,] + I_z \text{ of } ②$$

$$I_z = 2\,\{[\frac{1}{18}\frac{m}{6}b^2\,] + \frac{m}{6}[\,b^2 + (\frac{b}{3})^2\,]\} + \frac{1}{12}(\frac{2}{3}\,m\,)(\,2b\,)^2$$

$$I_z = \frac{11}{18}\,\mathrm{m}\,b^2$$

PROBLEM 18.153 – solution



I_{xz} of ② $= 0$

For the whole body :

$$I_{xz} = 2\,[\, I_{xz} \text{ of } ① \,]$$

$$I_{xz} = 2\,[\, 0 + \frac{m}{6}(-b)(-\frac{b}{6})]$$

$$I_{xz} = \frac{1}{18}\, m\, b^2$$

I_{yz} of ② $= 0$

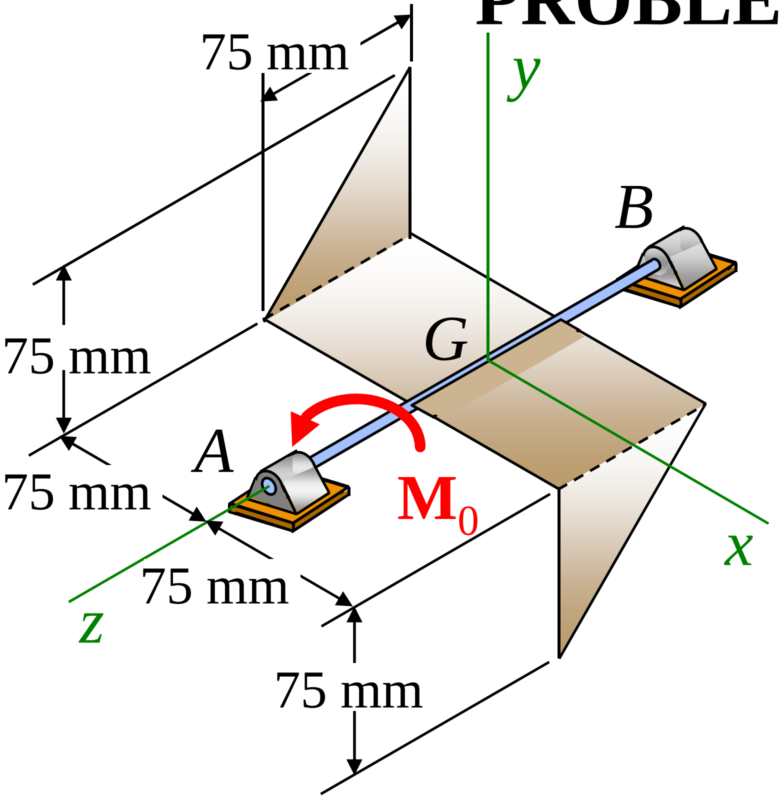
For the whole body :

$$I_{yz} = 2\,[\, I_{yz} \text{ of } ① \,]$$

$$I_{yz} = 2\,\{[\, -\frac{1}{36}\frac{m}{6} b^2\,] + \frac{m}{6}\,[(-\frac{b}{6})(\frac{b}{3})\,]\}$$

$$I_{yz} = -\frac{1}{36}\, m\, b^2$$

PROBLEM 18.153 - solution

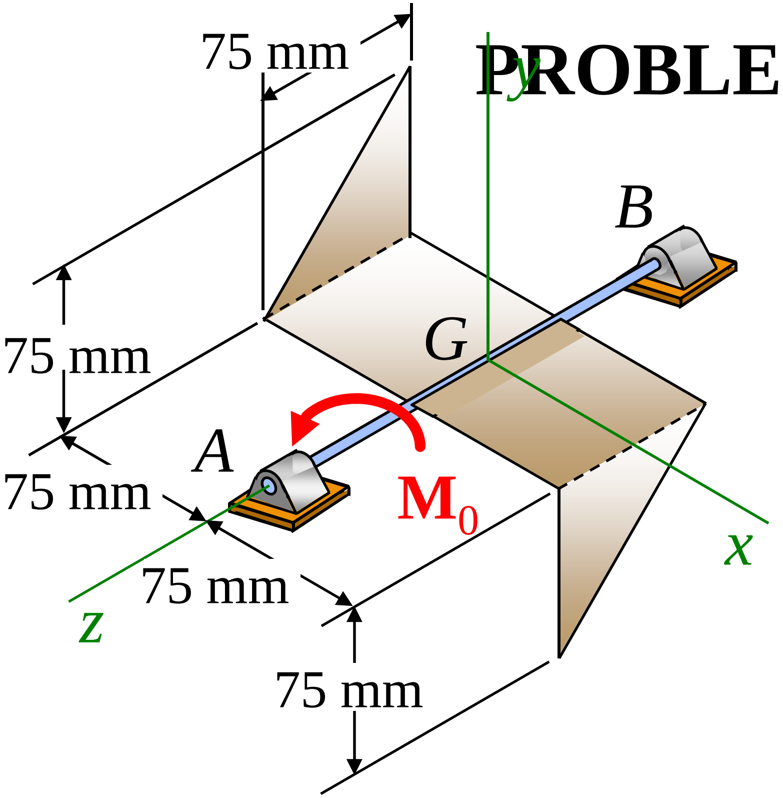


2. Determine the angular velocity w of the body and the angular velocity Ω of the rotating frame: **ω** is the angular velocity of the body with respect to a fixed frame of reference. The vector **ω** may be resolved into components along the rotating axes. **Ω** is the angular velocity of the rotating frame. If the rotating frame is rigidly attached to the body, **Ω** = **ω** .

$$\omega_x = \omega_y = 0, \qquad \omega_z = \omega$$

$$\boldsymbol{\omega} = \omega\,\mathbf{k}$$

$$\boldsymbol{\Omega} = \boldsymbol{\omega}$$



PROBLEM 18.153 - solution

3. Determine the angular momentum of the body:

The angular momentum $\mathbf{H}_G$ of a rigid body about point G can be expressed in terms of the components of its angular velocity **w** and its moments and products of inertia.

$$\vec{H}_G = \begin{bmatrix} I_{xx} & -P_{xy} & -P_{xz} \\ & I_{yy} & -P_{yz} \\ & & I_{zz} \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ w \end{Bmatrix} = \begin{Bmatrix} -P_{xz}w \\ -P_{yz}w \\ I_{zz}w \end{Bmatrix}$$

Angular momentum about G :

$$(H_G)_x = -I_{xz}\,\omega_z = -\frac{1}{18}\,m\,b^2\omega$$

$$(H_G)_y = -I_{yz}\,\omega_z = \frac{1}{36}\,m\,b^2\omega$$

$$(H_G)_z = I_z\,\omega_z = \frac{11}{18}\,m\,b^2\omega$$

Recall:

$$I_{xz} = \frac{1}{18}m\,b^2 \quad I_z = \frac{11}{18}m\,b^2 \quad I_{yz} = -\frac{1}{36}\,m\,b^2$$

$$\mathbf{H}_G = \frac{1}{36}\,m\,b^2\omega\,(-2\,\mathbf{i} + \mathbf{j} + 22\,\mathbf{k})$$

PROBLEM 18.153 - solution

4. Compute the rate of change of angular momentum : The rate of change of $\mathbf{H}_G$ with respect to a fixed frame is given by

$$\dot{\mathbf{H}}_G = (\dot{\mathbf{H}}_G)_{Oxyz} + \boldsymbol{\Omega} \times \mathbf{H}_G$$

where $(\dot{\mathbf{H}}_G)_{Oxyz}$ is the rate of change of $\mathbf{H}_G$ with respect to the rotating frame, and $\boldsymbol{\Omega}$ is the angular velocity of the rotating frame. If the rotating frame is rigidly attached to the body, $\boldsymbol{\Omega}$ is equal to $\boldsymbol{\omega}$, the angular velocity of the body.

$$\dot{\mathbf{H}}_G = (\dot{\mathbf{H}}_G)_{oxyz} + \boldsymbol{\Omega} \times \mathbf{H}_G$$

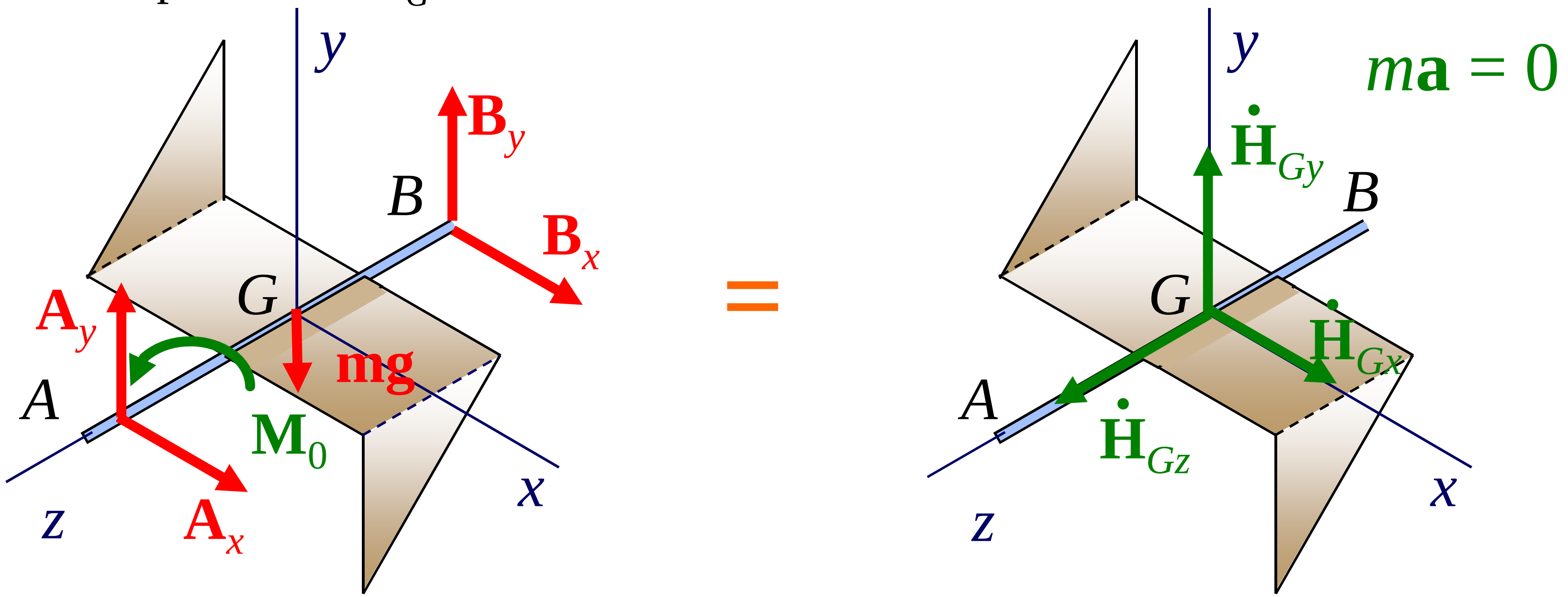
$$\dot{\vec{H}}_G = \begin{Bmatrix} -P_{xz}\dot{w} \\ -P_{yz}\dot{w} \\ I_{zz}\dot{w} \end{Bmatrix} + \begin{Bmatrix} 0 \\ 0 \\ w \end{Bmatrix} \times \begin{Bmatrix} -P_{xz}w \\ -P_{yz}w \\ I_{zz}w \end{Bmatrix} = \begin{Bmatrix} -P_{xz}\dot{w} + P_{yz}w^2 \\ -P_{yz}\dot{w} - P_{xz}w^2 \\ I_{zz}\dot{w} \end{Bmatrix}$$

Immediately after the couple is applied $\boldsymbol{\Omega} = 0$

$$\dot{\mathbf{H}}_G = \frac{1}{36}\, m\, b^2\, \alpha\, (-2\,\mathbf{i} + \mathbf{j} + 22\,\mathbf{k})$$

PROBLEM 18.153 - solution

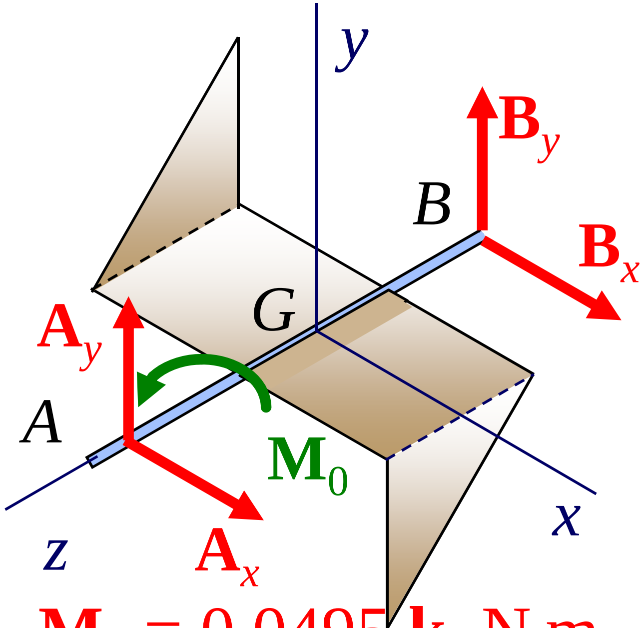
5. Draw the free-body-diagram equation: The diagram shows that the system of the external forces exerted on the body is equivalent to the vector $m\mathbf{a}$ applied at G and the couple vector $\dot{\mathbf{H}}_G$.



6. Write equations of motion: Six independent scalar equations can be written from

$$\Sigma\,\mathbf{F} = m\,\mathbf{a}, \qquad \Sigma\,\mathbf{M}_G = \dot{\mathbf{H}}_G$$

PROBLEM 18.153 - solution



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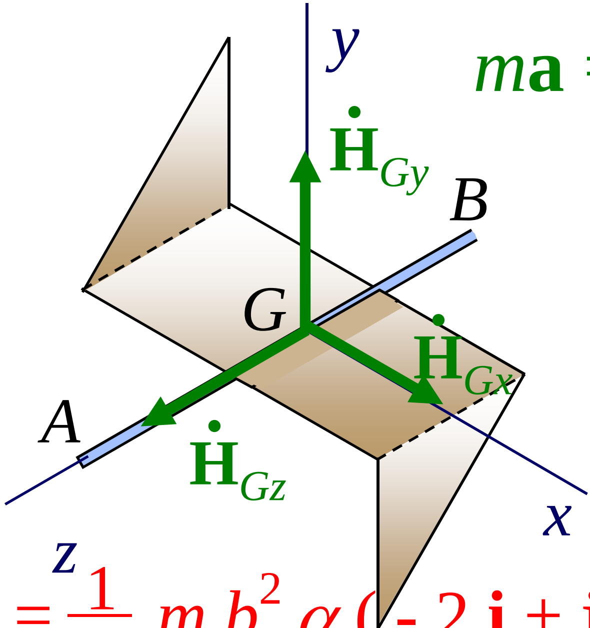


$m\mathbf{a} = 0$

$$\mathbf{M}_0 = 0.0495\,\mathbf{k}\ \ \mathrm{N{\cdot}m}$$

$$\mathbf{H}_G = \frac{1}{36}\, m\, b^2\, \alpha\,(-2\,\mathbf{i} + \mathbf{j} + 22\,\mathbf{k})$$

$m = 0.6$ kg, $b = 0.075$ m, $\alpha = 24$ rad/s^2

Recall: 1st Koenig Theorem

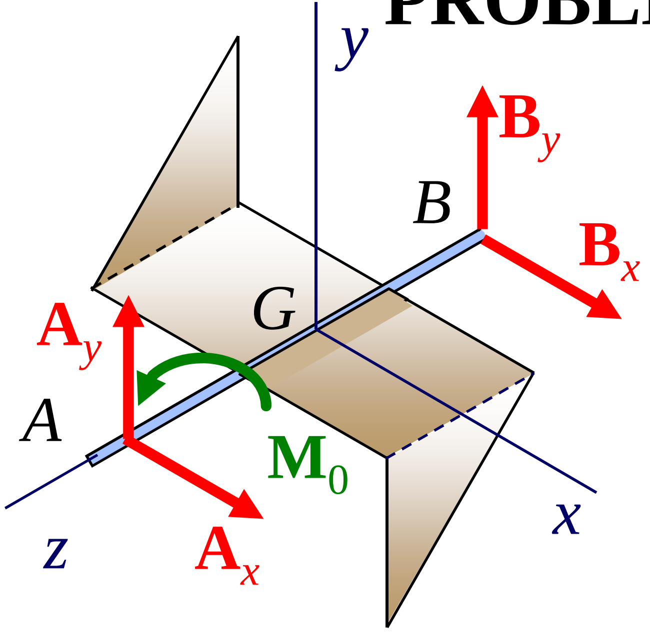
$$\vec{H}_B = B\vec{G} \times \vec{V}_G M + \vec{H}_G$$

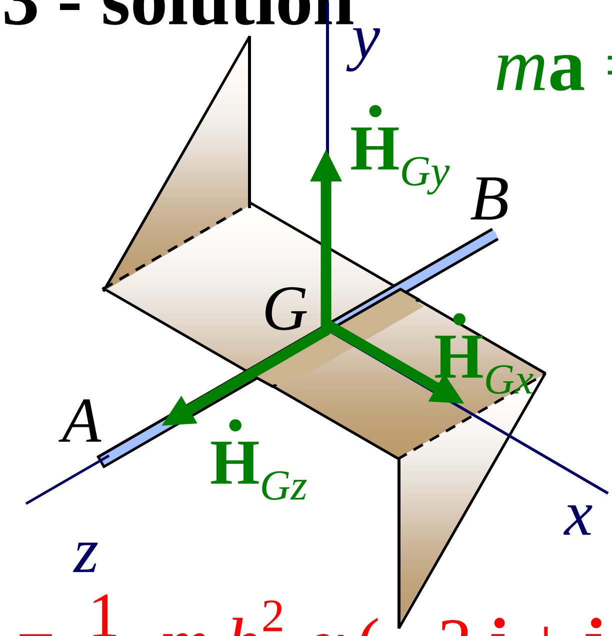
Equating moments about *B* : $\quad \Sigma\, \mathbf{M}_B = \dot{\mathbf{H}}_B$

$$B\vec{A} \times \vec{R}_A + B\vec{G} \times \vec{P} + \vec{M} = \dot{\vec{H}}_B$$

$$\begin{Bmatrix} 0 \\ 0 \\ 0.15 \end{Bmatrix} \times \begin{Bmatrix} A_x \\ A_y \\ 0 \end{Bmatrix} + \begin{Bmatrix} 0 \\ 0 \\ 0.075 \end{Bmatrix} \times \begin{Bmatrix} 0 \\ -mg \\ 0 \end{Bmatrix} + \begin{Bmatrix} 0 \\ 0 \\ 0.0495 \end{Bmatrix} = \begin{Bmatrix} -2 \times 1/36 mb^2 \dot{w} \\ 1/36 mb^2 \dot{w} \\ 22 \times 1/36 mb^2 \dot{w} \end{Bmatrix} \Leftrightarrow \begin{Bmatrix} A_y = 29.73\,[N] \\ A_x = 0.015\,[N] \\ \dot{w} = 24\,[rad/s^2] \end{Bmatrix}$$

PROBLEM 18.153 - solution





$m\mathbf{a} = 0$

$\mathbf{M}_0 = 0.0495\ \mathbf{k}\ \ \text{N·m}$

$\dot{\mathbf{H}}_G = \dfrac{1}{36}\, m\, b^2\, \alpha\, (-2\,\mathbf{i} + \mathbf{j} + 22\,\mathbf{k})$

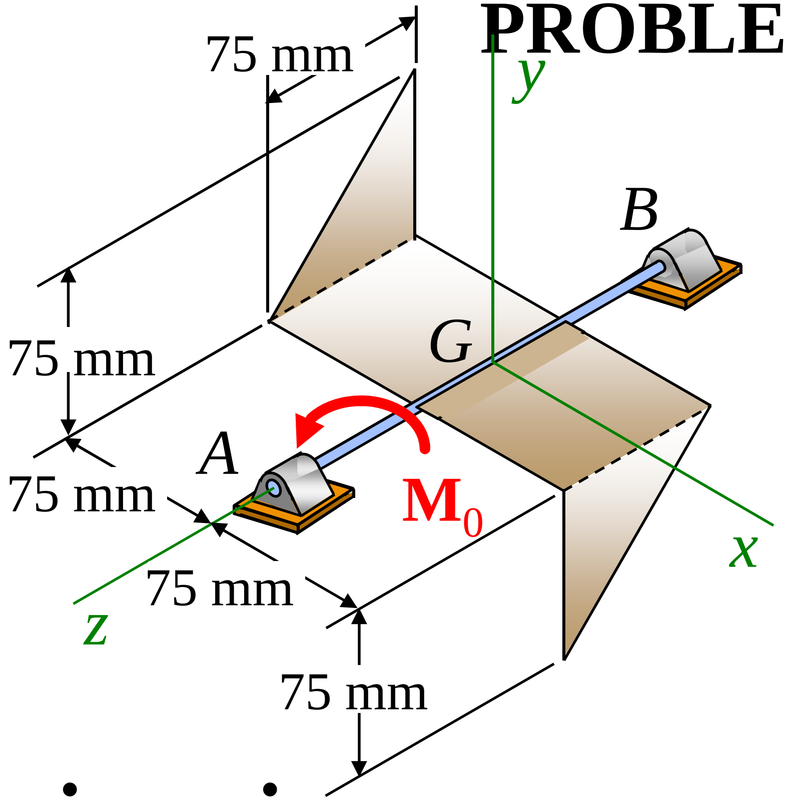
Equating forces : $\Sigma\, \mathbf{F} = m\mathbf{a}$

$$\begin{cases} A_x + B_x = m \times 0 \\ A_y + B_y - mg = m \times 0 \end{cases} \Leftrightarrow \begin{cases} B_x = -0.015\,[N] \\ B_y = -23.844\,[N] \end{cases}$$

(a) Dynamic reactions at *A* and *B* :

$\mathbf{A} = 0.015\ \mathbf{i} + 0.03\ \mathbf{j}\ \ \text{N}$ $\qquad$ $\mathbf{B} = -\ 0.015\ \mathbf{i} -\ 0.03\ \mathbf{j}\ \ \text{N}$

PROBLEM 18.153 - solution



(b) 0.6 s after the couple is applied :

$\alpha = 24 \text{ rad/s}^2$ (constant)

$\omega = 24\,t = 24(0.6) = 14.4 \text{ rad/s}$

$\boldsymbol{\omega} = \omega\,\mathbf{k} = 14.4\,\mathbf{k} \text{ rad/s}$

Compute the rate of change of angular momentum.

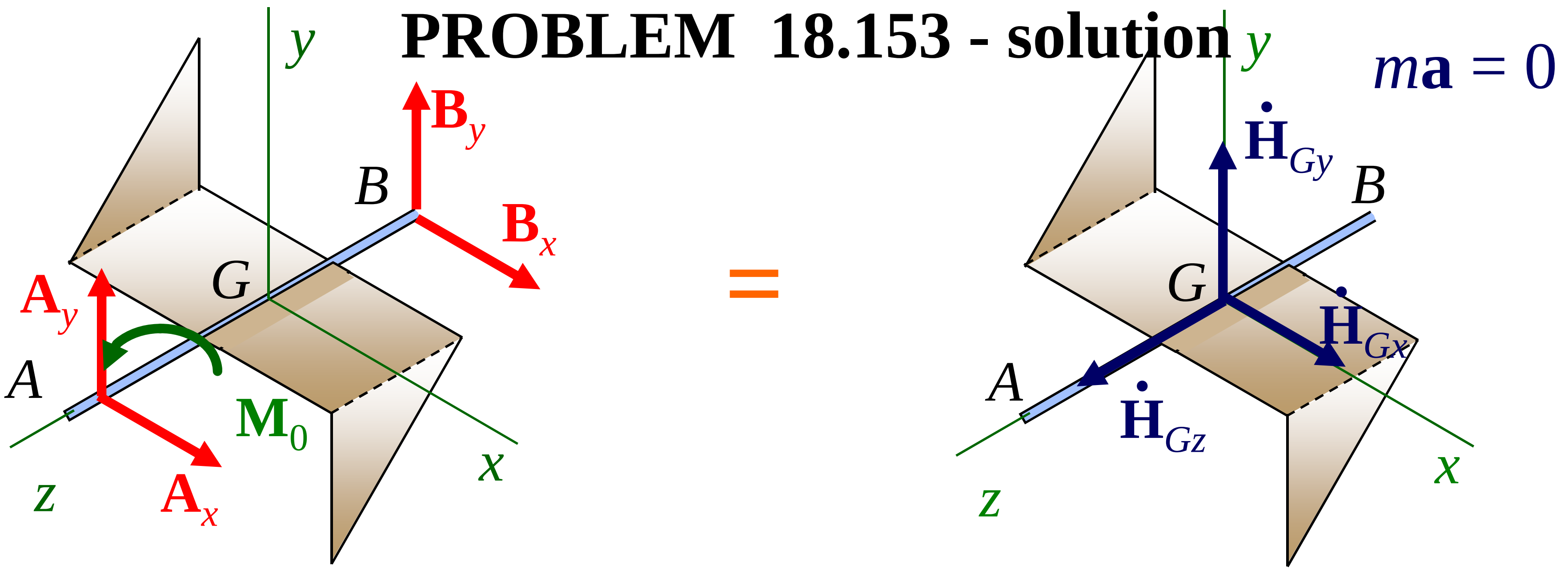
$$\mathbf{H}_G = \frac{1}{36} m\, b^2 \omega\,(-2\,\mathbf{i} + \mathbf{j} + 22\,\mathbf{k})$$

$$\dot{\mathbf{H}}_G = (\dot{\mathbf{H}}_G)_{oxyz} + \boldsymbol{\Omega} \times \mathbf{H}_G$$

$$\dot{\mathbf{H}}_G = \frac{1}{36} m\, b^2 \alpha\,(-2\,\mathbf{i} + \mathbf{j} + 22\,\mathbf{k}) + (\omega\,\mathbf{k}) \times \left(\frac{1}{36} m\, b^2 \omega\,(-2\,\mathbf{i} + \mathbf{j} + 22\,\mathbf{k}) \right)$$

$$\dot{\mathbf{H}}_G = \frac{1}{36} m\, b^2 [(-2\alpha - \omega^2)\,\mathbf{i} + (\alpha - 2\,\omega^2)\,\mathbf{j} + 22\alpha\,\mathbf{k}]$$

PROBLEM 18.153 - solution



$\mathbf{M}_0 = 0.0495\,\mathbf{k}\;\; \text{N·m}$ $\qquad \dot{\mathbf{H}}_G = \frac{1}{36} m\, b^2[(-2\alpha - \omega^2)\mathbf{i} + (\alpha - 2\omega^2)\mathbf{j} + 22\alpha\,\mathbf{k}]$

Write equations of motion. Moments about B :

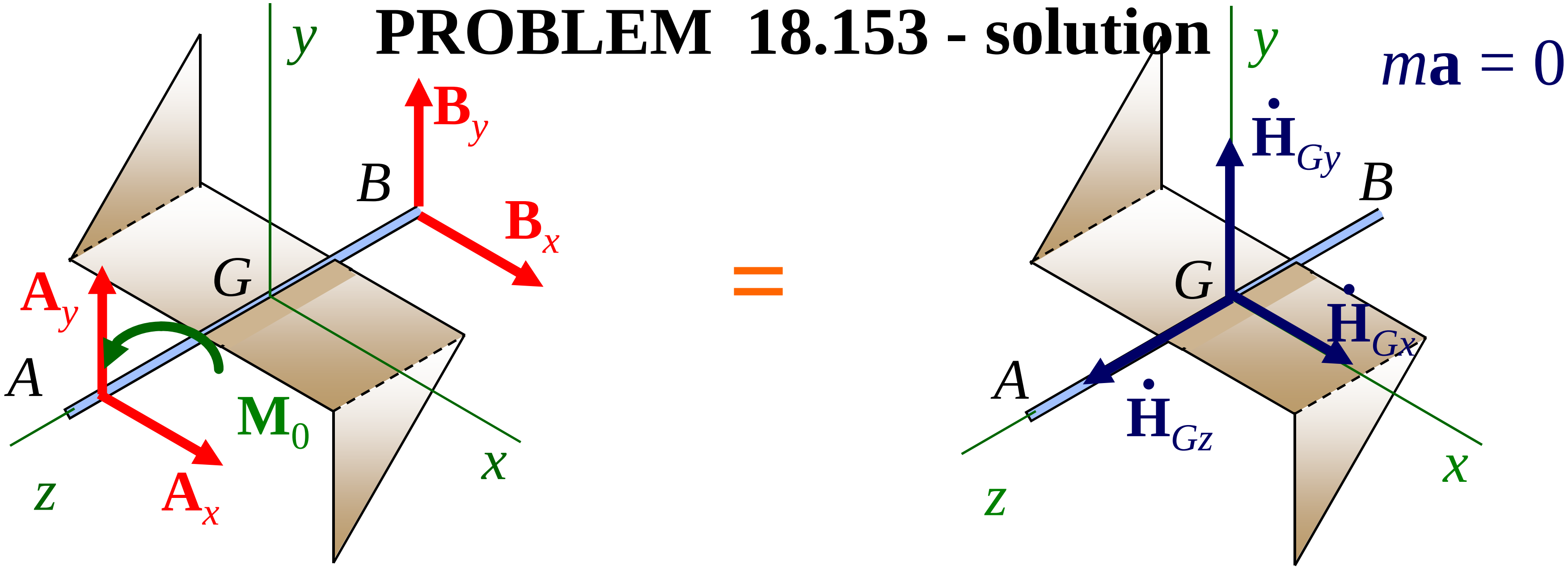
m = 0.6 kg, b = 0.075 m,
α = 24 rad/s^2, ω = 14.4 rad/s

$$\Sigma\, \mathbf{M}_B = \dot{\mathbf{H}}_B$$

y Component (+ ↑) : $A_x(0.15) = \frac{1}{36} m\, b^2(\alpha - 2\omega^2)$, $\quad A_x = -0.244$ N

x Component (+ ⟶) : $-A_y(0.15) = \frac{1}{36} m\, b^2(-2\alpha - \omega^2)$, $A_y = 0.1596$ N

PROBLEM 18.153 - solution



Write equations of motion. Equating forces : $\Sigma\,\mathbf{F} = m\mathbf{a}$

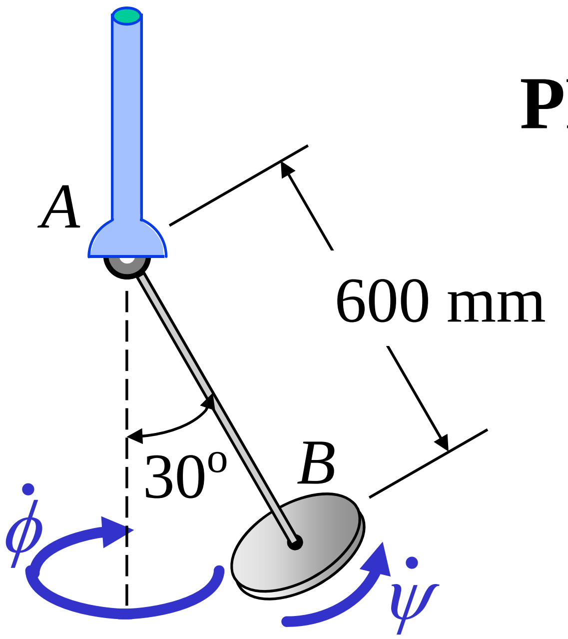
$$A_x = -0.244 \text{ N}, \quad A_y = 0.1596 \text{ N}$$

x Component (+ $\longrightarrow$) : $A_x + B_x = 0 \quad B_x = 0.244 \text{ N}$

y Component (+ $\uparrow$) : $A_y + B_y = 0 \quad B_y = -0.1596 \text{ N}$

Dynamic reactions at A and B after 0.6 s :

$$\mathbf{A} = -0.244\,\mathbf{i} + 0.1596\,\mathbf{j} \text{ N} \qquad \mathbf{B} = 0.244\,\mathbf{i} - 0.1596\,\mathbf{j} \text{ N}$$

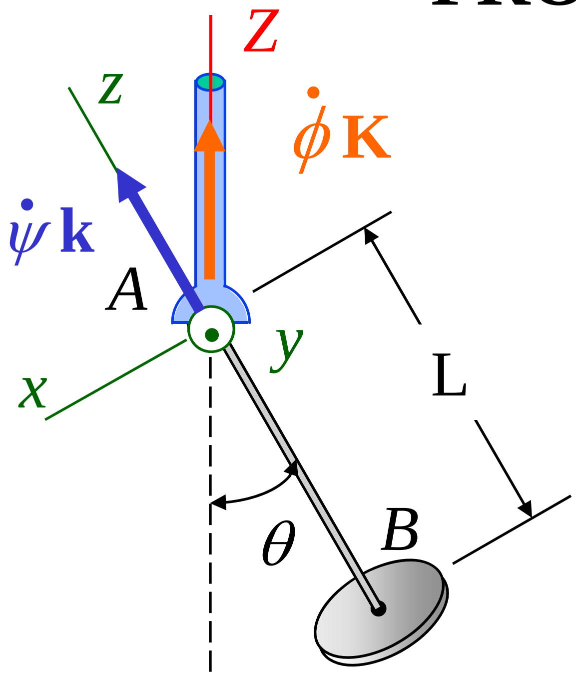


PROBLEM 18.157

A 2-kg disk of 150-mm diameter is attached to the end of a rod *AB* of negligible mass which is supported by a ball-and-socket joint at *A*. If the disk is observed to process about the vertical in the sense indicated at a constant rate of 36 rpm, determine the rate of spin $\dot{\psi}$ of the disk about *AB*.

1. Determine the angular velocity w of the body and the angular velocity W of the rotating frame: ω is the angular velocity of the body with respect to a fixed frame of reference. The vector ω may be resolved into components along the rotating axes. The angular velocity is often obtained by adding two components of angular velocities ω_1 and ω_2. Ω is the angular velocity of the rotating frame. If the rotating frame is rigidly attached to the body, $\Omega = \omega$.

PROBLEM 18.157 - solution



Determine the angular velocity ω of the body.

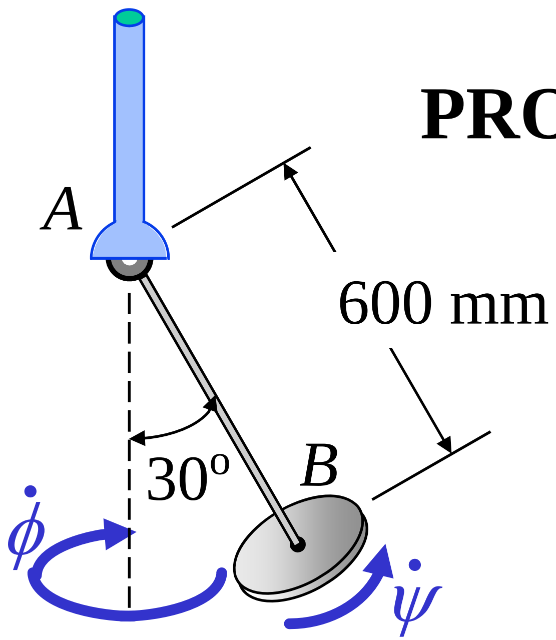
$$\dot{\phi} = -\ 36 \text{ rpm} = -\ 3.770 \text{ rad/s}$$

$$\boldsymbol{\omega} = -\dot{\phi}\sin\theta\,\mathbf{i} + (\dot{\psi} + \dot{\phi}\cos\theta)\,\mathbf{k}$$

Determine the angular velocity Ω of the rotating frame.

$$\boldsymbol{\Omega} = -\dot{\phi}\sin\theta\,\mathbf{i} + \dot{\phi}\cos\theta\,\mathbf{k}$$

2. Determine the mass moments and products of inertia of the body: For a three dimensional body these are the quantities I_x, I_y, I_z, I_{xy}, I_{xz}, and I_{yz}, where *xyz* is the rotating frame. If the rotating frame is centered at *G* (mass center) and is in the direction of the principal axes of inertia (*Gx'y'z'*), then the products of inertia are zero and $\bar{I}_x$, $\bar{I}_y$, and $\bar{I}_z$, are the principal centroidal moments of inertia.



PROBLEM 18.157 – solution

Determine the mass moments of inertia.

$$I_x = \frac{1}{4} m\, r^2 + m\, L^2$$

$$I_x = \frac{1}{4}\, 2\,(\,0.075\,)^2 + 2\,(\,0.6\,)^2 = 0.7228 \;\; kg{\cdot}m^2$$

$$I_z = \frac{1}{2} m\, r^2 = \frac{1}{2}\, 2\,(\,0.075\,)^2 = 0.005625 \;\; kg{\cdot}m^2$$

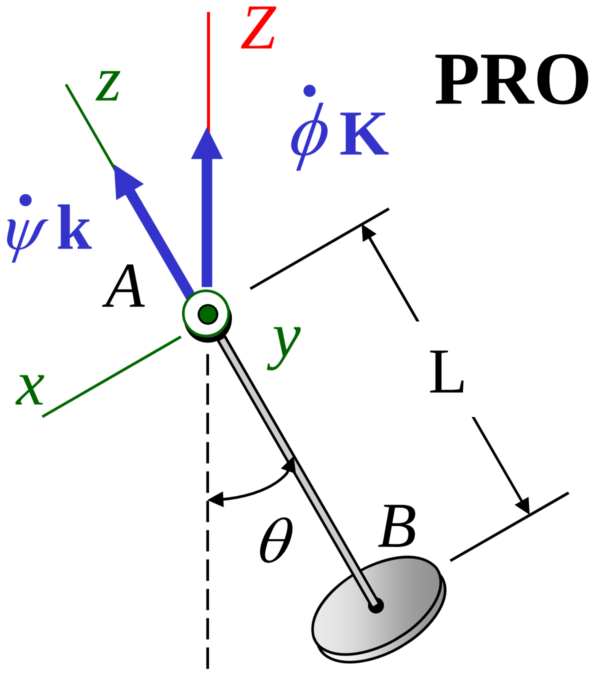
θ = 30°
L = 600 mm

3. Determine the angular momentum of the body: The angular momentum $\mathbf{H}_G$ of a rigid body about point *A* can be expressed in terms of the components of its angular velocity **w** and its moments and products of inertia.

$$(\,H_A\,)_x = +\, I_x\, \omega_x - I_{xy}\, \omega_y - I_{xz}\, \omega_z$$

$$(\,H_A\,)_y = -\, I_{yx}\, \omega_x + I_y\, \omega_y - I_{yz}\, \omega_z$$

$$(\,H_A\,)_z = -\, I_{zx}\, \omega_x - I_{zy}\, \omega_y + I_z\, \omega_z$$



PROBLEM 18.157 – solution

3. Determine the angular momentum of the body.

$$\boldsymbol{\omega} = -\,\dot{\phi}\sin\theta\,\mathbf{i} + (\,\dot{\psi} + \dot{\phi}\cos\theta\,)\,\mathbf{k}$$

Angular momentum about A :

$$\mathbf{H}_A = I_x\,\omega_x\,\mathbf{i} + I_z\,\omega_z\,\mathbf{k}$$

$$\mathbf{H}_A = -\,I_x\,\dot{\phi}\sin\theta\,\mathbf{i} + I_z\,(\,\dot{\psi} + \dot{\phi}\cos\theta\,)\,\mathbf{k}$$

4. Compute the rate of change of angular momentum.

$$\boldsymbol{\Omega} = -\dot{\phi}\sin\theta\,\mathbf{i} + \dot{\phi}\cos\theta\;\mathbf{k}$$

$$\dot{\mathbf{H}}_A = (\,\dot{\mathbf{H}}_A\,)_{oxyz} + \boldsymbol{\Omega}\times\mathbf{H}_A$$

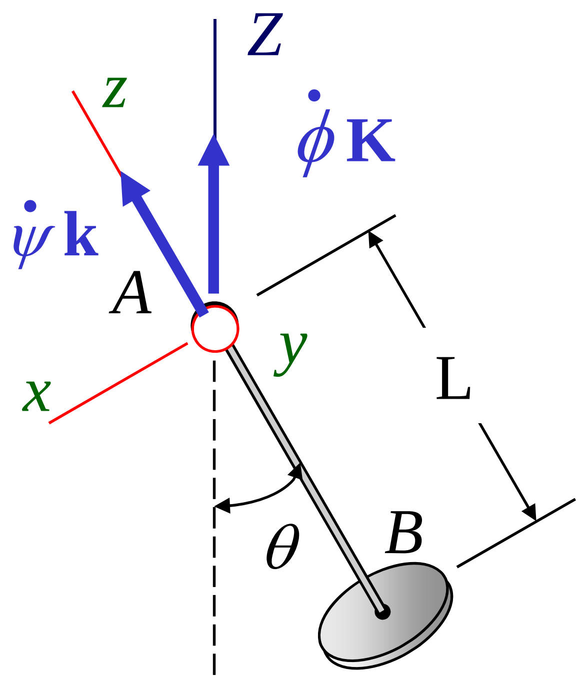
$(\,\dot{\mathbf{H}}_A\,)_{oxyz} = 0,$ since $\dot{\psi} =$ constant

$$\dot{\mathbf{H}}_A = \boldsymbol{\Omega}\times\mathbf{H}_A = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -\dot{\phi}\sin\theta & 0 & \dot{\phi}\cos\theta \\ -\,I_x\,\dot{\phi}\sin\theta & 0 & I_z\,(\dot{\psi} + \dot{\phi}\cos\theta\,) \end{vmatrix}$$

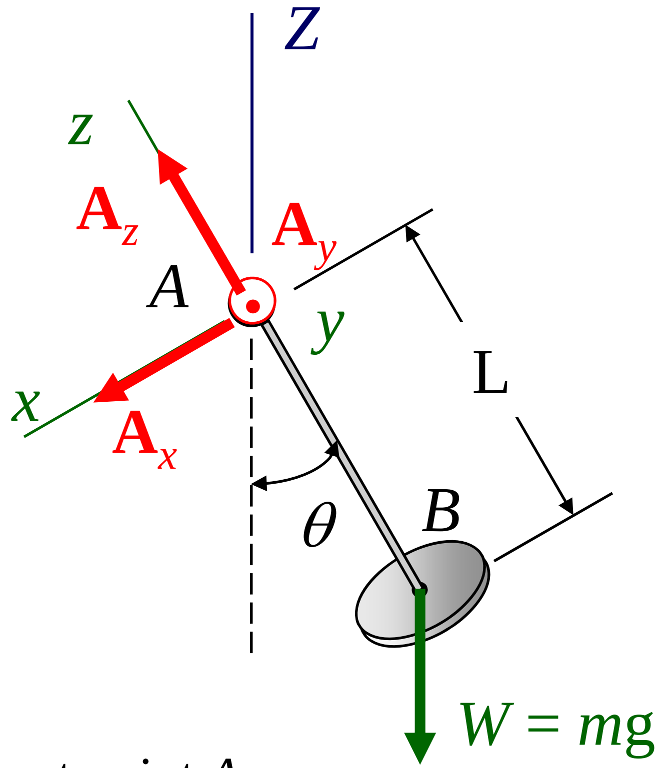
$$\dot{\mathbf{H}}_A = \phi\sin\theta\,[\,I_z\,(\dot{\psi} + \dot{\phi}\cos\theta\,) - I_x\,\dot{\phi}\cos\theta\,]\,\mathbf{j}$$

PROBLEM 18.157 – solution

5. Draw the free-body-diagram: The diagram shows the system of the external forces exerted on the body.

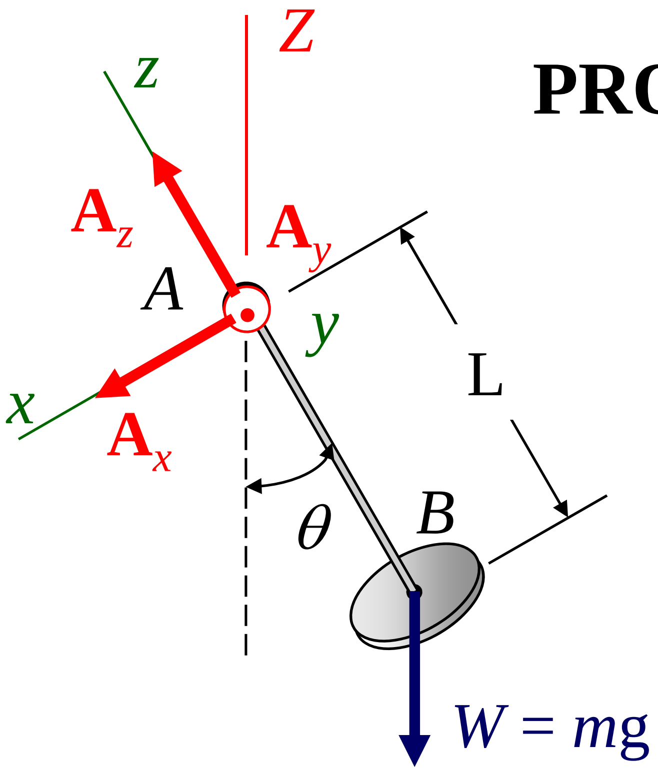


Note:
The y axis and $\mathbf{A}_y$ are in a direction perpendicular (out) to the plane of the figure.



6. Write equation of motion: For a body rotating about point A :

$$\Sigma\, \mathbf{M}_A = \dot{\mathbf{H}}_A$$



PROBLEM 18.157 – solution

Write equation of motion.

Recall:

$$\dot{\mathbf{H}}_A = \dot{\phi}\sin\theta\,[\, I_z\,(\dot{\psi} + \dot{\phi}\cos\theta\,) - I_x\,\dot{\phi}\cos\theta\,]\,\mathbf{j}$$

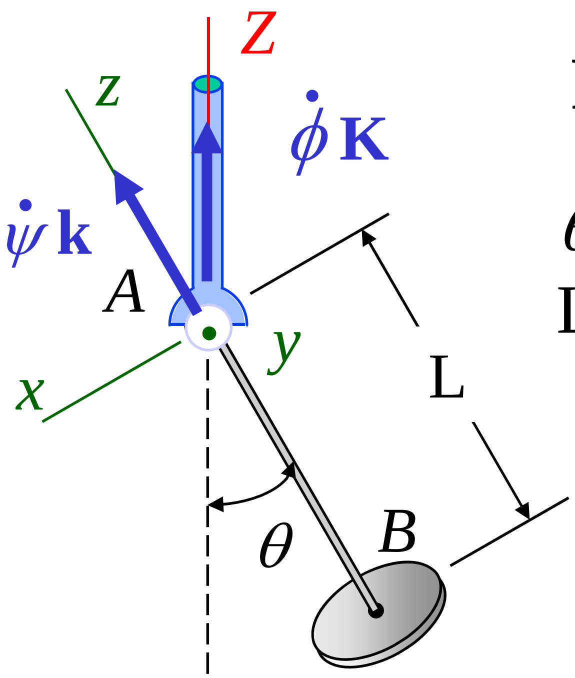
Sum of moments about A :

$$\Sigma\,\mathbf{M}_A = \dot{\mathbf{H}}_A :$$

$$(-\,\mathrm{L}\,\mathbf{k}\,) \times (\,-mg\,\mathbf{K}\,) = \dot{\phi}\sin\theta\,[\, I_z\,(\dot{\psi} + \dot{\phi}\cos\theta\,) - I_x\,\dot{\phi}\cos\theta\,]\,\mathbf{j}$$

$$-\,mg\,\mathrm{L}\,\sin\theta\,\mathbf{j} = \dot{\phi}\sin\theta\,[\, I_z\,(\dot{\psi} + \dot{\phi}\cos\theta\,) - I_x\,\dot{\phi}\cos\theta\,]\,\mathbf{j}$$

$$\dot{\psi} = \frac{I_x - I_z}{I_z}\,\dot{\phi}\cos\theta - \frac{m\,g\,\mathrm{L}}{I_z\,\dot{\phi}}$$



PROBLEM 18.157 – solution

$\theta = 30^o$

L = 600 mm

$I_x = 0.7228 \ \ kg{\cdot}m^2$

$I_z = 0.005625 \ \ kg{\cdot}m^2$

$\dot{\phi} = -\ 3.77 \ \ rad/s$

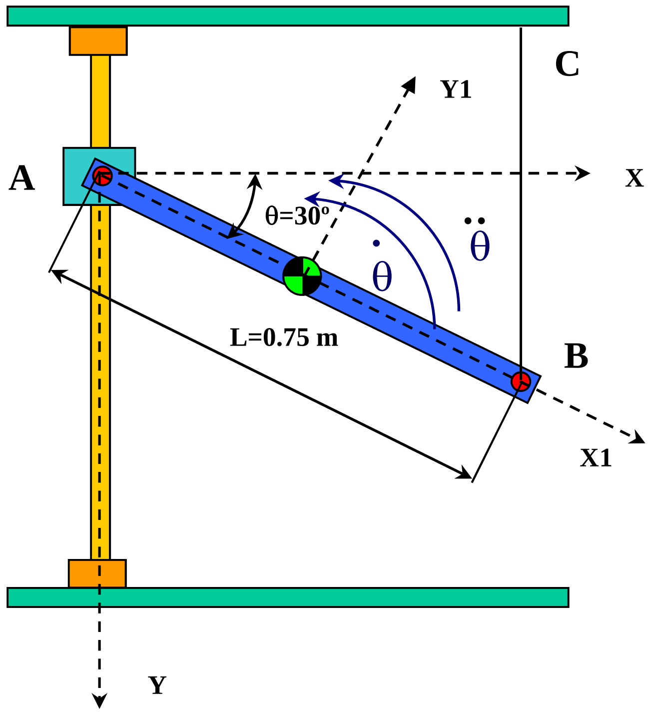
$m = 2$ kg

$$\dot{\psi} = \frac{I_x - I_z}{I_z}\ \dot{\phi}\cos\theta - \frac{m\,g\,L}{I_z\ \dot{\phi}}$$

$$\dot{\psi} = \frac{0.7228 - 0.005625}{0.005625}\ (-\ 3.77\ )\cos 30^o - \frac{(2)(9.81)(0.6)}{0.005625\ (-\ 3.77)}$$

$$\dot{\psi} = 138.9 \ \ rad/s$$

TEST EXERCISE – Thematic exercise 17



Problem:

The extreme point of a uniform bar AB, with a mass of 8 [kg] is connected to a slide vertical frictionless and massless cursor.
The other extreme point is connected to a vertical cable BC.
If the bar is released from rest, for the position shown, determine:
a) The angular acceleration of the bar;
b)The dynamic instant reaction.

Solution: ***By Newton's second law***

$$\begin{cases} \sum \vec{F} = m\vec{a}_G \\ \sum \vec{M} = \dot{\vec{H}}_G \end{cases}$$

Angular momentum determination:

$$\vec{H}_G\Big|_{System\ S1} = [I_G]\{w\} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{12}mL^2 & 0 \\ 0 & 0 & \frac{1}{12}mL^2 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ -\dot{\theta} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ -\frac{1}{12}mL^2\dot{\theta} \end{Bmatrix}$$

TEST EXERCISE – solution

Time derivative of momentum determination (Dynamic momentum):

$$\dot{\vec{H}}_G\Big|_{System\ S0} = \begin{Bmatrix} 0 \\ 0 \\ -\frac{1}{12}mL^2\ddot{\theta} \end{Bmatrix} + \Omega_{System\ S1} \times \begin{Bmatrix} 0 \\ 0 \\ -\frac{1}{12}mL^2\dot{\theta} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ -\frac{1}{12}mL^2\ddot{\theta} \end{Bmatrix} + \begin{Bmatrix} 0 \\ 0 \\ -\dot{\theta} \end{Bmatrix} \times \begin{Bmatrix} 0 \\ 0 \\ -\frac{1}{12}mL^2\dot{\theta} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ -\frac{1}{12}mL^2\ddot{\theta} \end{Bmatrix}$$

Cinematic analysis: Mass center acceleration

$$\vec{a}_G = \vec{a}_A + \dot{\vec{W}} \times \vec{AG} + \vec{W} \times \left(\vec{W} \times \vec{AG}\right)$$

$$\vec{a}_G = \begin{Bmatrix} a_{Gx} \\ a_{Gy} \\ 0 \end{Bmatrix} = \begin{Bmatrix} 0 \\ a_{Ay} \\ 0 \end{Bmatrix} + \begin{Bmatrix} 0 \\ 0 \\ -\ddot{\theta} \end{Bmatrix} \times \begin{Bmatrix} L/2\frac{\sqrt{3}}{2} \\ L/2\frac{1}{2} \\ 0 \end{Bmatrix} + \begin{Bmatrix} 0 \\ 0 \\ -\dot{\theta} \end{Bmatrix} \times \left[\begin{Bmatrix} 0 \\ 0 \\ -\dot{\theta} \end{Bmatrix} \times \begin{Bmatrix} L/2\frac{\sqrt{3}}{2} \\ L/2\frac{1}{2} \\ 0 \end{Bmatrix} \right]$$

$$\vec{a}_G = \begin{Bmatrix} 0 \\ a_{Ay} \\ 0 \end{Bmatrix} + \begin{Bmatrix} \ddot{\theta}\,L/2\frac{1}{2} \\ -\ddot{\theta}\,L/2\frac{\sqrt{3}}{2} \\ 0 \end{Bmatrix} + \begin{Bmatrix} 0 \\ 0 \\ -\dot{\theta} \end{Bmatrix} \times \begin{Bmatrix} +\dot{\theta}\,L/2\frac{1}{2} \\ -\dot{\theta}\,L/2\frac{\sqrt{3}}{2} \\ 0 \end{Bmatrix} = \begin{Bmatrix} \ddot{\theta}\,L/2\frac{1}{2} - \dot{\theta}^2\,L/2\frac{\sqrt{3}}{2} \\ a_{Ay} - \ddot{\theta}\,L/2\frac{\sqrt{3}}{2} - \dot{\theta}^2\,L/2\frac{1}{2} \\ 0 \end{Bmatrix} = \begin{Bmatrix} \ddot{\theta}\,L/2\frac{1}{2} \\ a_{Ay} - \ddot{\theta}\,L/2\frac{\sqrt{3}}{2} \\ 0 \end{Bmatrix}$$

TEST EXERCISE – solution

Cinematic analysis: Mass center acceleration

$$\vec{a}_B = \vec{a}_A + \dot{\vec{W}} \times \vec{A}B + \vec{W} \times \left(\vec{W} \times \vec{A}B\right)$$

$$\begin{Bmatrix} a_{Bx} \\ a_{By} \\ 0 \end{Bmatrix} = \begin{Bmatrix} 0 \\ a_{Ay} \\ 0 \end{Bmatrix} + \begin{Bmatrix} 0 \\ 0 \\ -\ddot{\theta} \end{Bmatrix} \times \begin{Bmatrix} L\frac{\sqrt{3}}{2} \\ L\frac{1}{2} \\ 0 \end{Bmatrix} + \begin{Bmatrix} 0 \\ 0 \\ -\dot{\theta} \end{Bmatrix} \times \left[\begin{Bmatrix} 0 \\ 0 \\ -\dot{\theta} \end{Bmatrix} \times \begin{Bmatrix} L\frac{\sqrt{3}}{2} \\ L\frac{1}{2} \\ 0 \end{Bmatrix} \right]$$

$$\begin{Bmatrix} a_{Bx} \\ 0 \\ 0 \end{Bmatrix} = \begin{Bmatrix} 0 \\ a_{Ay} \\ 0 \end{Bmatrix} + \begin{Bmatrix} \ddot{\theta}L\frac{1}{2} \\ -\ddot{\theta}L\frac{\sqrt{3}}{2} \\ 0 \end{Bmatrix} + \begin{Bmatrix} 0 \\ 0 \\ -\dot{\theta} \end{Bmatrix} \times \begin{Bmatrix} +\dot{\theta}L\frac{1}{2} \\ -\dot{\theta}L\frac{\sqrt{3}}{2} \\ 0 \end{Bmatrix} = \begin{Bmatrix} \ddot{\theta}L\frac{1}{2} - \dot{\theta}^2 L\frac{\sqrt{3}}{2} \\ a_{Ay} - \ddot{\theta}L\frac{\sqrt{3}}{2} - \dot{\theta}^2 L\frac{1}{2} \\ 0 \end{Bmatrix}$$

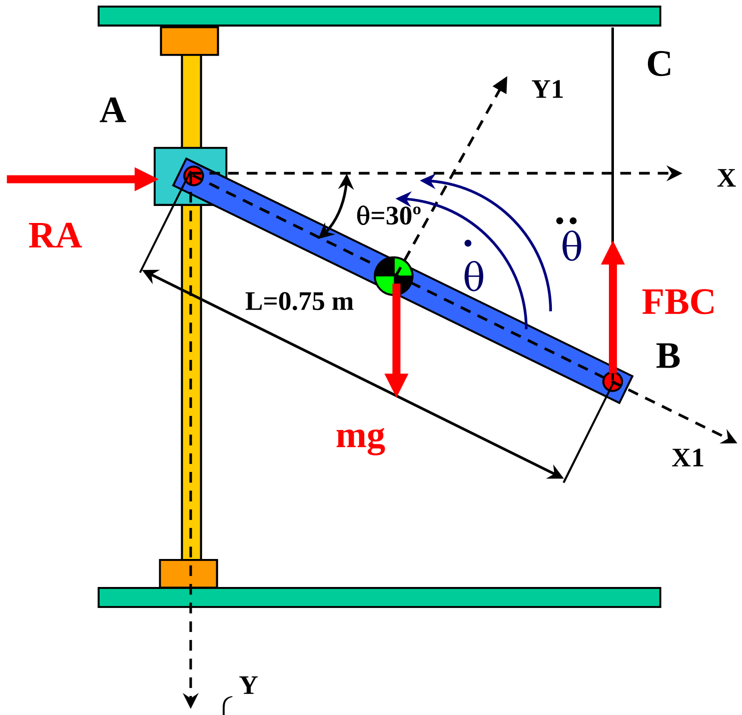
Second equation leads to:

$$0 = a_{Ay} - \ddot{\theta}L\frac{\sqrt{3}}{2} \Rightarrow a_{Ay} = \ddot{\theta}L\frac{\sqrt{3}}{2}$$

Substitution in to previous equation :

$$\begin{Bmatrix} a_{Gx} \\ a_{Gy} \\ 0 \end{Bmatrix} = \begin{Bmatrix} \ddot{\theta}\, L/2\frac{1}{2} - \dot{\theta}^2\, L/2\frac{\sqrt{3}}{2} \\ a_{Ay} - \ddot{\theta}\, L/2\frac{\sqrt{3}}{2} - \dot{\theta}^2\, L/2\frac{1}{2} \\ 0 \end{Bmatrix} = \begin{Bmatrix} \ddot{\theta}\, L/2\frac{1}{2} \\ \ddot{\theta}L\frac{\sqrt{3}}{2} - \ddot{\theta}\, L/2\frac{\sqrt{3}}{2} \\ 0 \end{Bmatrix} = \begin{Bmatrix} \ddot{\theta}\, L/2\frac{1}{2} \\ \ddot{\theta}L\frac{\sqrt{3}}{4} \\ 0 \end{Bmatrix}$$

TEST EXERCISE – solution



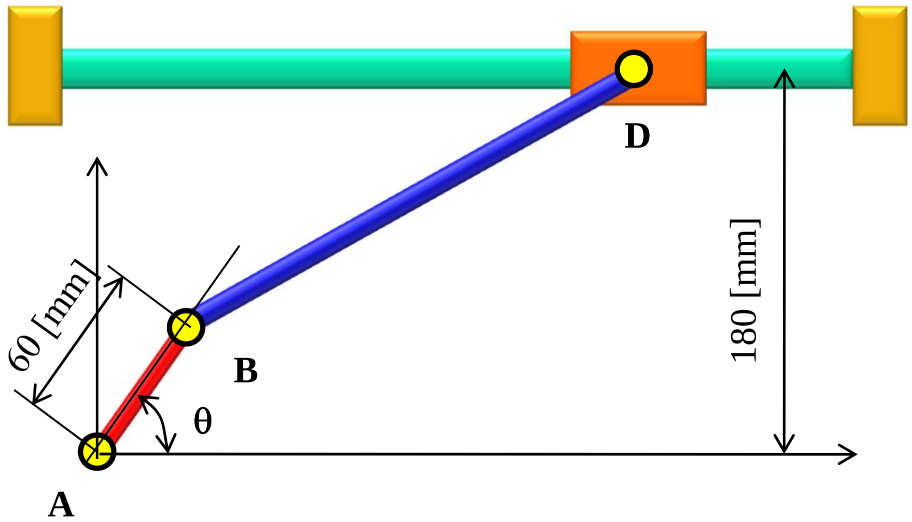
$$\begin{cases} \sum \vec{F} = m\vec{a}_G \\ \sum \vec{M} = \dot{\vec{H}}_G \end{cases} \Leftrightarrow \begin{cases} RA = m\ddot{\theta}\,{}^{L}\!/_{4} \\ -FBC + mg = m\ddot{\theta}\,{}^{L\sqrt{3}}\!/_{4} \\ RA\left(\frac{L}{4}\right) - FBC\left({}^{L\sqrt{3}}\!/_{4}\right) = -\frac{1}{12}mL^2\ddot{\theta} \end{cases}$$

$$\begin{cases} \sum \vec{F} = m\vec{a}_G \\ \sum \vec{M} = \dot{\vec{H}}_G \end{cases} \Leftrightarrow \begin{cases} RA = m\ddot{\theta}\,{}^{L}\!/_{4} \\ FBC = mg - m\ddot{\theta}\,{}^{L\sqrt{3}}\!/_{4} \\ m\ddot{\theta}\,{}^{L}\!/_{4}\left(\frac{L}{4}\right) - \left(mg - m\ddot{\theta}\,{}^{L\sqrt{3}}\!/_{4}\right)\left({}^{L\sqrt{3}}\!/_{4}\right) = -\frac{1}{12}mL^2\ddot{\theta} \end{cases} \Leftrightarrow \begin{cases} - \\ - \\ \ddot{\theta}\frac{L^2}{16} - g\,{}^{L\sqrt{3}}\!/_{4} + \ddot{\theta}\frac{L^2 3}{16} = -\frac{1}{12}L^2\ddot{\theta} \end{cases}$$

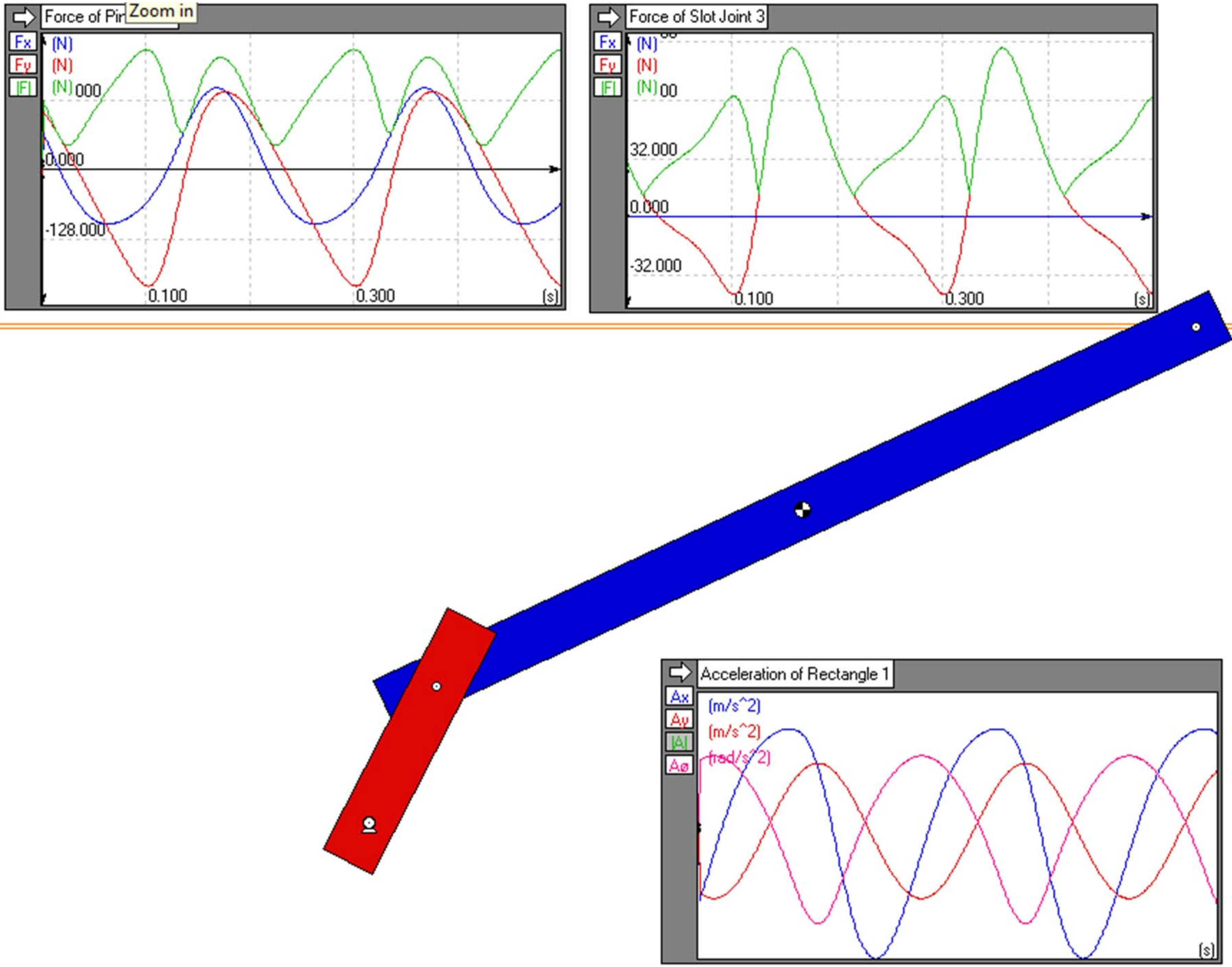
$$\Leftrightarrow \begin{cases} - \\ - \\ -g\,{}^{L\sqrt{3}}\!/_{4} = \ddot{\theta}\left(-\frac{L^2 3}{16} - \frac{L^2}{16} - \frac{1}{12}L^2\right) \end{cases} \Leftrightarrow \begin{cases} - \\ - \\ -g\,{}^{L\sqrt{3}}\!/_{4} = \ddot{\theta}\left(-\frac{64}{12\times 16}L^2\right) \end{cases} \Leftrightarrow \begin{cases} RA = m * \frac{g6\sqrt{3}}{8L} * \frac{L}{4} = 25.46[N] \\ FBC = \ldots \\ \ddot{\theta} = \frac{g6\sqrt{3}}{8L} = 16.97[rad/s2] \end{cases}$$

EXERCISE 16.119

- The 300 [mm] uniform rod BD of mass 3 [kg] is connected, as shown, to crank AB and to a collar D of negligible mass, which can slide freely along a horizontal rod. Knowing that crank AB rotates counter clockwise at the constant rate of 300[rpm], determine the dynamic reaction at support D, when $\theta=0^{o}$.



EXERCISE 16.119 – NUMERICAL SOLUTION



ENERGY METHODS

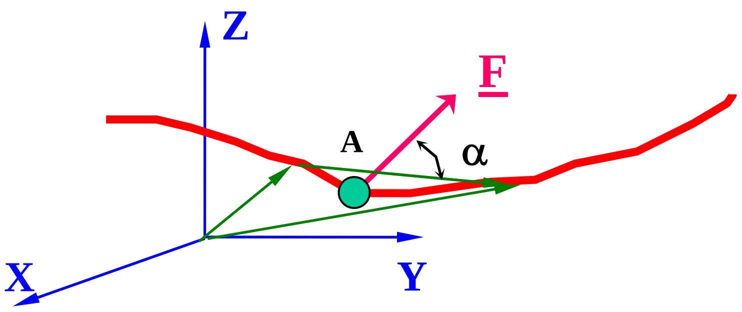
The ***principle of work and energy*** for a rigid body is expressed in the form

$$T_1 + U_{1\rightarrow 2} = T_2$$

where T_1 and T_2 represent the initial and final values of the kinetic energy of the rigid body and $U_{1\rightarrow 2}$ the work of the *external forces* acting on the rigid body.

The ***work of a force F*** applied at a point A is:

$$U_{1\rightarrow 2} = \int_{s_1}^{s_2} (F \cos \alpha)\, ds$$



$$dw = \vec{F} \bullet d\vec{r} = F.Cos\alpha.ds$$

where F is the magnitude of the force, α the angle it forms with the direction of motion of A, and s the variable of integration measuring the distance travelled by A along its path.

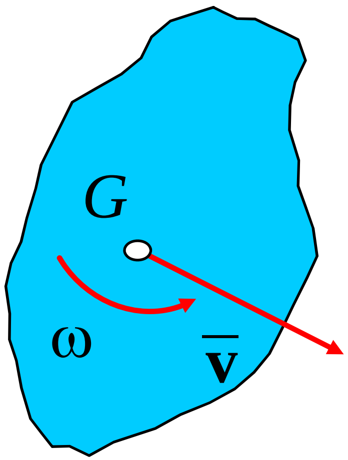
WORK OF A COUPLE

The ***work of a couple of moment*** **M** applied to a rigid body during a rotation in θ of the rigid body is:

$$U_{1\rightarrow 2} = \int_{\theta_1}^{\theta_2} M\, d\theta$$

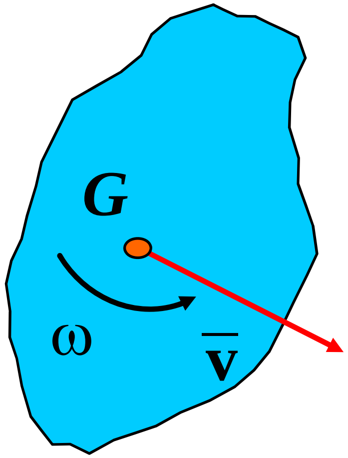
The kinetic energy of a rigid body in plane motion is calculated according to the 3rd Koenig theorem:

$$T = \frac{1}{2} m\bar{v}^{2} + \frac{1}{2} \bar{I}\omega^2$$



where $\bar{v}$ is the velocity of the mass centre *G* of the body, w the angular velocity of the body, and $\bar{I}$ its moment of inertia about an axis through *G* perpendicular to the plane of reference.

KINETIC ENERGY



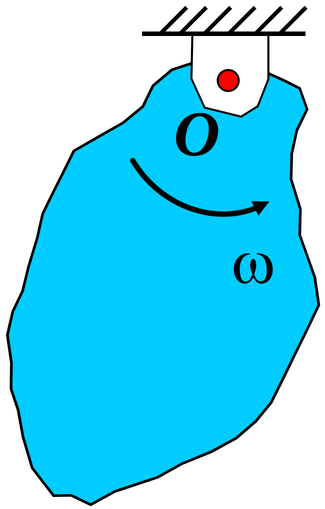
$$T = \frac{1}{2} m\bar{v}^2 + \frac{1}{2} \bar{I}\omega^2$$

The kinetic energy of a rigid body in plane motion may be separated into two parts: (1) the kinetic energy $\frac{1}{2} m\bar{v}^2$ associated with the motion of the mass center G of the body, and (2) the kinetic energy $\frac{1}{2}\bar{I}w^2$ associated with the rotation of the body about G.

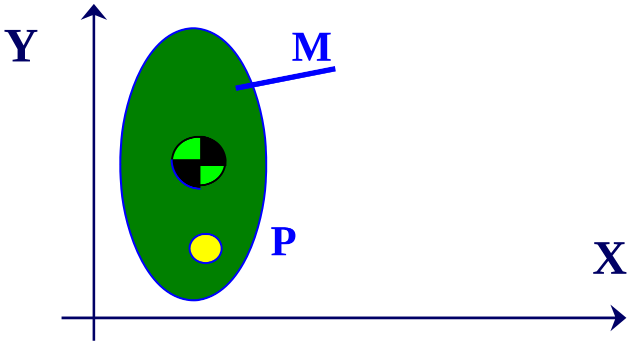
For a rigid body rotating about a fixed axis through O with an angular velocity w,

$$T = \frac{1}{2} I_O\omega^2$$

where I_O is the moment of inertia of the body about the considered fixed axis.



KINETIC ENERGY – PLANE MOTION



Velocity of a generic point of the body

$$\vec{v}_P = \vec{v}_G + \vec{w} \times \vec{G}P$$

Kinetic energy of the body

$$E_c = \frac{1}{2} M v_G^2 + \int_M \frac{1}{2}\left(\vec{w} \times \vec{G}P\right)^2 dm$$

$$= \frac{1}{2} M v_G^2 + \frac{1}{2}[I_G] w^2$$

Special case: Body movement about a fixed point:

$$E_c = \frac{1}{2}[I_O] w^2$$

PRINCIPLE OF ENERGY CONSERVATION

When a rigid body, or a system of rigid bodies, moves under the action of **conservative forces**, the ***principle of work and energy*** may be expressed in the form

$$T_1 + U_{1\to 2} = T_2 \qquad \Leftrightarrow \qquad T_1 + V_1 = T_2 + V_2$$

which is referred to as the ***principle of conservation of energy***. This principle may be used to solve problems involving conservative forces such as the force of gravity or the force exerted by a spring.

The ***Work of a conservative force*** is numerically equal to minus the potential variation associated to the force. The work is independent of the way of the force application point.

$$dw = -dU$$

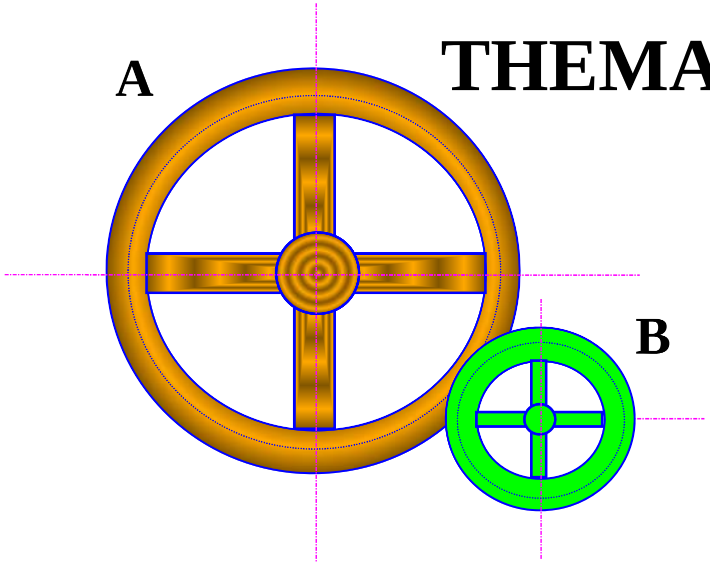
Examples:

$$W_1^2 = E_{p1} - E_{p2} \text{ , with } E_{pi} = mgh_i \qquad \text{Gravitational force}$$

$$W_1^2 = E_{e1} - E_{e2} \text{ , with } E_{ei} = \frac{1}{2}Kx_i^2 \qquad \text{Elastic force}$$

$$W_1^2 = E_{i1} - E_{i2} \text{ , with } E_{ei} = \frac{1}{2}m\dot{x}_i^2 \qquad \text{Inertia force}$$

THEMATIC EXERCISE - 18



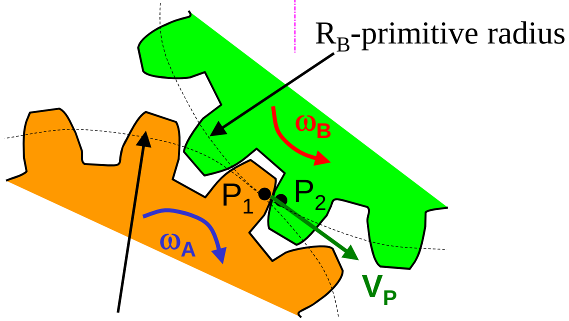
The gear chain A has a mass of 10 (kg), a gyration radius of 200 [mm] and a primitive radius of 250 [mm].

The gear B has a mass of 3 [kg], a gyration radius of 80 [mm] and a primitive radius of 100 [mm].

The system is at rest when a couple M equal to 6 [Nm] is applied to gear B. Neglecting the friction, determine:

a) The number of revolutions till it's velocity has achieved 600 [r.p.m.];

b) The average tangential force that B exerts on gear A.



a) Applying the principle of work and energy to the gear chain (A+B):

$$T_1 + U_1^2 = T_2 \Leftrightarrow 0 + U_1^2 = (T_2^A + T_2^B)$$

$$T_2^A = \frac{1}{2} I_G^A (w^A)^2 = \frac{1}{2}\left(m^A \left(K^A\right)^2\right)(w^A)^2 = \frac{1}{2}\left(10 \times 0{,}2^2\right)(w^A)^2$$

$$T_2^B = \frac{1}{2} I_G^B (w^B)^2 = \frac{1}{2}\left(m^B \left(K^B\right)^2\right)\left(\frac{600 \times 2\pi}{60}\right)^2 = \frac{1}{2}\left(3 \times 0{,}08^2\right)\left(\frac{600 \times 2\pi}{60}\right)^2$$

Recall:
Moment of inertia of a ring about perpendicular axis ($I=MR^2$).

THEMATIC EXERCISE

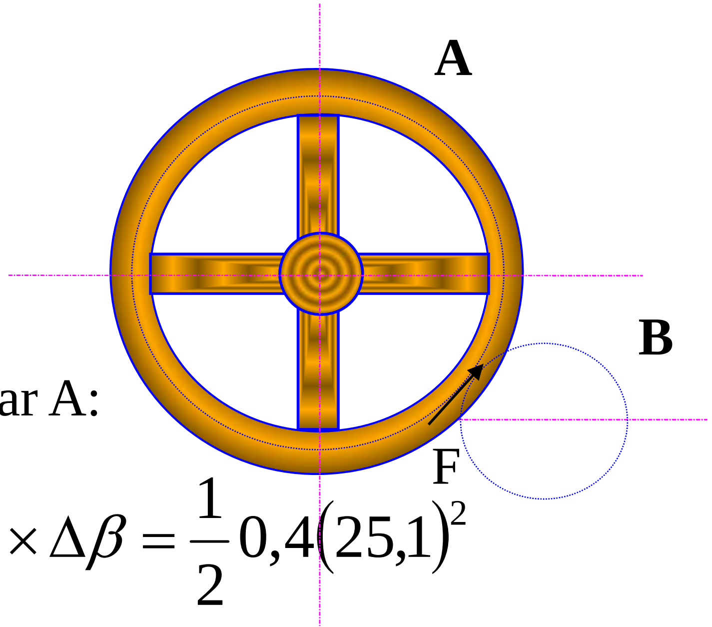
A kinematics approach:

$$\vec{v}_I^A = \vec{v}_I^B \Leftrightarrow w^A \times 0{,}25 = w^B \times 0{,}1 \Leftrightarrow w^A = 0{,}4w^B$$

The impulse energy:

$$U_1^2 = M\Delta\theta = 6\Delta\theta$$

So: $\Delta\theta = 27{,}37(rad) = 4{,}356(revolution\ s)$



The ***principle of work and energy*** applied to the gear A:

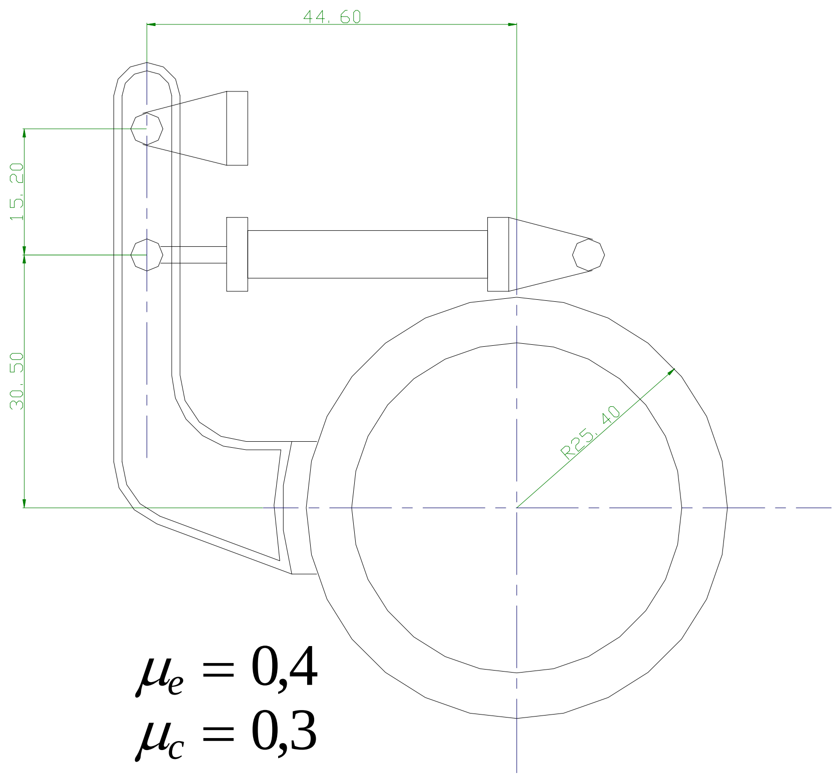
$$T_1 + U_1^2 = T_2 \Leftrightarrow 0 + U_1^2 = (T_2^A) \Leftrightarrow F \times 0{,}25 \times \Delta\beta = \frac{1}{2}0{,}4(25{,}1)^2$$

The hypothesis of no slipping:

$$\Delta S^A = \Delta S^B \Leftrightarrow r^A\Delta\beta = r^B\Delta\theta \Leftrightarrow \Delta\beta = 10{,}95(rad)$$

Solution: F=46(N)

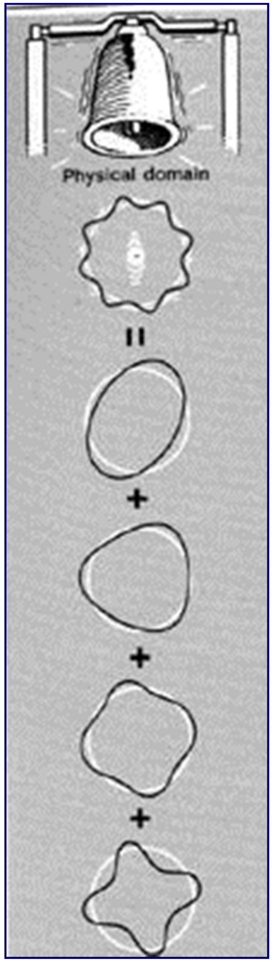
EXERCISE 17.10



$$\mu_e = 0,4$$
$$\mu_c = 0,3$$

A rotating body is breaking by means of a geometric defined arm, as represented in the figure. The rotating element has a radius of 254 [mm], and a inertial moment of 18,3 [kgm^2]. Knowing that the initial angular velocity is equal to 180 [rpm], anticlockwise, determine the force that must be applied by the hydraulic jack to stop the system after 50 revolutions.

VIBRATIONS



Vibrations are consequence of particular processes, where dynamic forces excite the structures or elements.

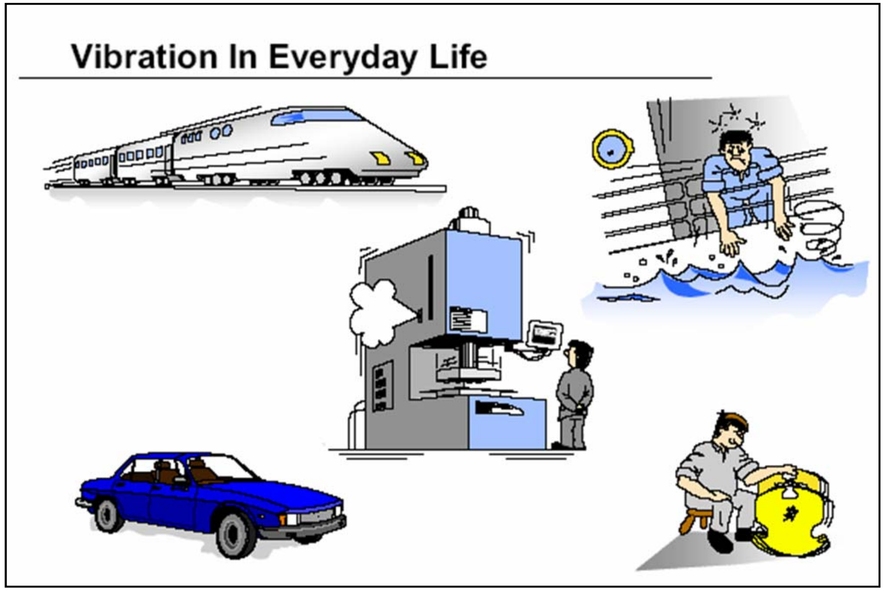
In machines, cars and buildings, those effects may lead to a decreasing in efficiency, bad function, lose of control and severe irreversible problems.

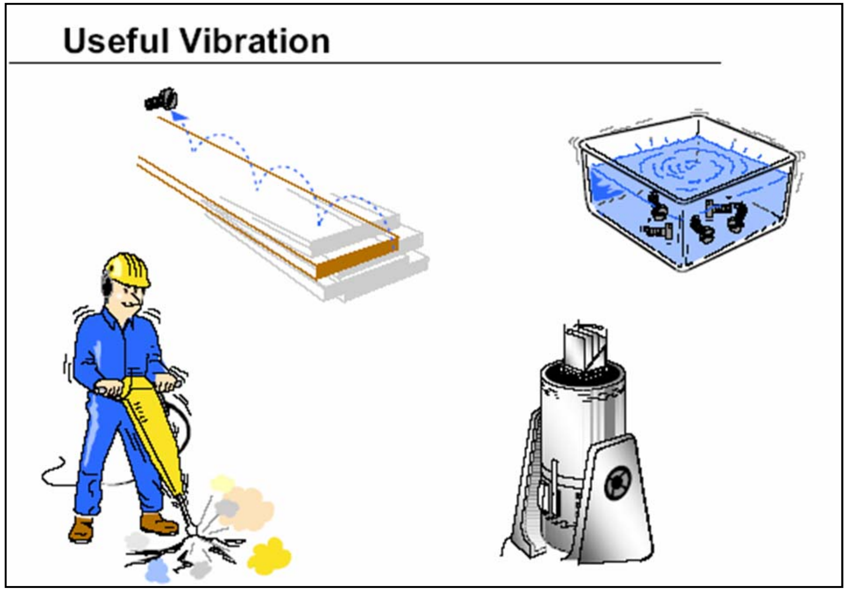
Vibrations and noise are related. The noise is part of vibrational energy transformed on to air pressure variation.

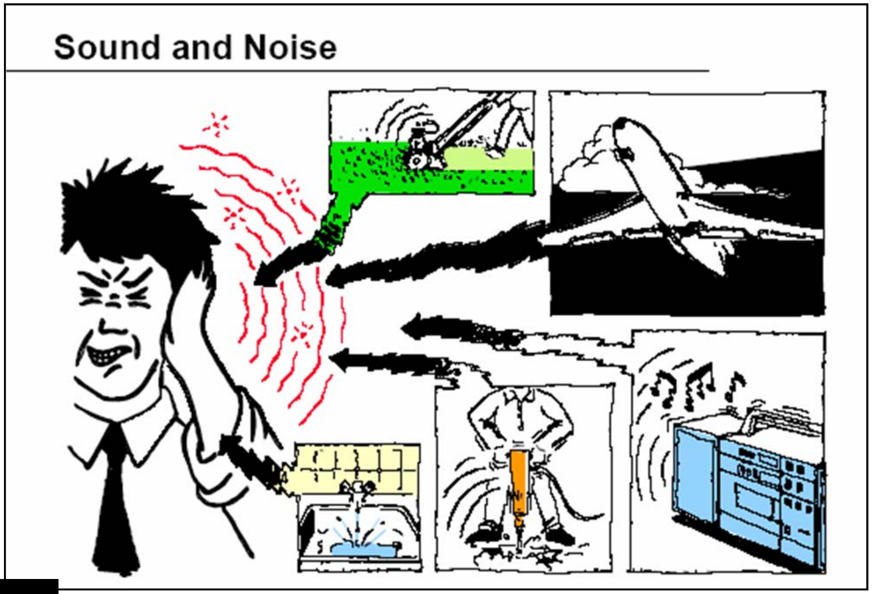
The major problems in vibration occur by resonance phenomenon. This problem may occur when the dynamic forces excite the natural frequencies or modes of vibration in the neighbourhood of the structural element.

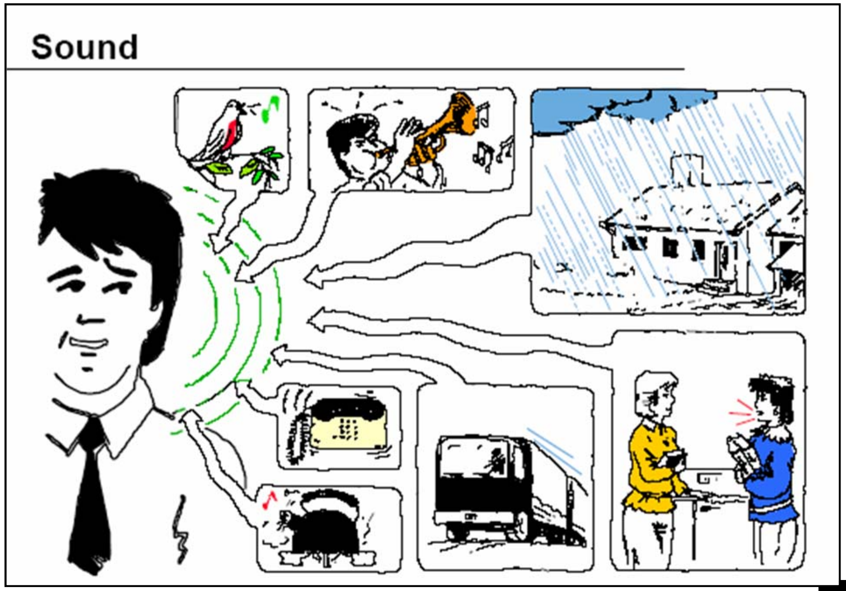
Modes and natural frequencies are characteristics from the geometric body.

VIBRATIONS – (UN) HELPFUL CASES

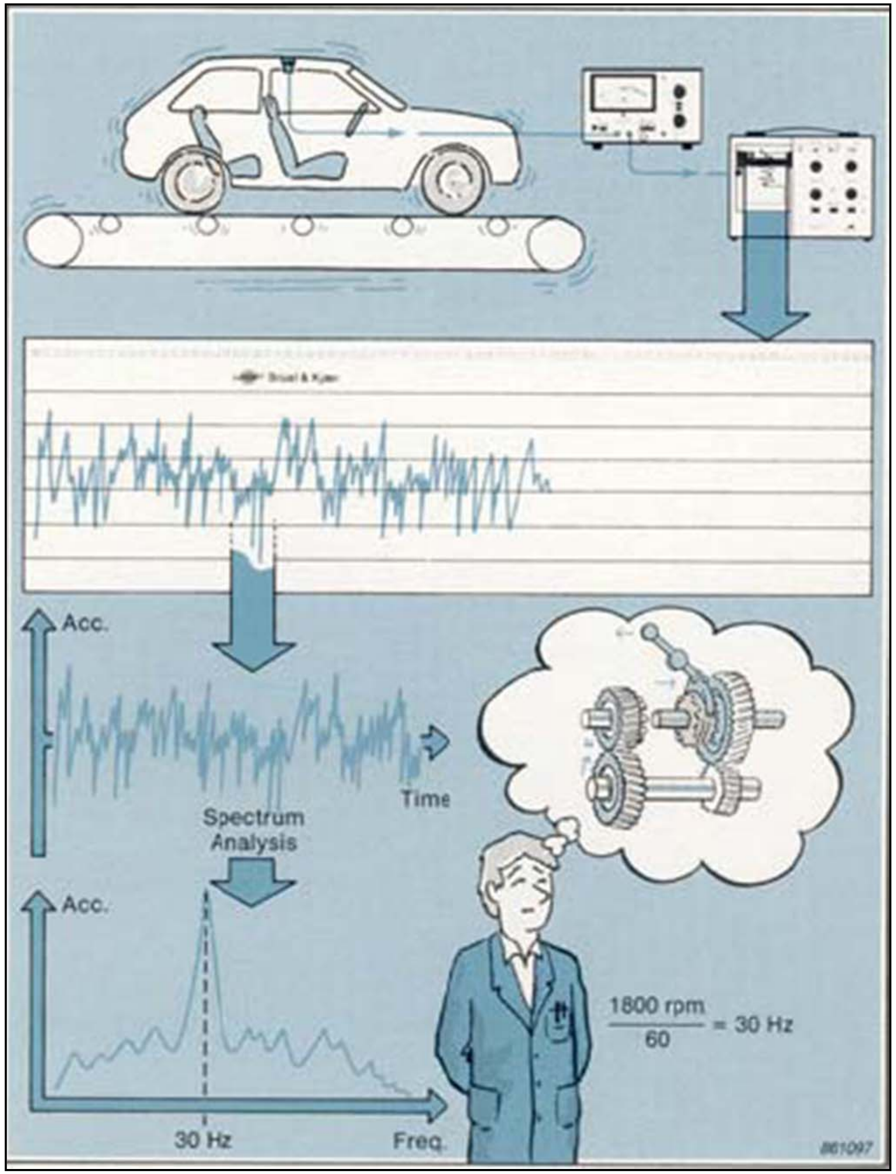








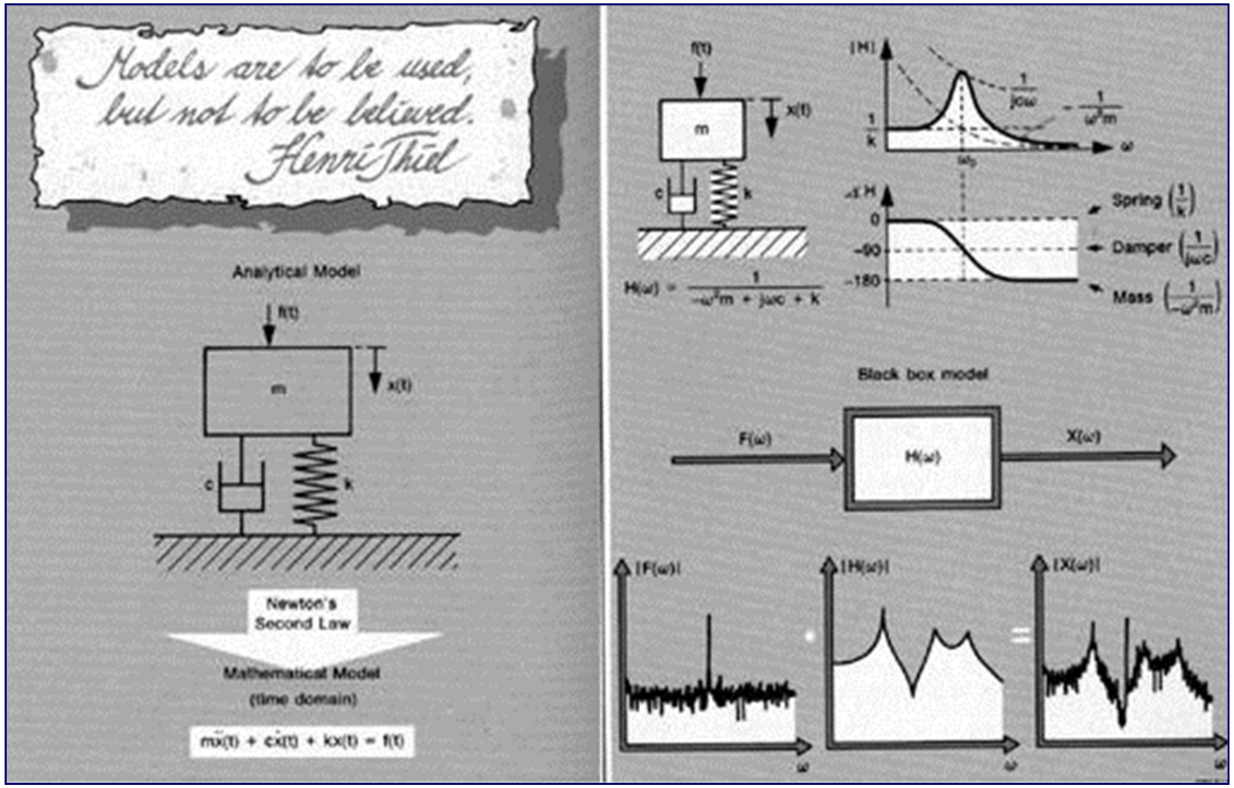
PROBLEM DETECTION IN VIBRATIONS



During tests, the results observation in time domain does not give so much useful information, however, if that information is treated in the frequency domain, may help to identify the energy concentration for some specific frequencies.

From kinematics analysis, the mechanical engineer may identify the mechanical element with its different natural frequencies, identifying the problem source.

MATHEMATICAL MODEL

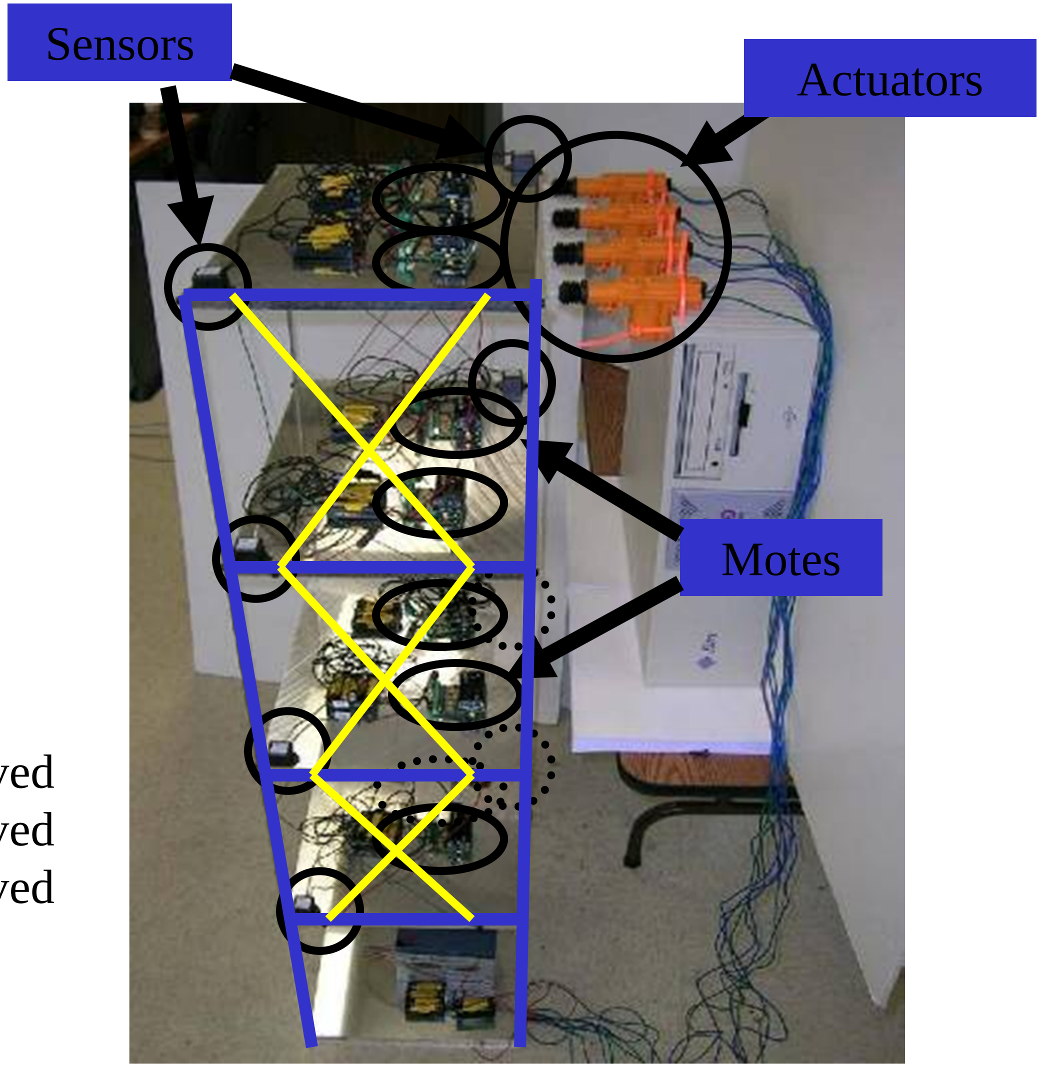


Mathematical models may be useful to simulate the behaviour of the element, when submitted to exterior actions and modifications to dynamic characteristics, promoted by geometric changes.

EXPERIMENTAL MODEL (Univ. Southern California)

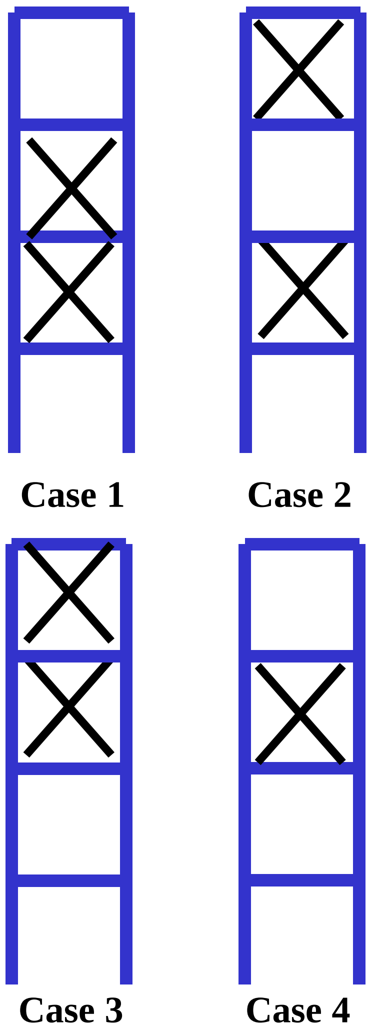
- ***Building Details***
  - 48 inches high, 4 floors, 60 lbs
  - Floors –1/2 x 12 x 18 aluminum plates
  - steel 1/2 x 1/8 inch steel columns
  - 5.5 lb/inch spring braces
  - 4 actuators on the top floor
  - 8 motes, 2/floor
  - dual axis, 200Hz, 2 starGates

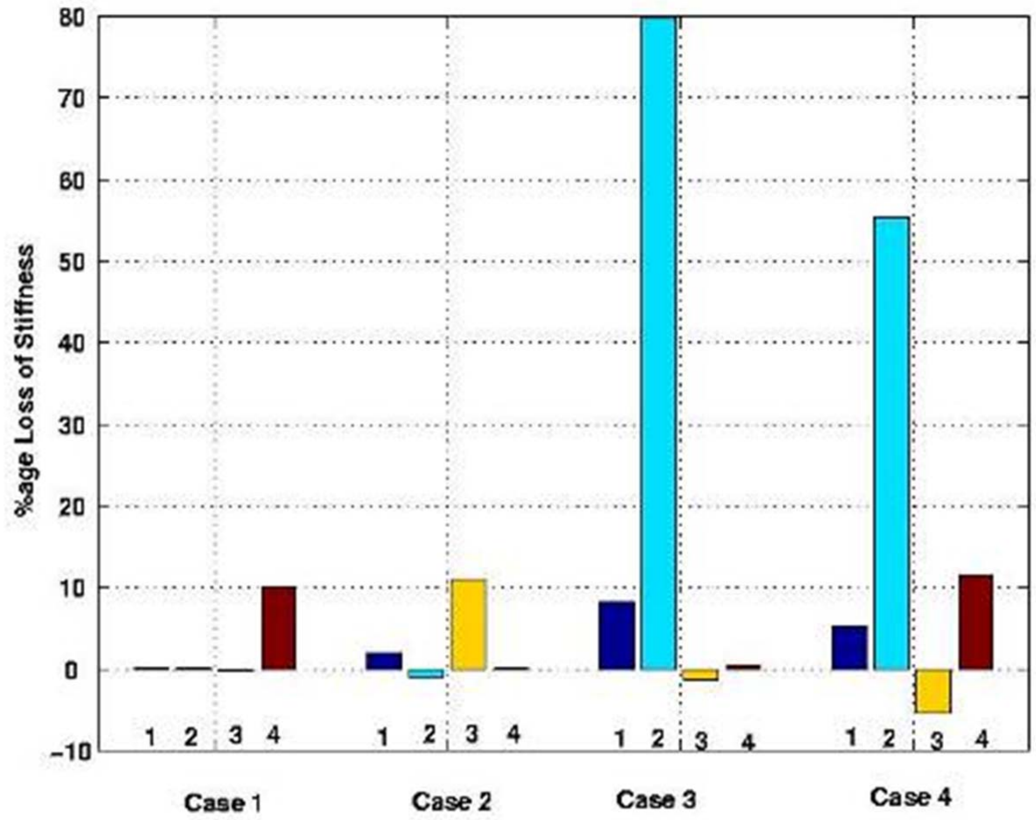
- ***4 Test Cases***
  - Case 1: braces from floor 4 removed
  - Case 2: braces from floor 3 removed
  - Case 3: braces from floor 2 removed
  - Case 4: braces from floor 2 and 4 removed



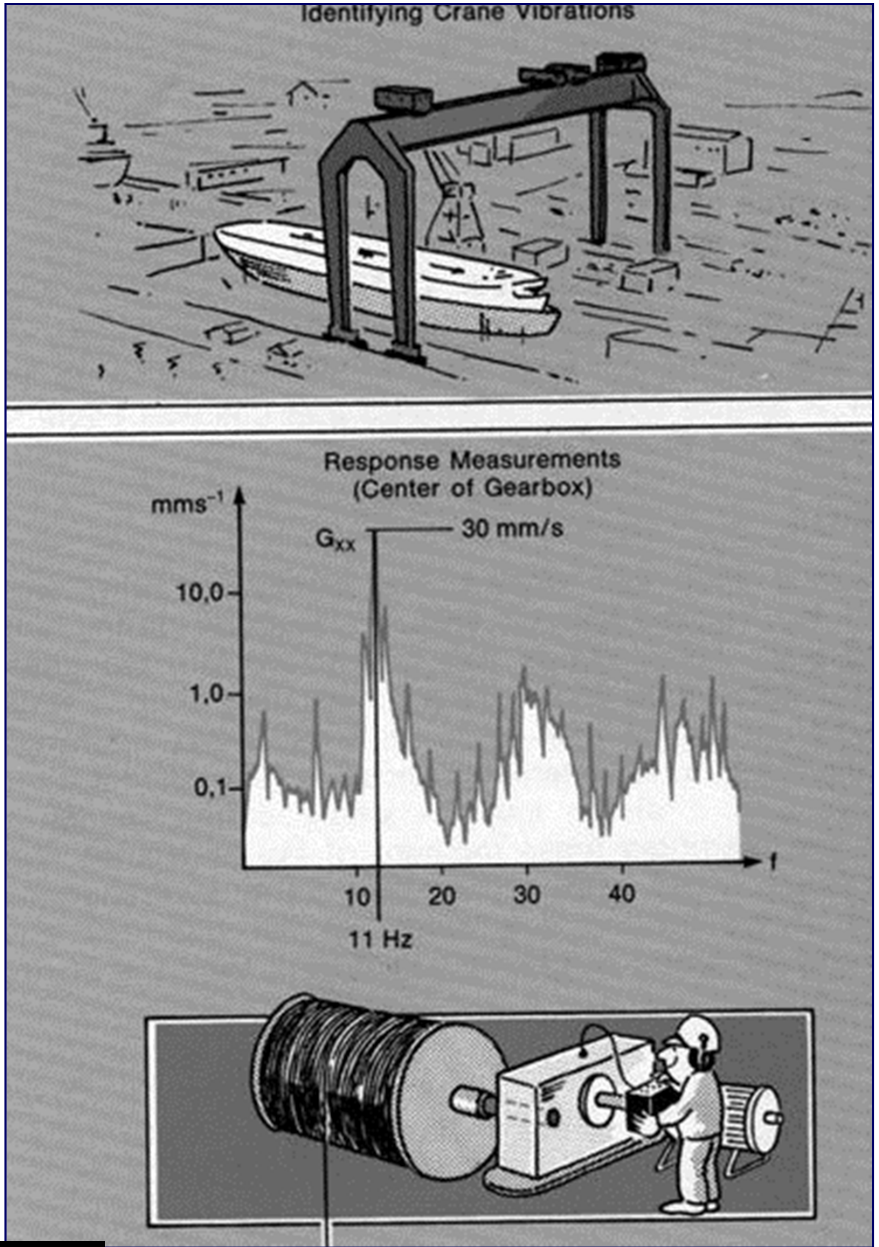
EXPERIMENTAL MODEL (Univ. Southern California)

- Stiffness reduction due to the removal of bracing systems to the frame structure.





CASE STUDY

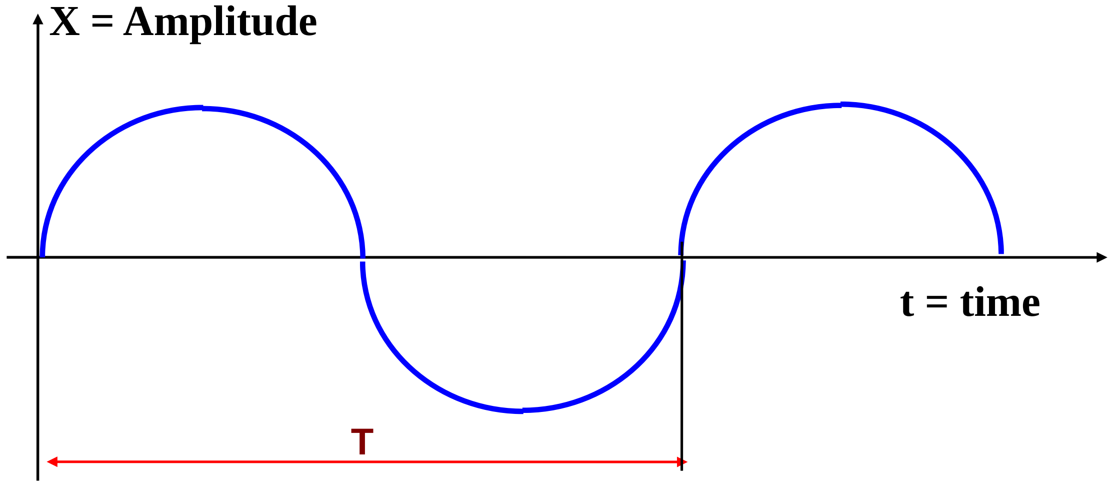


Problem: The portal frame presented in the picture, presents a vibration problem, when moving the load. The maintenance could stop and verify the source of the problem, or let go on till collapse. This dilemma and also the uncertain questions, if the force level generated by the gear motor was very high or if those level forces were amplified by resonance of the gear motor structure, let the engineer think about the problem and try to do a diagnostic.

Analysis: An experimental measurement (accelerometer) produces a spectrum like in the picture.

Solution: A piece of the gear train was collapsed and produce instability on the rotation procedure.

PERIODIC PHENOMENON



Characteristics:

T = period

f = frequency = 1/T (Hz)

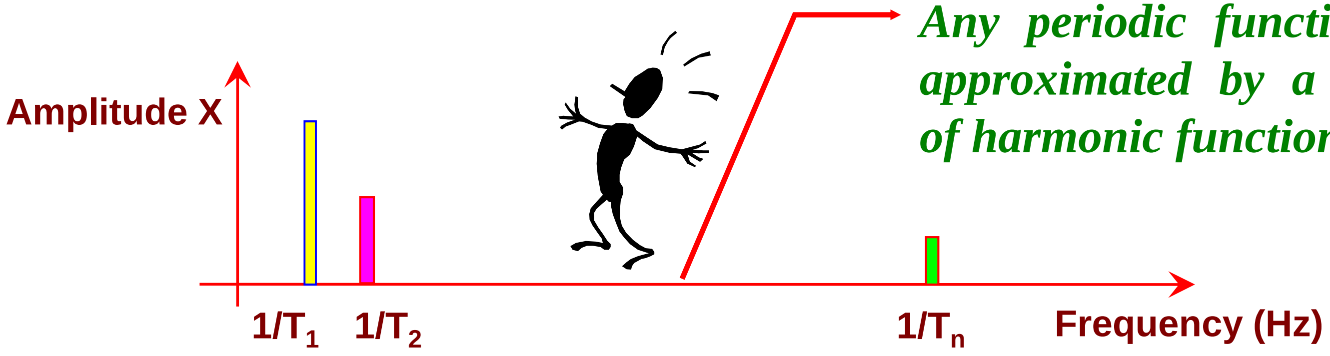
w = angular frequency = $2\pi f$

ϕ= phase

Answer in time domain: $x = X \sin(wt + \phi)$

Answer in frequency domain:

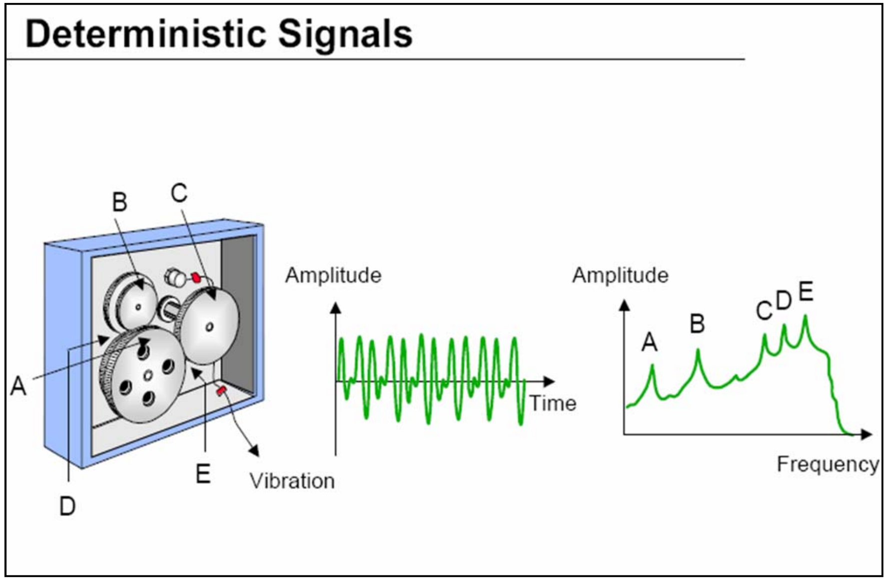
$F(t) = X_0 + X_1 \sin(w_1 t + \phi_1) + X_2 \sin(w_2 t + \phi_2) + \ldots + X_n \sin(w_n t + \phi_n)$

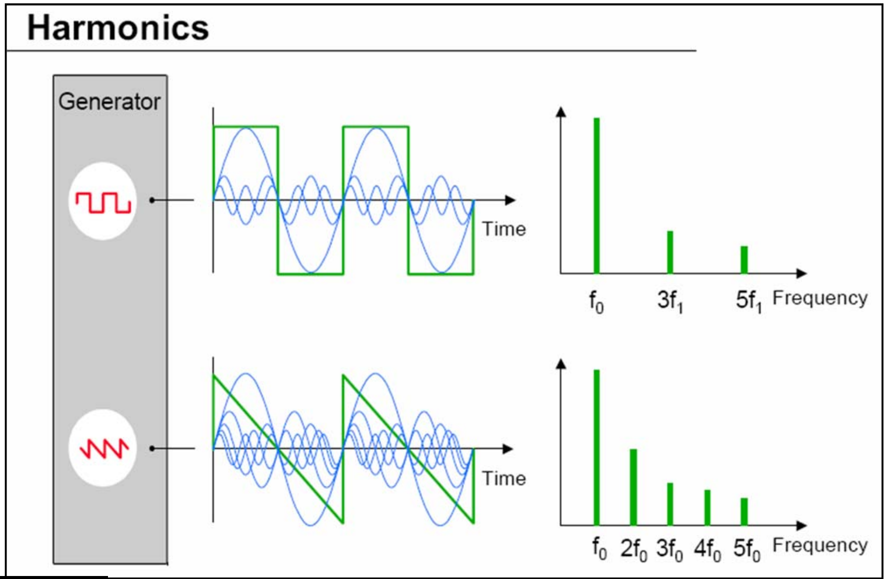


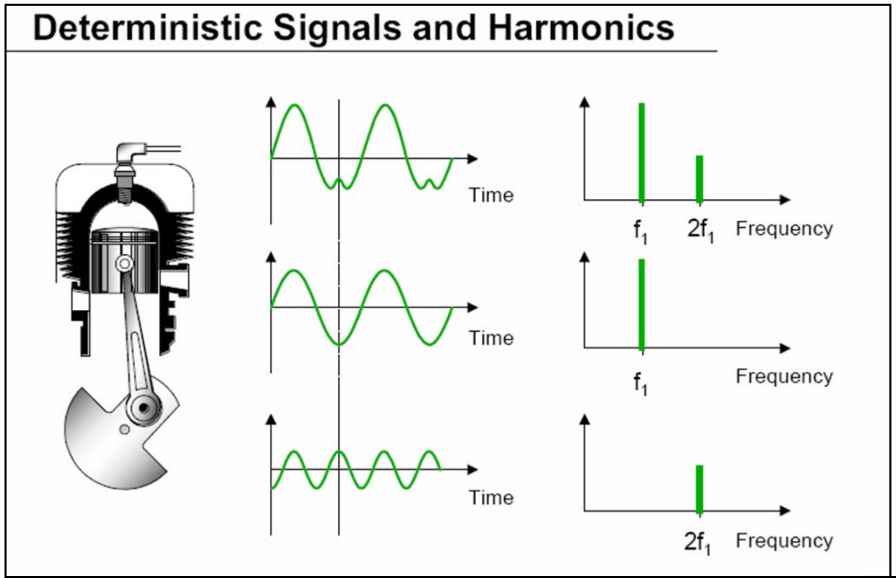
Any periodic function may be approximated by a summation of harmonic functions.

SIGNALS AND HARMONICS

- The motion of a mechanical system can consist of a single component at a single frequency or it can consist of several components occurring at different frequencies simultaneously, as for example with the piston motion of an internal combustion engine.

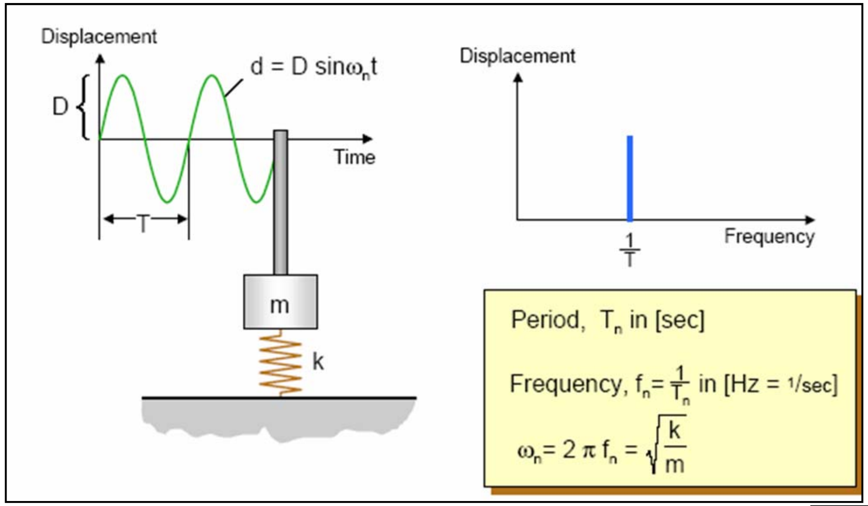


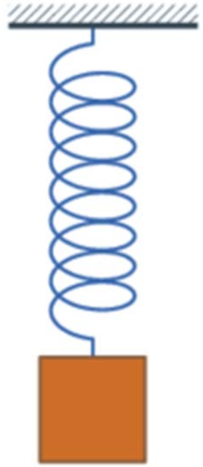


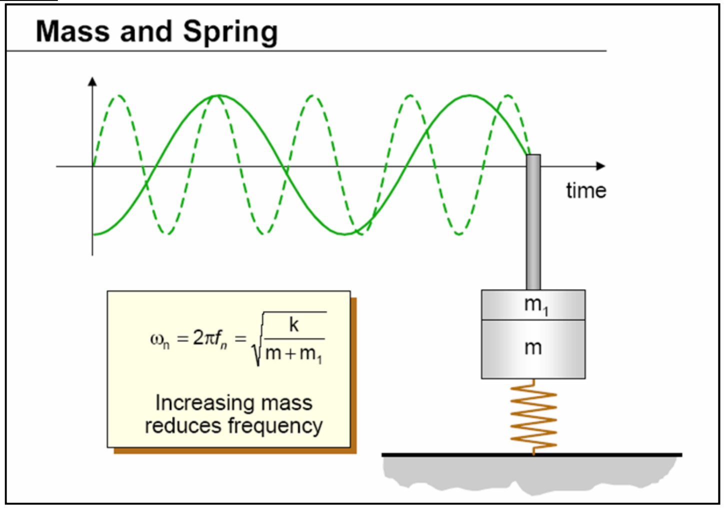


SIMPLEST FORM OF VIBRATION SYSTEM

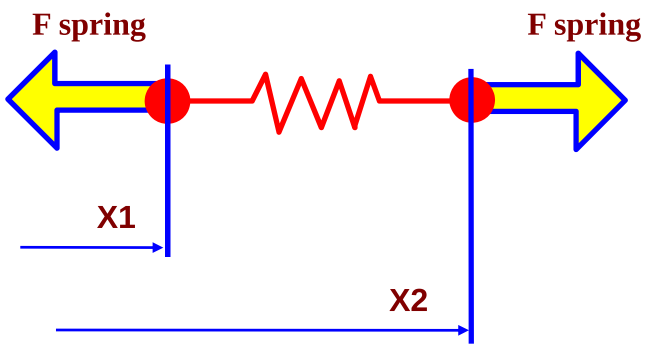
- Once a (theoretical) system of a mass and a spring is set in motion it will continue this motion with constant frequency and amplitude. The system is said to oscillate with a sinusoidal waveform.

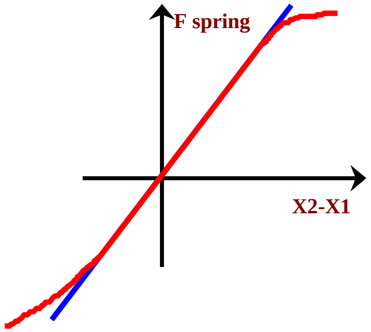


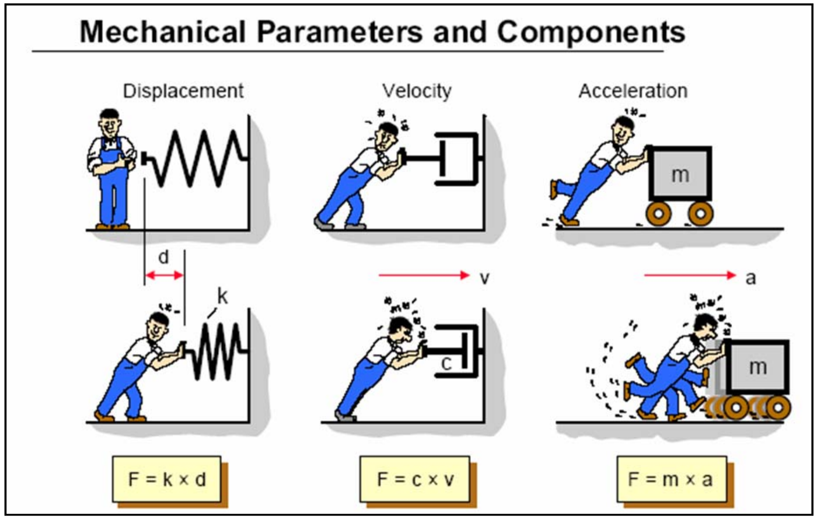


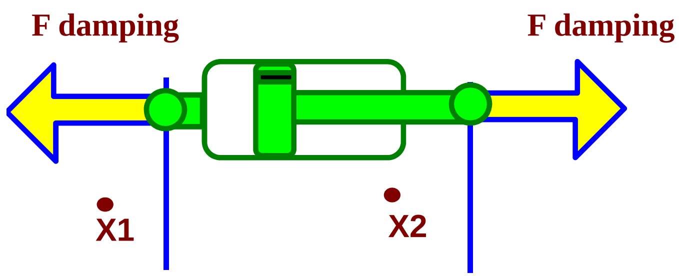


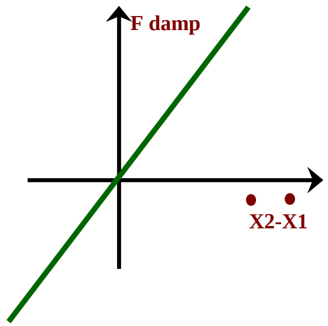
MECHANICAL DISCRETE SYSTEMS

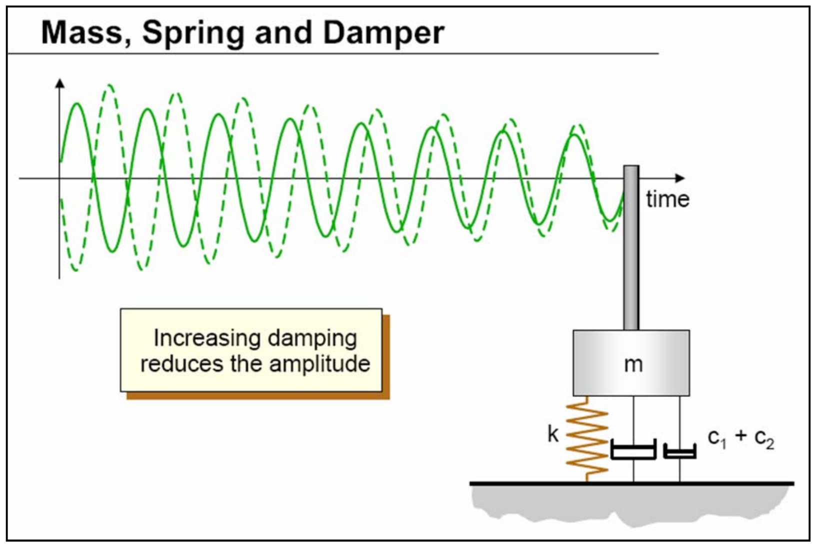


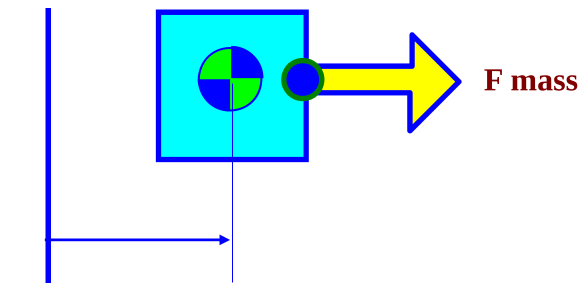


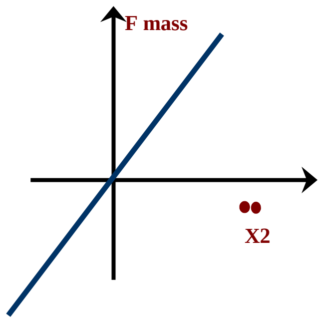




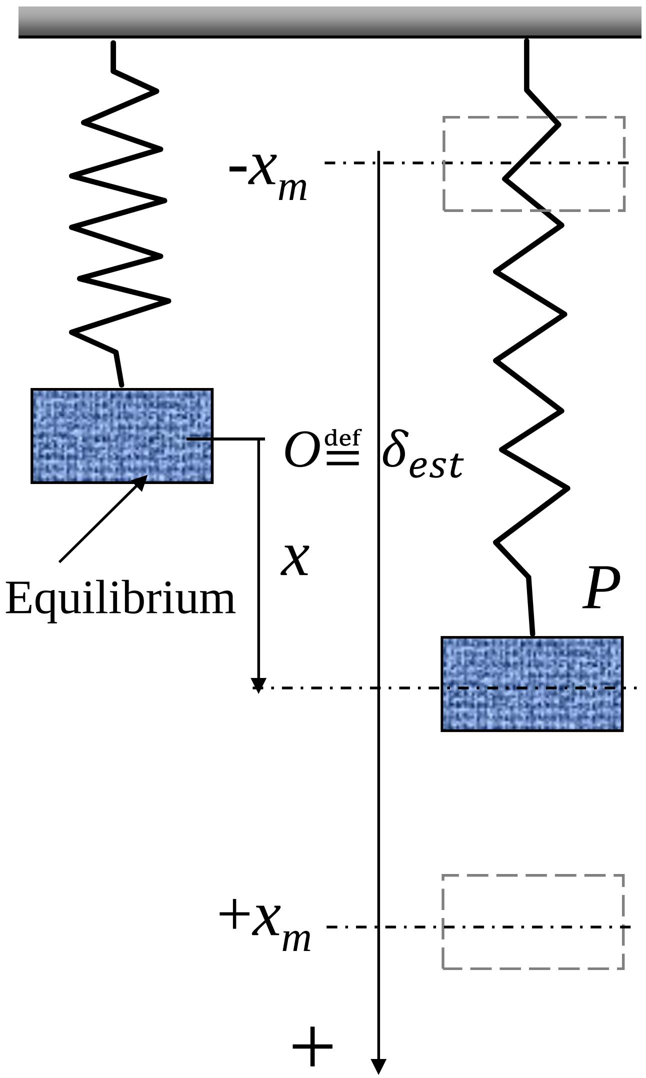








MECHANICAL VIBRATIONS



Consider the ***free vibration of a particle***, i.e., the motion of a particle *P* subjected to a restoring force proportional to the displacement of the particle - such as the force exerted by a spring.

If the displacement *x* of the particle *P* is measured from its equilibrium position *O*, the resultant **F** of the forces acting on *P* (including its weight) has a magnitude *kx* and is directed toward *O*.

Applying Newton's second law ($F = ma$) with $a = \ddot{x}$, the differential equation of motion is:

$$m\ddot{x} + kx = 0$$

Recall statics and dynamics:

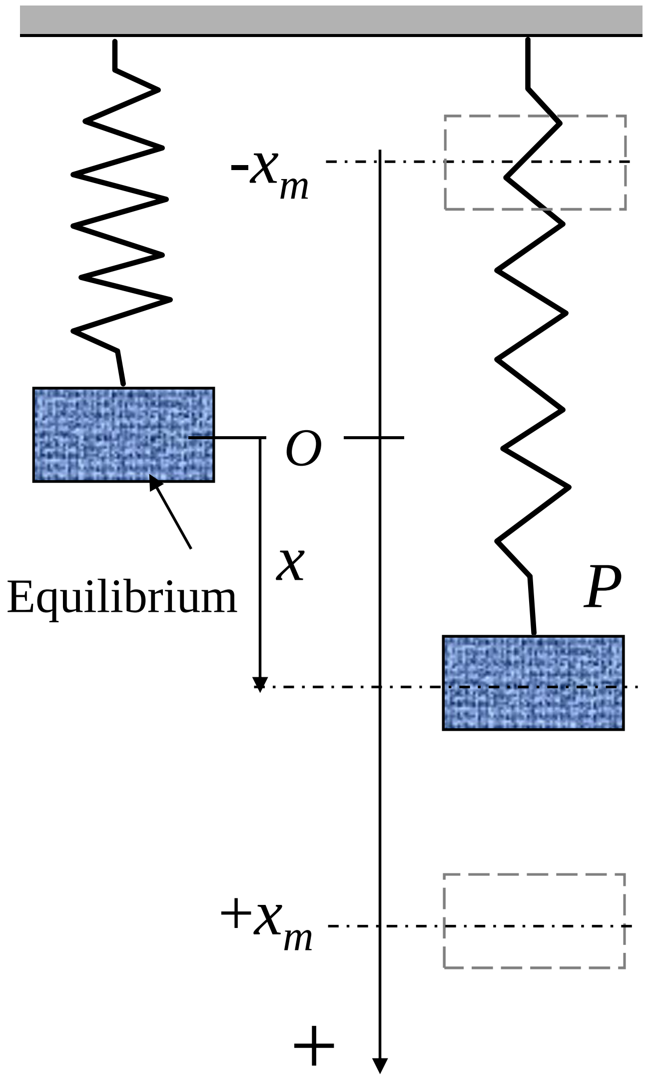
$\sum \vec{F} = \vec{0}$

$\sum \vec{F} = m\vec{a}$

mg-kδ_{est}=0

mg-k(δ_{est}+x)=m$\ddot{x}$

MECHANICAL VIBRATIONS



$$m\ddot{x} + kx = 0$$

setting $\omega_n^2 = k/m$

$$\ddot{x} + \omega_n^2 x = 0$$

The motion defined by this expression is called ***simple harmonic motion.***

The solution of this equation, which represents the displacement of the particle *P* is expressed as:

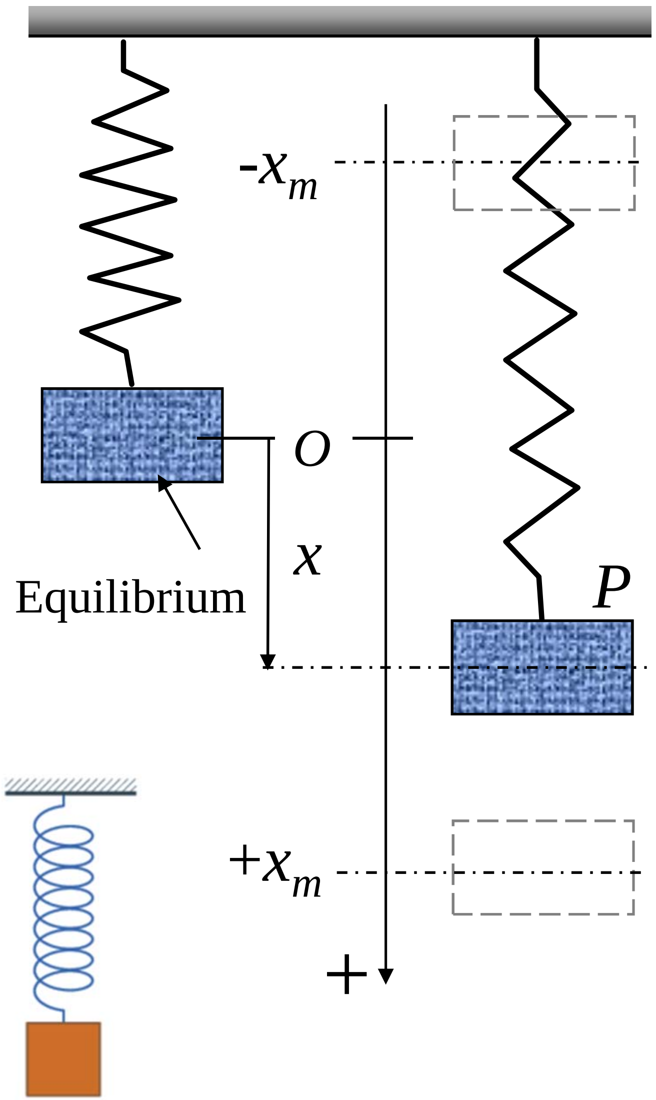
$$x = x_m \sin(\omega_n t + \phi)$$

Where: x_m = amplitude of the vibration

$w_n = \sqrt{k/m}$ = natural circular frequency

ϕ = phase angle

MECHANICAL VIBRATIONS



Recalling:

$$\ddot{x} + \omega_n^2 x = 0$$

$$x = x_m \sin(\omega_n t + \phi)$$

The ***period of the vibration*** (i.e., the time required for a full cycle) and its *frequency* (i.e., the number of cycles per second) are expressed as:

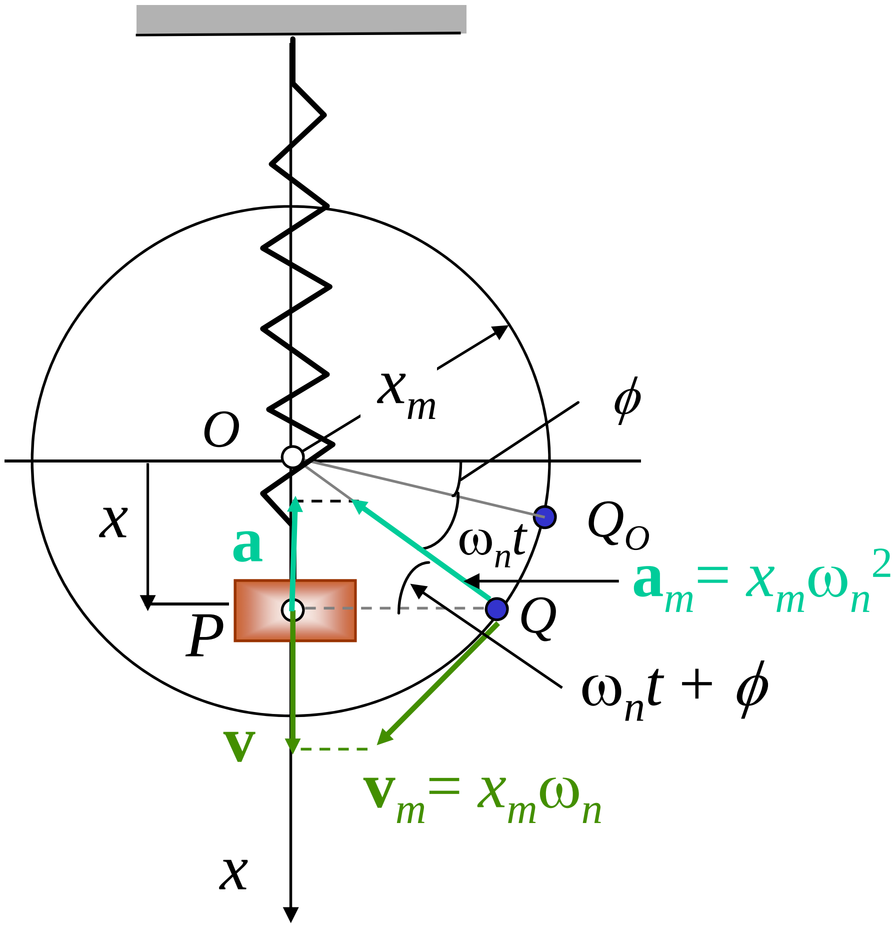
$$\text{Period} = T_n = \frac{2\pi}{\omega_n}$$

$$\text{Frequency} = f_n = \frac{1}{\tau_n} = \frac{\omega_n}{2\pi}$$

The velocity and acceleration of the particle are obtained by differentiating *x*, and their maximum values are:

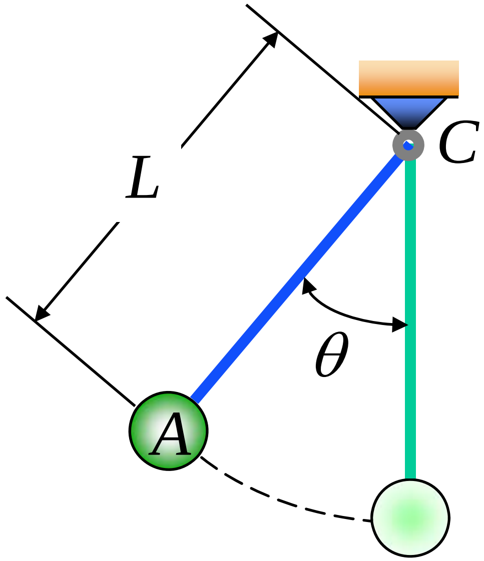
$$v_m = x_m \omega_n \qquad a_m = x_m \omega_n^2$$

OSCILLATORY MOTION



The ***oscillatory motion*** of the particle *P* may be represented by the projection on the *x* axis of the motion of a point *Q* describing an auxiliary circle of radius x_m with the constant angular velocity ω_n.

The instantaneous values of the velocity and acceleration of *P* may then be obtained by projecting on the *x* axis the vectors $\mathbf{v}_m$ and $\mathbf{a}_m$ representing, respectively, the velocity and acceleration of *Q*.



SIMPLE PENDULUM

While the motion of a ***simple pendulum*** is not truly a simple harmonic motion, the formulas given above may be used with $\omega_n{}^2 = g/L$ to calculate the period and frequency of the *small oscillations* of a simple pendulum.

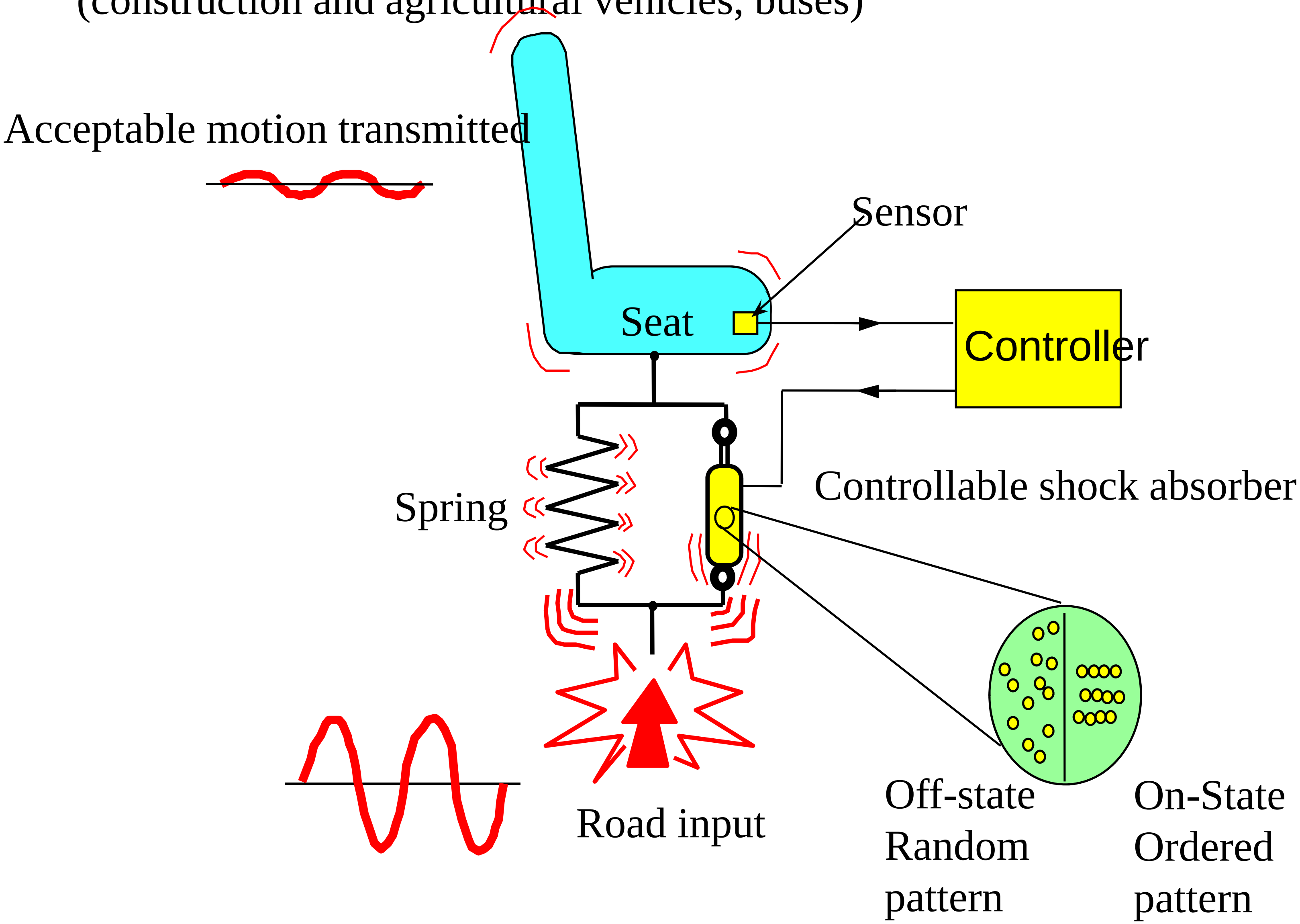
The *free vibrations of a rigid body* may be analyzed by choosing an appropriate variable, such as a distance x or an angle θ, to define the position of the body, drawing a diagram expressing the equivalence of the external and effective forces, and writing an equation relating the selected variable and its second derivative. If the equation obtained is of the form

$$\ddot{x} + \omega_n{}^2 x = 0 \qquad \text{or} \qquad \ddot{\theta} + \omega_n{}^2 \theta = 0$$

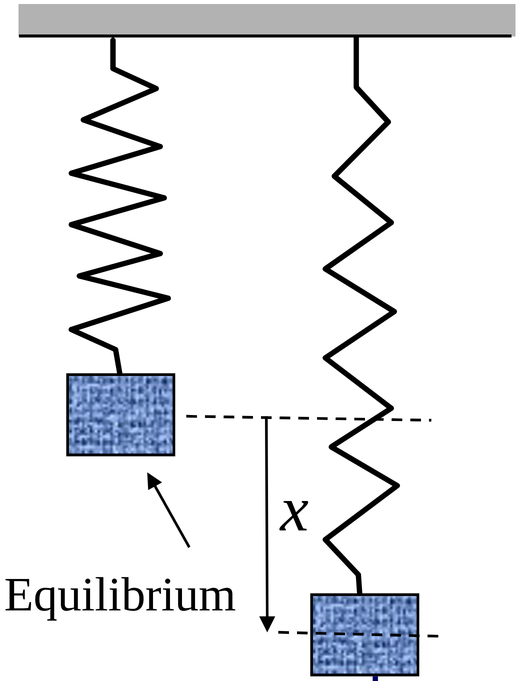
the vibration considered is a simple harmonic motion and its period and frequency may be obtained.

FORCED VIBRATION WITH A SINGLE D.O.F.

- Single Degree of Freedom System - Heavy Duty Vehicle Suspended Seats (construction and agricultural vehicles, buses)



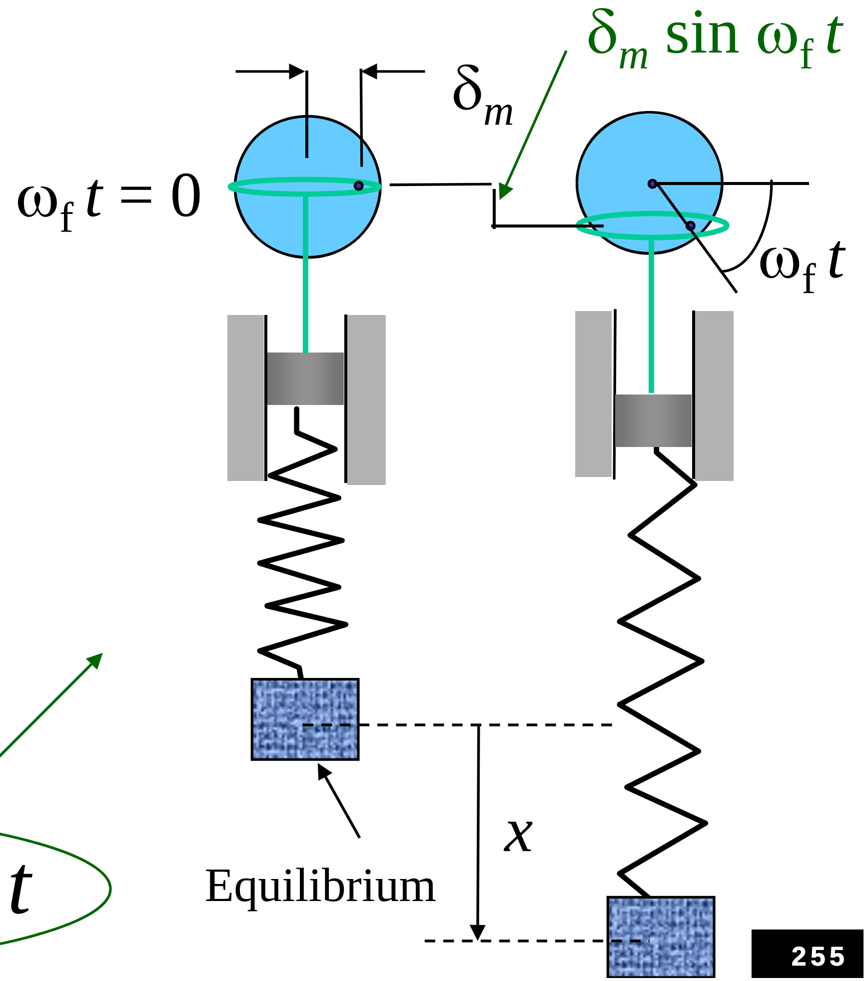
FORCED VIBRATION



$\mathbf{P} = P_m \sin \omega_f t$

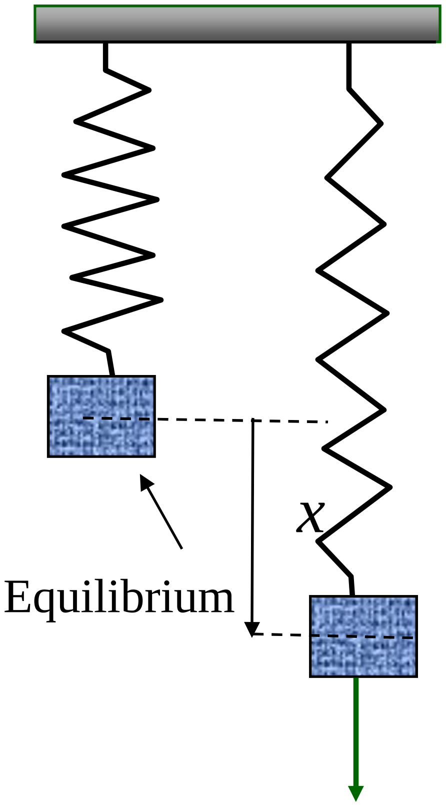
The ***forced vibration*** of a mechanical system occurs when the system is subjected to a periodic force or when it is elastically connected to a support which has an alternating motion. The differential equation describing each system is presented below.

$$m\ddot{x} + kx = P_m \sin \omega_f t$$



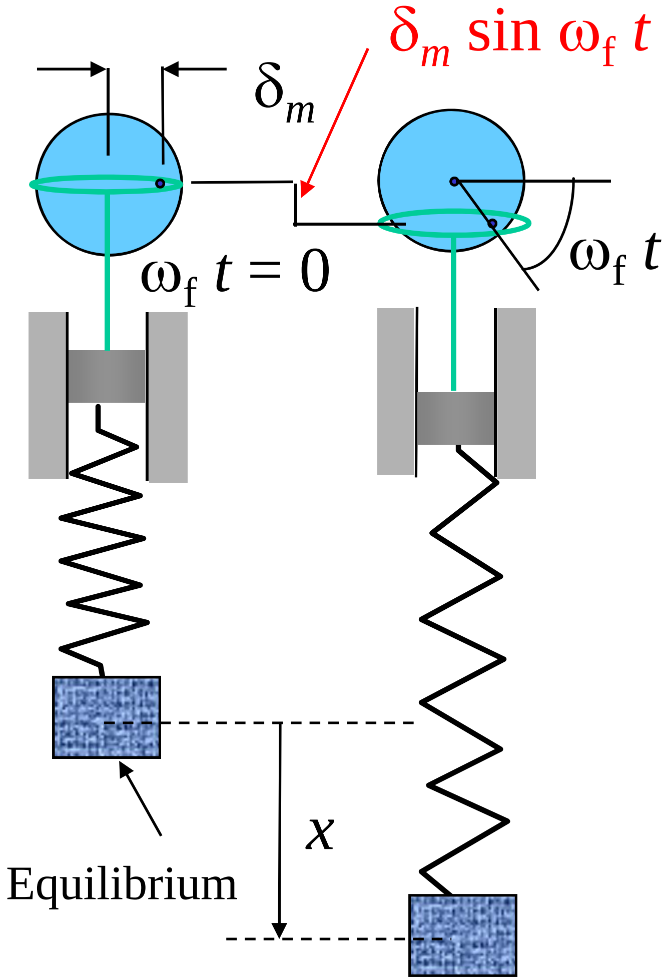
$$m\ddot{x} + kx = k\delta_m \sin \omega_f t$$

FORCED VIBRATION



$$m\ddot{x} + kx = P_m \sin \omega_f t$$

$$m\ddot{x} + kx = k\delta_m \sin \omega_f t$$



$\mathbf{P} = P_m \sin \omega_f t$

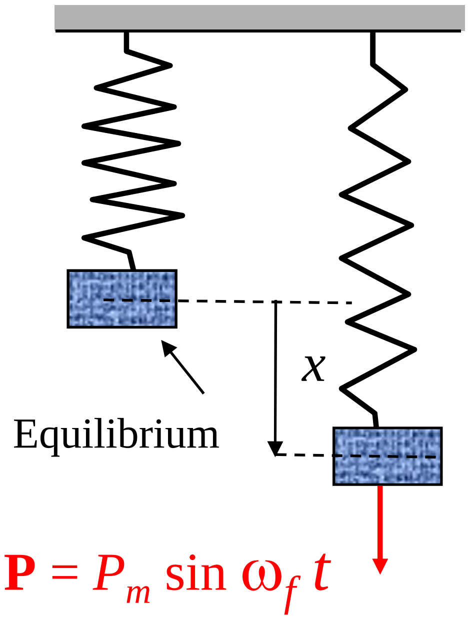
The solution of these equations is obtained by adding <u>a particular solution of the form:</u>

$$x_{\text{part}} = x_m \sin \omega_f t$$

<u>to the general solution.</u>

- ***The <u>general solution</u> of the corresponding homogeneous equation represents a transient free vibration which may generally be neglected.***
- ***The <u>particular solution</u> represents the steady-state vibration of the system.***

MAGNIFICATION FACTOR

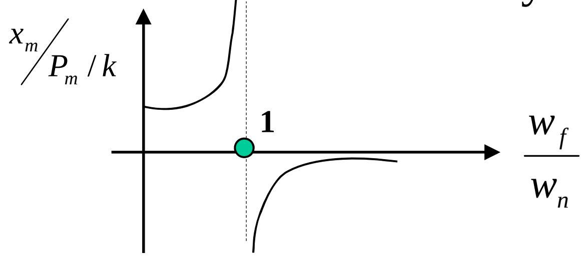


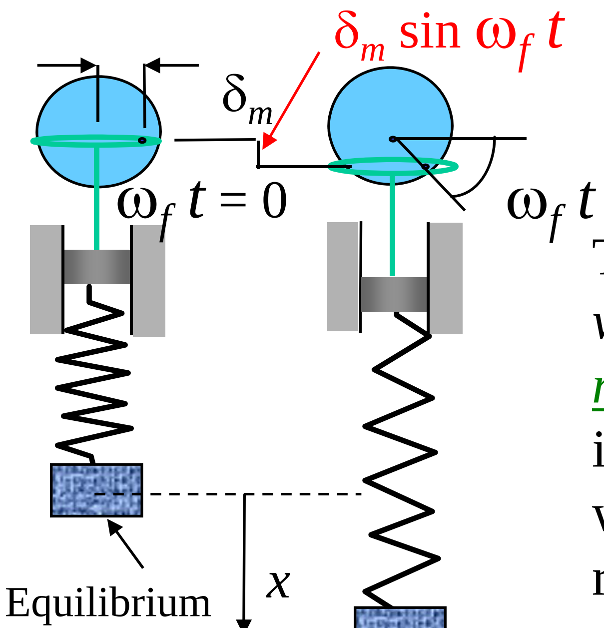
$$m\ddot{x} + kx = P_m \sin \omega_f t \qquad m\ddot{x} + kx = k\delta_m \sin \omega_f t$$

-Substituting x_{part}, $\dot{x}_{part,}$ $\ddot{x}_{part}$ into each of the differential equation, and recalling the definition of natural frequency ($k/m = w_n^2$).

-Dividing the amplitude x_m of the steady-state vibration by P_m/k in the case of a periodic force, or by δ_m in the case of an oscillating support, the <u>*magnification factor*</u> of the vibration is defined by:

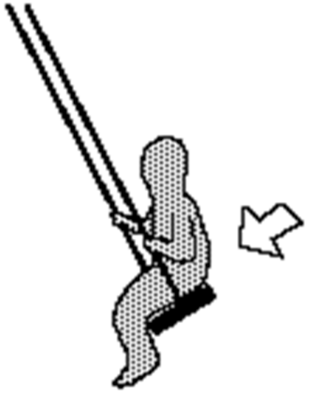
$$\frac{x_m}{P_m/k} = \frac{x_m}{\delta_m} = \frac{1}{1 - \left(\dfrac{w_f}{w_n}\right)^2}$$



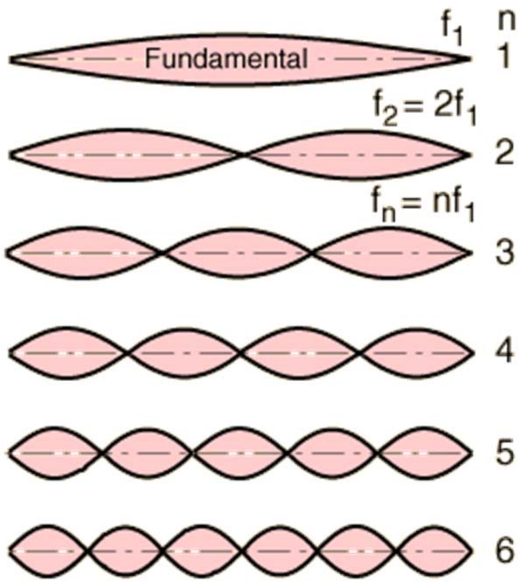


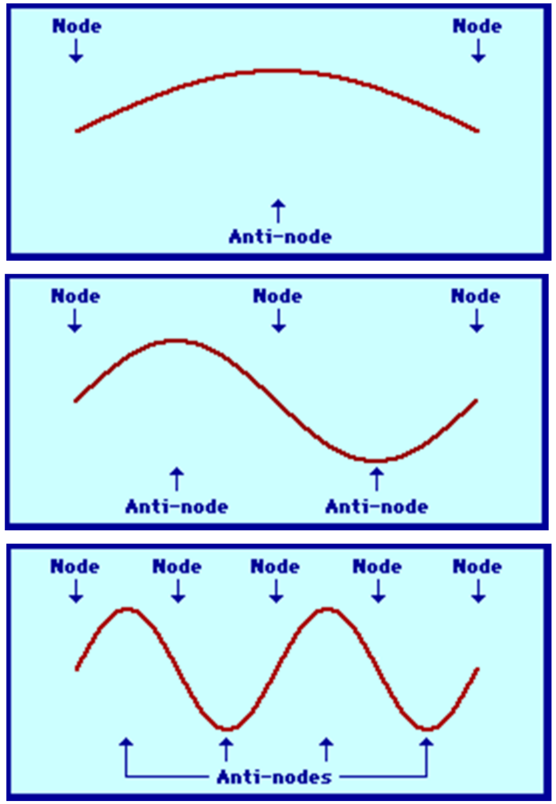
The amplitude x_m of the forced vibration *becomes infinite when* $\omega_f = \omega_n$, i.e., <u>*when the forced frequency is equal to the natural frequency of the system*</u>. The impressed force or impressed support movement is then said to be in <u>***resonance***</u> with the system. Actually the amplitude of the vibration remains finite, due to damping forces.

RESONANCE VIBRATIONS



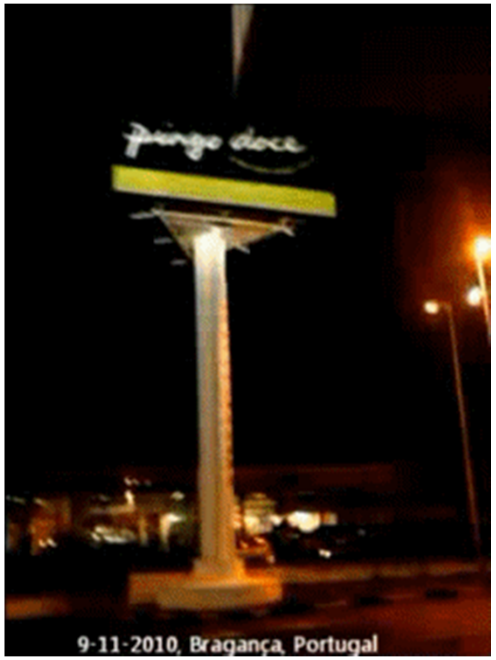
- Swinging a child in a playground is an easy job because you are helped by its natural frequency.
- With a tiny push on the swing each time it comes back to you, you can continue to build up the amplitude of swing.
- If someone try to force it to swing a twice that frequency, it will find it very difficult.





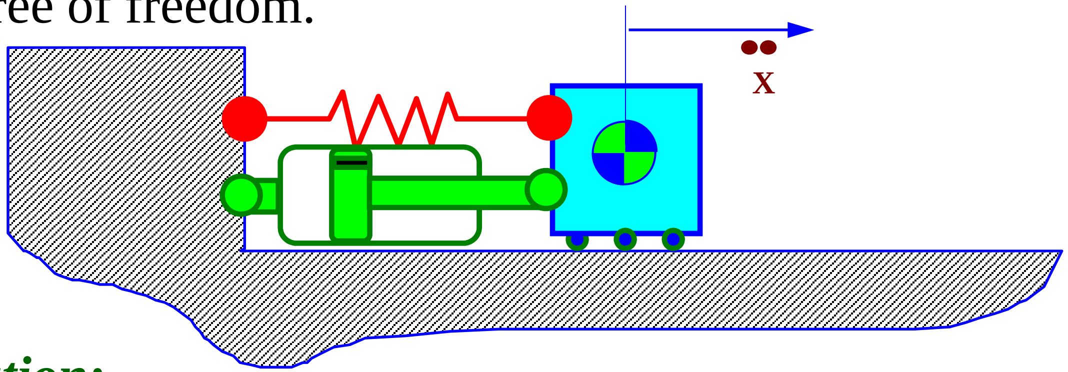
RESONANCE VIBRATIONS

- Old story: “Napolean army cross bridge with organized step, being this step frequency equal to the bridge natural frequency, increasing the amplitude of vibration, producing collapse.
- Real Story: Tacoma, USA, November, the 7th of 1940. The wind movement excite bridge natural frequencies (first the longitudinal modes and then the torsional modes, most critical).
- Real Story: Pingo Doce supermarket, Bragança, Portugal, 9-11-2010. The wind speed excite natural frequency (torsional mode).



FREE VIBRATIONS WITH DAMPING

***Academic example*:** Mass, spring and shock absorber system in a horizontal plane, with one degree of freedom.



Dynamic equation:

$$\sum F = m\ddot{X} \Leftrightarrow \{ -KX - C\dot{X} = m\ddot{X}$$

Movement equation:

$$m\ddot{X} + C\dot{X} + KX = 0$$

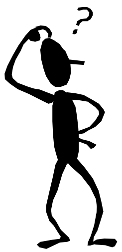
Possible solution:

$$X = Ae^{st}$$

A – constant to be determined

S – Characteristic of the present system

SOLUTION VERIFICATION



$$X = Ae^{st}$$

$$\dot{X} = A(s)e^{st}$$

$$\ddot{X} = A(s^2)e^{st}$$

Being a solution, the movement equation should be verified.

$$mA(s^2)e^{st} + CA(s)e^{st} + KAe^{st} = 0$$

Any solution $Ae^{st} \neq 0$ different from zero, leads to:

$$ms^2 + Cs + K = 0$$

Second order polynomial, may produce two independent solutions:

$$\begin{matrix} s_1 \\ s_2 \end{matrix} = \frac{-C}{2m} \pm \sqrt{\left(\frac{C}{2m}\right)^2 - \frac{K}{m}}$$

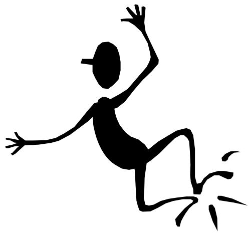
$$X_1 = A_1 e^{s_1 t}$$

$$X_2 = A_2 e^{s_2 t}$$

SOLUTION VERIFICATION

Any combination of those two solutions, may be considered also a solution

$$X(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}$$



Notes:

1- The constants A_1 and A_2 may be calculated as a function of the initial conditions.

2- The characteristics S_1 and S_2 are function of the body geometry .

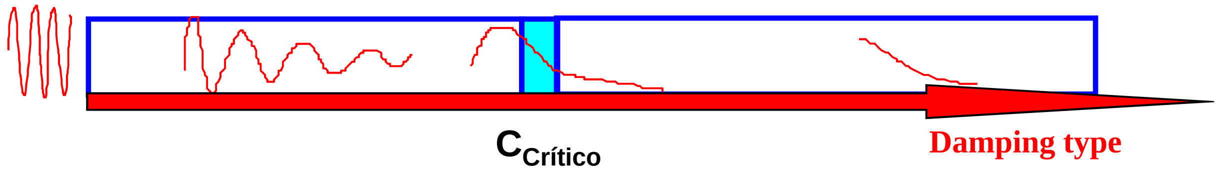
Solution Analysis: Characteristics of S_1 and S_2

$$\left(\frac{C}{2m}\right)^2 - \frac{K}{m} > 0$$ Real solutions, non oscillatory behaviour

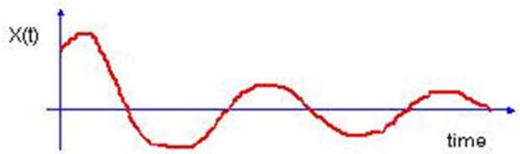
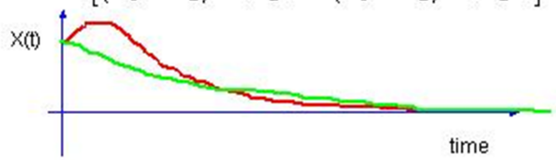
$$\left(\frac{C}{2m}\right)^2 - \frac{K}{m} = 0$$ Double real solutions, non oscillatory behaviour

$$\left(\frac{C}{2m}\right)^2 - \frac{K}{m} < 0$$ Complex solutions, oscillatory behaviour

DAMPING TYPE



$$\left(\frac{C_{Critico}}{2m}\right)^2 = \frac{K}{m} \Leftrightarrow C_{Critico} = 2m\sqrt{\frac{K}{m}} \Leftrightarrow C_{Critico} = 2m(W_n)$$

Damping type	Characteristics	Answer
Under damping	Damping coefficient $\beta = \frac{C}{C_{Critico}} = \frac{C}{2\sqrt{mK}}$ $\begin{matrix} s_1 \\ s_2 \end{matrix} = \frac{-C}{2m} \pm \sqrt{\left(\frac{C}{2m}\right)^2 - \frac{K}{m}} = -\beta W \pm \sqrt{(-1)\left[\left(\frac{K}{m}\right) - \left(\frac{C}{2m}\right)^2\right]}$ $= -\beta W \pm Wi\sqrt{(1-\beta^2)}$ Natural frequency $W_d = W\left(\sqrt{1-\beta^2}\right)$	$X(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}$ $= A_1 e^{(-\beta W + iw_d)t} + A_2 e^{(-\beta W - iw_d)t}$ $= e^{-\beta Wt}\left[\left(A_1 + A_2\right)\cos(W_d t) + i\left(A_1 - A_2\right)\sin(W_d t)\right]$ 
Critical damping		
Over damping	Damping coefficient $\beta = \frac{C}{C_{Critico}} = \frac{C}{2\sqrt{mK}}$ $\begin{matrix} s_1 \\ s_2 \end{matrix} = \frac{-C}{2m} \pm \sqrt{\left(\frac{C}{2m}\right)^2 - \frac{K}{m}} = -\beta W \pm \sqrt{\left[\left(\frac{K}{m}\right) - \left(\frac{C}{2m}\right)^2\right]}$ $= -\beta W \pm W\sqrt{(\beta^2 - 1)}$ Natural frequency $W_d = W\left(\sqrt{1-\beta^2}\right)$	$X(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}$ $= A_1 e^{(-\beta W + w_d)t} + A_2 e^{(-\beta W - w_d)t}$ $= e^{-\beta Wt}\left[\left(A_1 + A_2\right)ch(W_d t) + \left(A_1 - A_2\right)sh(W_d t)\right]$ 

three different cases of damping, namely,
(1) *heavy damping*, when $c > c_c$,
(2) *critical damping*, when $c = c_c$,
(3) *light damping*, when $c < c_c$.

In the first two cases, the system when disturbed tends to return to its equilibrium position without oscillation. In the third case, the motion is vibratory with diminishing amplitude.

DAMPING

The equation of motion describing the *damped free vibrations* of a system with *viscous damping* is

$$m\ddot{x} + c\dot{x} + kx = 0$$

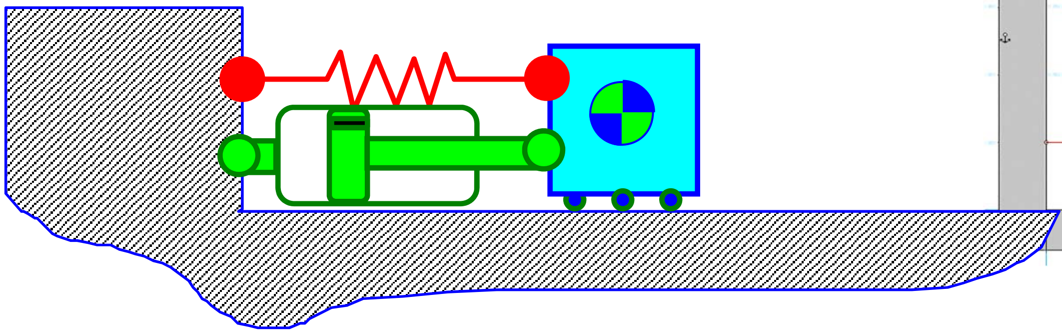
where c is a constant called the ***coeficient of viscous damping***. Defining the *critical damping coefficient* c_c as

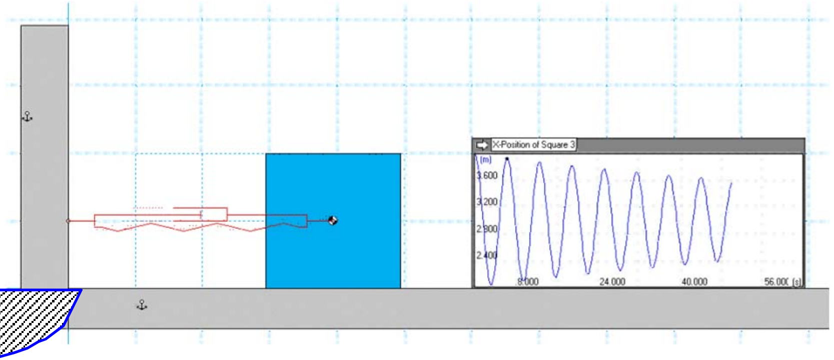
$$c_c = 2m\sqrt{\frac{k}{m}} = 2m\omega_n$$

where ω_n is the natural frequency of the system in the absence of damping, we distinguish three different cases of damping, namely:

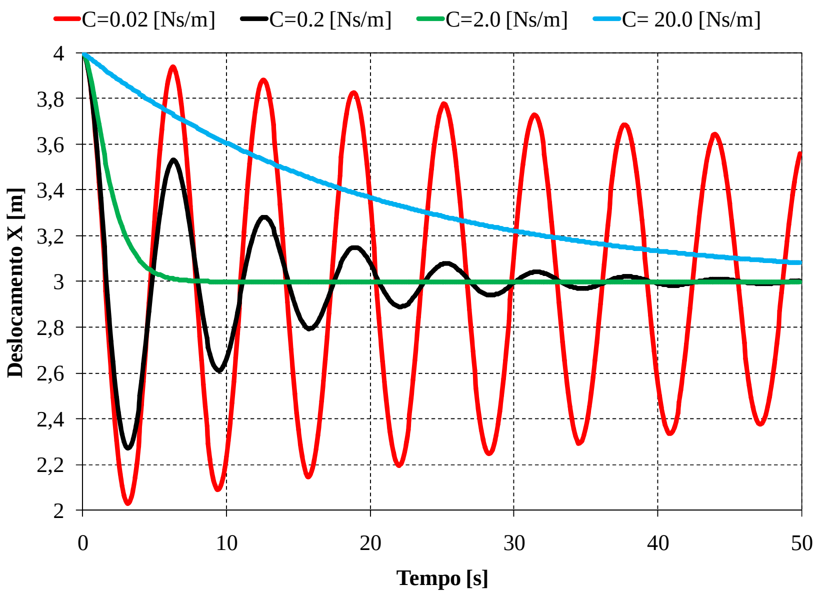
(1)heavy damping, when $c > c_c$,
(2)critical damping, when $c = c_c$,
(3)light damping, when $c < c_c$.

FREE VIBRATIONS WITH DAMPING – EXERCISE

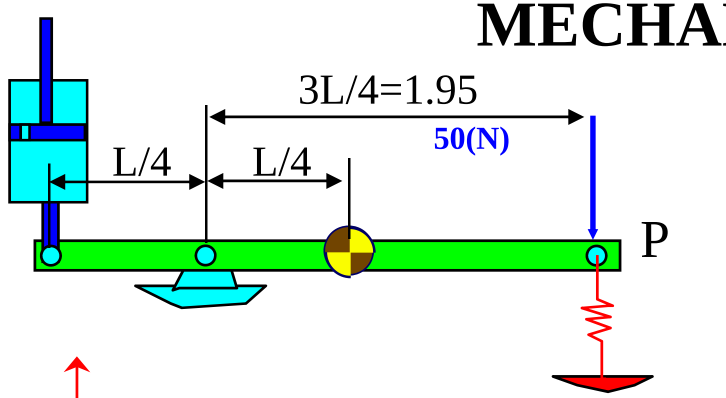


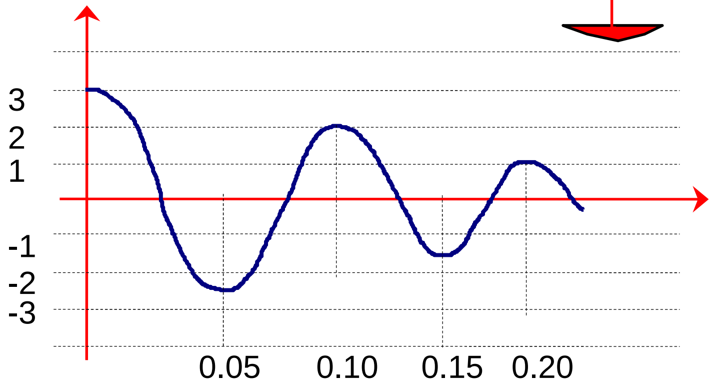


Mass= 1[kg]
Spring constant = 1 [N/m]
Damping coefficient = variable, from 0.02, 0.2, 2, 20



THEMATIC EXERCISE 19 - FUNDAMENTALS OF MECHANICAL VIBRATIONS

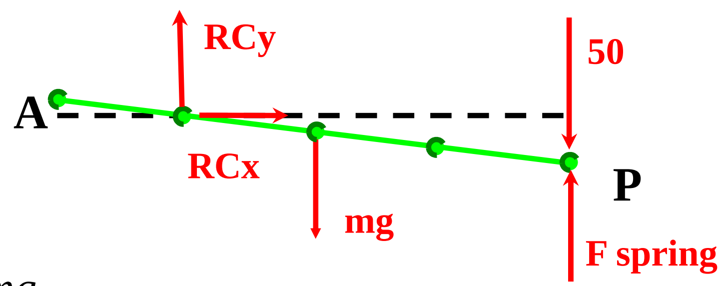




The beam shown in the left figure has a mass of 31 [kg] and a length of 2.6 [m].
A force of 50 [N] is static applied at point “P” and remove after. The oscillations of that point “P” were observed in a spectral analyser, being possible to represent the acceleration.
Use the data on graph to determine the spring constant and the damping coefficient.

Static analysis:

$$\begin{cases}\sum \vec{F} = \vec{0} \\ \sum M_G = \vec{0}\end{cases} \Leftrightarrow \begin{cases} RCx = 0 \\ RCy - mg - 50 + k\delta_{est} = 0 \\ -\frac{l}{4}RCy - \frac{l}{2}50 + \frac{l}{2}k\delta_{est} = 0 \end{cases} \Leftrightarrow \begin{cases} RCx = 0 \\ RCy = \frac{2}{3}mg \\ \delta_{est} = \frac{50}{k} + \frac{mg}{3k} \end{cases}$$



EXERCISE - solution

Kinematic analysis: Determination of the mass centre acceleration and the relationship between two distinct points A and P.

$$\vec{a}_G = \vec{a}_O + \dot{\vec{w}} \times \vec{OG} + \vec{w} \times \left(\vec{w} \times \vec{OG} \right)$$

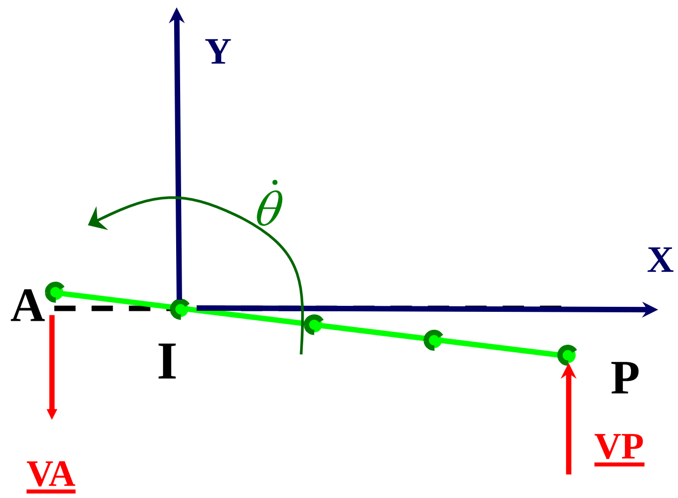
$$= \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} + \begin{Bmatrix} 0 \\ 0 \\ \ddot{\theta} \end{Bmatrix} \times \begin{Bmatrix} l/4 \\ 0 \\ 0 \end{Bmatrix} + \begin{Bmatrix} 0 \\ 0 \\ \dot{\theta} \end{Bmatrix} \times \left[\begin{Bmatrix} 0 \\ 0 \\ \dot{\theta} \end{Bmatrix} \times \begin{Bmatrix} l/4 \\ 0 \\ 0 \end{Bmatrix} \right] = \begin{Bmatrix} -\dot{\theta}^2 l/4 \\ \ddot{\theta}\, l/4 \\ 0 \end{Bmatrix}$$

$$\vec{V}_A = \vec{V}_I + \vec{w} \times \vec{IA}$$

$$= \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} + \begin{Bmatrix} 0 \\ 0 \\ \dot{\theta} \end{Bmatrix} \times \begin{Bmatrix} -l/4 \\ 0 \\ 0 \end{Bmatrix} = \begin{Bmatrix} 0 \\ -\dot{\theta}\, l/4 \\ 0 \end{Bmatrix}$$

$$\vec{V}_P = \vec{V}_I + \vec{w} \times \vec{IP}$$

$$= \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} + \begin{Bmatrix} 0 \\ 0 \\ \dot{\theta} \end{Bmatrix} \times \begin{Bmatrix} 3l/4 \\ 0 \\ 0 \end{Bmatrix} = \begin{Bmatrix} 0 \\ \dot{\theta} 3l/4 \\ 0 \end{Bmatrix}$$



EXERCISE – solution

Dynamic analysis: Newton's second law.

$$\begin{cases} \sum \vec{F} = m\vec{a}_G \\ \sum \vec{M}_G = \dot{\vec{H}}_G \end{cases} \Leftrightarrow -\begin{cases} RCx = m\left(-\dot{\theta}^2\, l/4\right) \\ RCy - mg + c\dot{x}_P - 50 + k\left(\delta_{est} - x_P\right) = m\left(\ddot{\theta}\, l/4\right) \\ -l/4\, RCy - l/2.50 - c\dot{x}_A\, l/2 + l/2\, k\delta_{est} - kx_P\, l/2 = 1/12\, ml^2\ddot{\theta} \end{cases}$$

$$\Leftrightarrow \begin{cases} RCx = -4/9\, m/l\, \dot{x}_P^2 \\ RCy = -100 - 2/3c\dot{x}_P + 2k\delta_{est} - 2kx_P - 4/9\, m\ddot{x}_P \\ \ddot{x}_P + 3/7\, c/m\, \dot{x}_P + 27k/7m\, x_P = 0 \end{cases}$$

Dynamic momentum determination: Centre of mass

$$\vec{H}_G\Big|_{SM} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1/12ml^2 & 0 \\ 0 & 0 & 1/12ml^2 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ \dot{\theta} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 1/12\, ml^2\dot{\theta} \end{Bmatrix}$$

$$\dot{\vec{H}}_G\Big|_{SM} = \begin{Bmatrix} 0 \\ 0 \\ 1/12\, ml^2\ddot{\theta} \end{Bmatrix} + \begin{Bmatrix} 0 \\ 0 \\ \dot{\theta} \end{Bmatrix} \times \begin{Bmatrix} 0 \\ 0 \\ 1/12\, ml^2\dot{\theta} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 1/12\, ml^2\ddot{\theta} \end{Bmatrix}$$

EXERCISE – solution

Graphical analysis: Data retrieved from the time domain result.

$$T = 0.1[s] \Rightarrow f = \frac{1}{T} = 10[Hz]$$

$$w_d = 2\pi f = 62.8[rad/s]$$

Logarithm decrement

$$\delta = \frac{\beta 2\pi}{\sqrt{1-\beta^2}} = Ln\left[\frac{x(t)}{x(t+T)}\right] = Ln\frac{2}{3} \Leftrightarrow \beta = 0.0644$$

The natural frequency

$$w_d = w\sqrt{1-\beta^2} \Leftrightarrow 62.8 = \sqrt{\frac{k_{eq}}{m_{eq}}}\sqrt{1-0.0644^2}$$

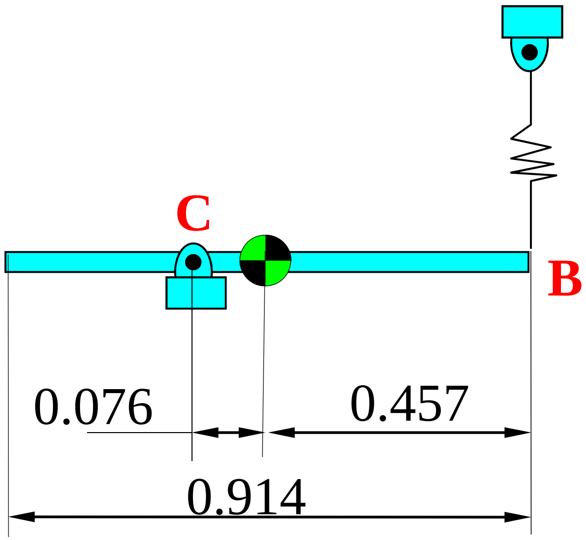
$$\Leftrightarrow 62.93 = \sqrt{\frac{27k}{7*31}} \Leftrightarrow k = 31828(N/m)$$

Damping coefficient

$$\beta = \frac{c_{eq}}{c_{cr}} = c_{eq}/\left(2\sqrt{k_{eq}m_{eq}}\right) = c_{eq}/\left(2\sqrt{27/7\, km}\right) \Leftrightarrow$$

$$\Leftrightarrow c_{eq} = 251.3 \Leftrightarrow c = 586.4[Ns/m]$$

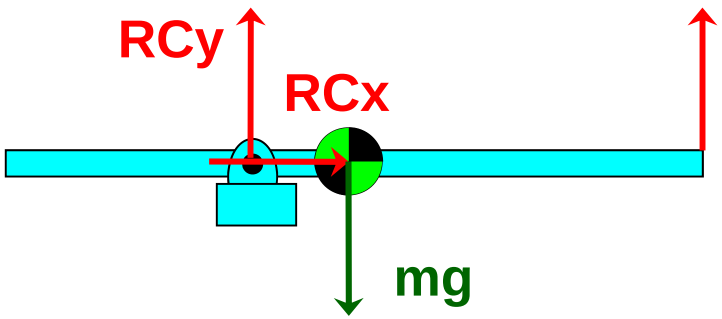
BEAM - SPRING EXERCISE



A straight homogeneous bar of 5.44 [kg] is connected to a spring with elastic constant equal to 525 (N/m). If the extremity B was moved down 12,7 (mm), and then released, determine:

a) The motion equation
b) Position, velocity and acceleration of point B.
c) The period of the oscillation.
d) The maximum velocity.
e) The reaction variation at point C.

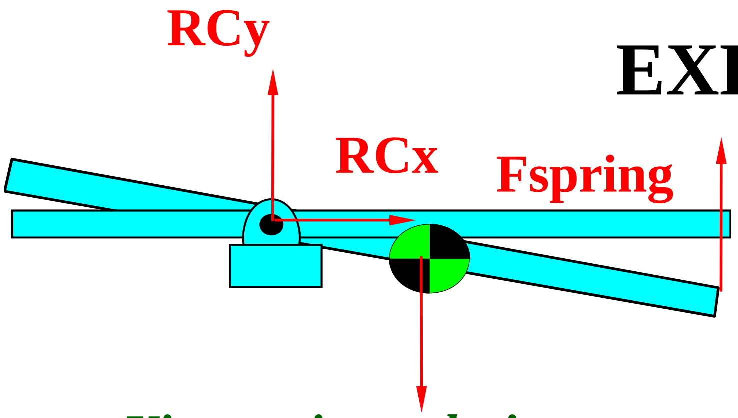
Static analysis:



$$\begin{cases} \sum Fx = 0 \\ \sum Fy = 0 \\ \sum M_Z^G = 0 \end{cases} \Leftrightarrow \begin{cases} RCx = 0 \\ RCy - mg + K\delta_{est} = 0 \\ -0.076 \times RCy + 0.457 \times K\delta_{est} = 0 \end{cases}$$

Rcx=0; RCy=45.75(N) $\delta_{est} = 0.014492(m)$

EXERCISE - solution



Due to the beam self weight, it should be oscillating from the position of self static equilibrium.

<u>Kinematic analysis:</u>

$$\vec{a}_G = \vec{a}_C + \dot{\vec{W}} \times C\vec{G} + W \times (W \times C\vec{G})$$

$$= \begin{Bmatrix} 0 \\ 0 \\ -\dot{\theta} \end{Bmatrix} \times \begin{Bmatrix} -0{,}076 \\ 0 \\ 0 \end{Bmatrix} + \begin{Bmatrix} 0 \\ 0 \\ -\dot{\theta} \end{Bmatrix} \times \begin{Bmatrix} 0 \\ 0 \\ -\dot{\theta} \end{Bmatrix} \times \begin{Bmatrix} -0{,}076 \\ 0 \\ 0 \end{Bmatrix}$$

$$= \begin{Bmatrix} 0{,}076\dot{\theta}^2 \\ 0{,}076\ddot{\theta} \\ 0 \end{Bmatrix}$$

EXERCISE – solution

Dynamic momentum:

$$\vec{H}_G = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{12}ml^2 & 0 \\ & 0 & \frac{1}{12}ml^2 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ -\dot{\theta} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ -\frac{1}{12}ml^2\dot{\theta} \end{Bmatrix}$$

$$\dot{\vec{H}}_G = \begin{Bmatrix} 0 \\ 0 \\ -\frac{1}{12}ml^2\ddot{\theta} \end{Bmatrix}$$

Dynamic analysis:

$$\sum \vec{F} = m.\vec{a}_{cm}$$

$$\sum \vec{M}_G = \dot{\vec{H}}_G$$

$$\begin{cases} RCx = 0{,}4134\dot{\theta}^2 \\ RCy = mg - 525(\delta_{est} + y_B) - 5{,}44 \times 0{,}076\ddot{\theta} \\ -0{,}076 \times \left[5{,}44 \times 9{,}81 - 525(0{,}01442 + y_B) - 5{,}44 \times 0{,}076\ddot{\theta}\right] + 525(0{,}01442 + y_B)0{,}457 = -\frac{1}{12}mL^2\ddot{\theta} \end{cases}$$

EXERCISE – solution

General differential equations:

$$0{,}41012\ddot{\theta} + 279{,}8y_B = 0$$

$$0{,}7694\ddot{y}_B + 279.8y_B = 0$$

$$0{,}41012\ddot{\theta} + 149{,}146\theta = 0$$

One possible solution:

$$y_B = A.\mathrm{Cos}(wt) + B\sin(wt)$$

or

$$\theta = C.\mathrm{Cos}(wt) + D.\sin(wt)$$

Boundary conditions:

$$\dot{y}_B(t=0) = 0$$
$$y_B(t=0) = 0{,}0127$$

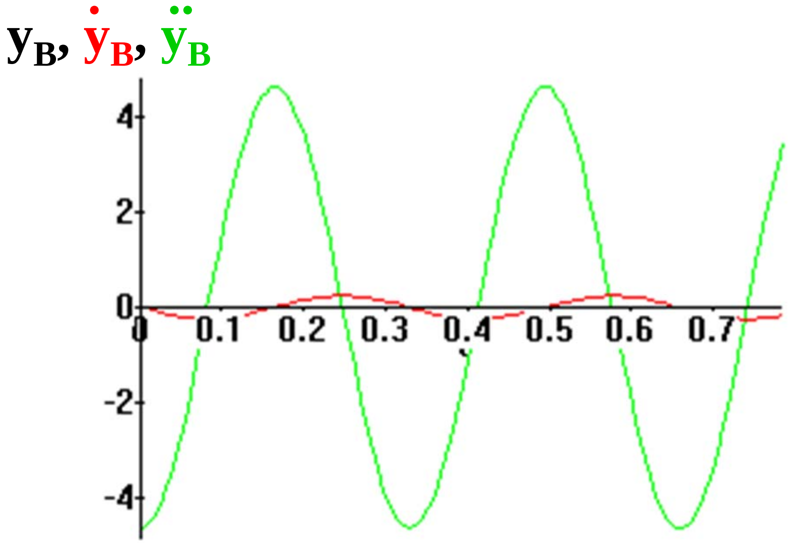
or

$$\dot{\theta}(t=0) = 0$$
$$\theta(t=0) = 1{,}36^{\circ} = 0{,}0238\,[rad]$$

EXERCISE – solution

Final solution:

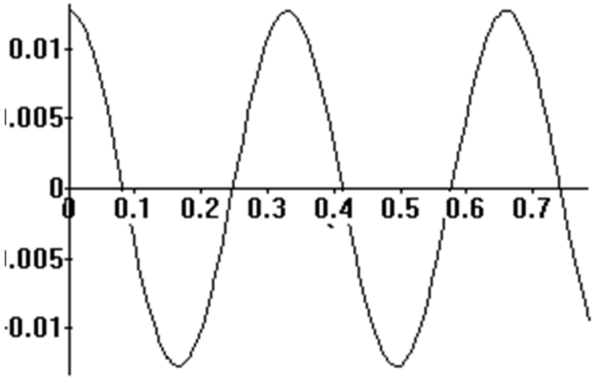
$$y_B = 0{,}0127 \times \cos(19.07 \times t)$$
$$\dot{y}_B = -0.242 \times \sin(19.07 \times t)$$
$$\ddot{y}_B = -4.618 \times \cos(19.07 \times t)$$



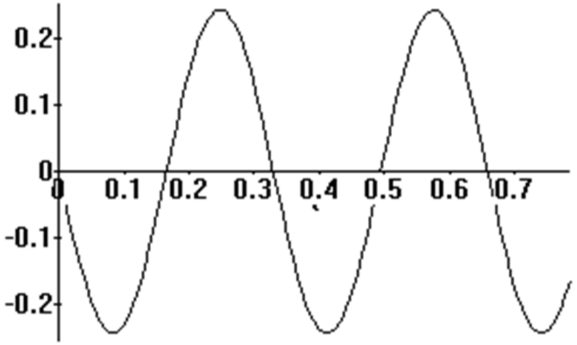
$$\theta(t) = 0.0238 \times \cos(19.07 \times t)$$
$$\dot{\theta}(t) = -0.4538 \times \sin(19.07 \times t)$$
$$\ddot{\theta}(t) = -8.655 \times \cos(19.07 \times t)$$

$$RCx = 0.4134 \times \left(-0.454 \times \sin((19.07 \times t)\right)^2$$
$$RCy = 45.75 - 525 \times (0.0127 \times \cos(19.07 \times t)) - 0.413(-8.655 \times \cos(19.07 \times t))$$

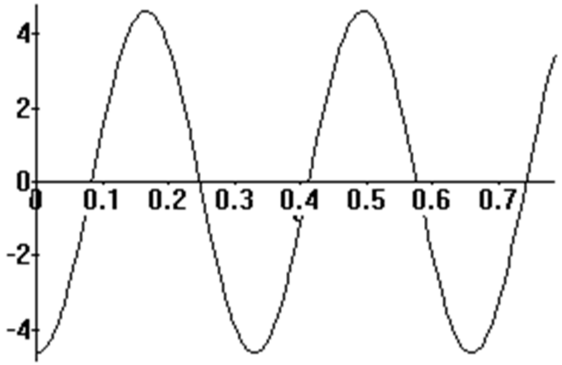
EXERCISE – solution



Position of point B

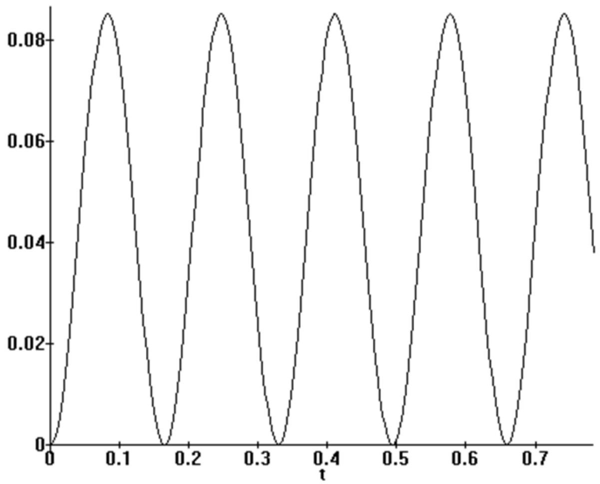


Velocity of point B

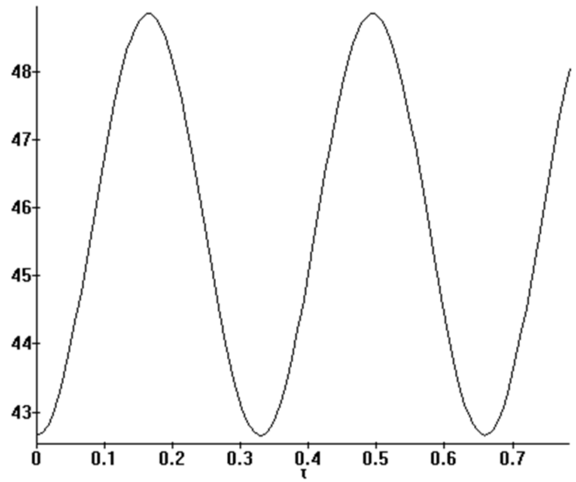


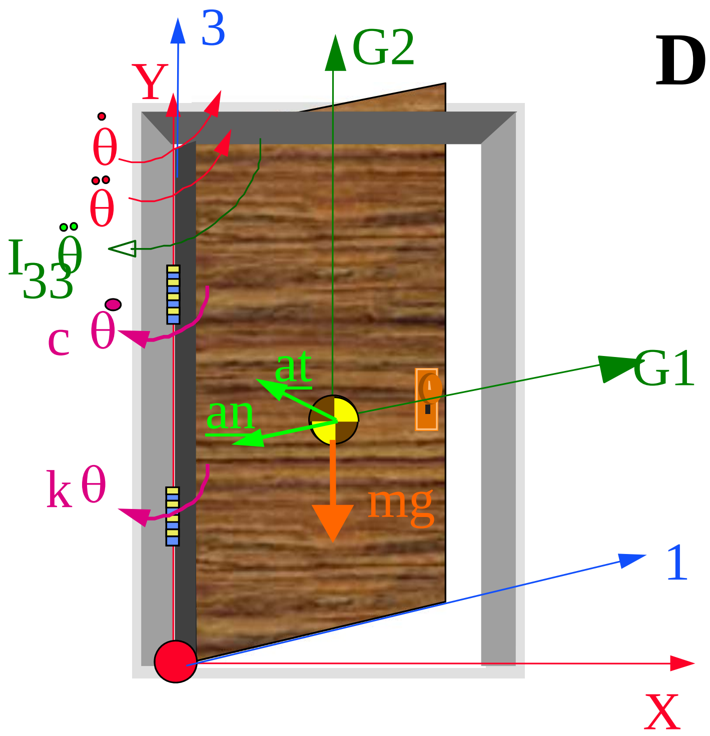
Acceleration of point B

R_{CX}



R_{CY}





DOOR EXERCISE

The door of a restaurant is equioed with a spring and damp system, so it can return to its original position after the push and pull procedure.

The door has a mass of 60 (kg) and a principal inertial moment of 7.2 (kgm^2), relative to G2.

The rotational spring has a constant of 25 (Nm/rad).

a) Determine the critical damping coefficient.

b) A man with a occupied arm and in a hurry, uses its foot to open the door. Calculate the angular velocity necessary to open the door 70º.

c) How long takes the door to reach the position of 5º.?

d) Repeat the questions a),b) and c) in the case the door possess a damping coeficient of 1.3.

DOOR EXERCISE – solution

Kinematic analysis:

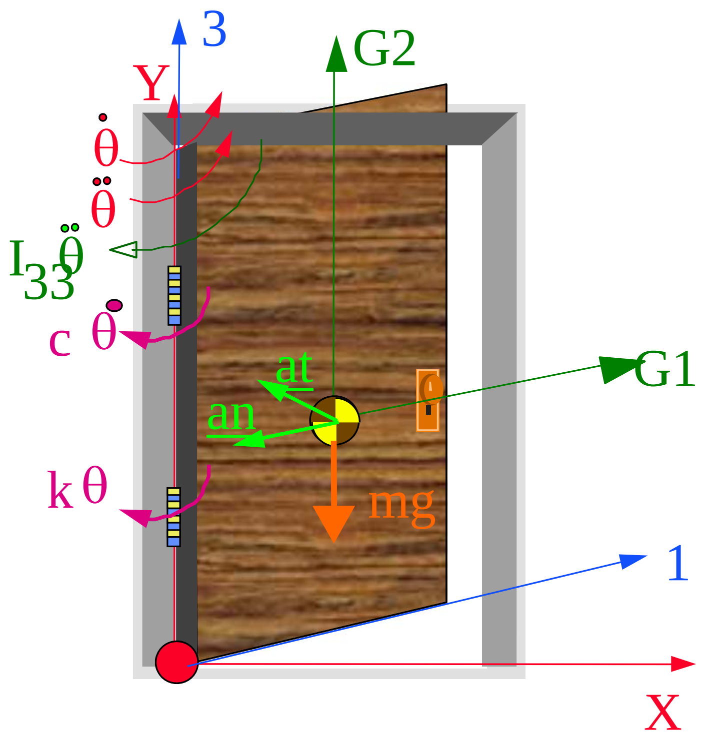
$$\vec{a}_G\Big|_{S12} = \vec{a}_0 + \dot{\vec{W}} \times O\vec{G} + W \times (W \times O\vec{G})$$

$$= \vec{0} + \begin{Bmatrix} 0 \\ 0 \\ \ddot{\theta} \end{Bmatrix} \times \begin{Bmatrix} 0.91/2 \\ 0 \\ 0 \end{Bmatrix} + \begin{Bmatrix} 0 \\ 0 \\ \dot{\theta} \end{Bmatrix} \times \left[\begin{Bmatrix} 0 \\ 0 \\ \dot{\theta} \end{Bmatrix} \times \begin{Bmatrix} 0.91/2 \\ 0 \\ 0 \end{Bmatrix} \right]$$

$$= \begin{Bmatrix} -\dot{\theta}(0.455) \\ \ddot{\theta}(0.455) \\ 0 \end{Bmatrix}$$

Dynamic momentum:

$$\vec{H}_G\Big|_{S12} = \begin{bmatrix} I_{G1G1} & 0 & 0 \\ 0 & I_{G2G2} & 0 \\ 0 & 0 & I_{G3G3} \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ \dot{\theta} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ I_{G3G3}\dot{\theta} \end{Bmatrix}$$



$$\dot{\vec{H}}_G\Big|_{S12} = \begin{Bmatrix} 0 \\ 0 \\ I_{G3G3}\ddot{\theta} \end{Bmatrix} + \begin{Bmatrix} 0 \\ 0 \\ \dot{\theta} \end{Bmatrix} \times \begin{Bmatrix} 0 \\ 0 \\ I_{G3G3}\dot{\theta} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ I_{G3G3}\ddot{\theta} \end{Bmatrix}$$

DOOR EXERCISE – solution

Dynamic momentum:

$$\vec{H}_O\Big|_{S12} = [I_O]\begin{Bmatrix}0\\0\\\dot{\theta}\end{Bmatrix} = \left[[I_G] + m\begin{bmatrix}0 & 0 & 0\\ 0 & (-0.455)^2+0 & 0\\ 0 & 0 & (-0.455)^2+0\end{bmatrix}\right]\begin{Bmatrix}0\\0\\\dot{\theta}\end{Bmatrix}$$

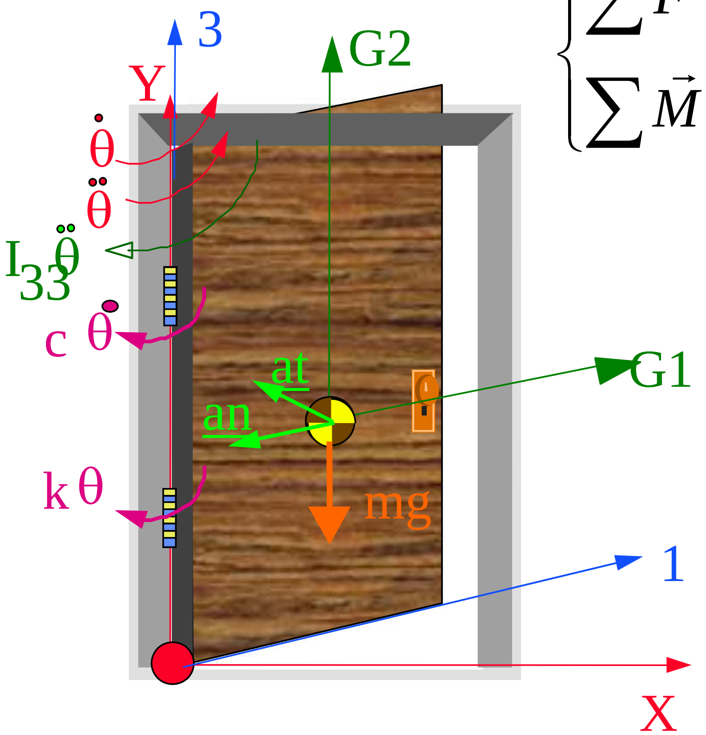
$$\vec{H}_O\Big|_{S12} = \begin{bmatrix}I_{G1G1} & 0 & 0\\ 0 & I_{G2G2}+m(-0.455)^2 & 0\\ 0 & 0 & I_{G3G3}+m(-0.455)^2\end{bmatrix}\begin{Bmatrix}0\\0\\\dot{\theta}\end{Bmatrix}$$

$$\dot{\vec{H}}_O\Big|_{S12} = \begin{Bmatrix}0\\0\\(I_{G3G3}+m0.455^2)\ddot{\theta}\end{Bmatrix} + \begin{Bmatrix}0\\0\\\dot{\theta}\end{Bmatrix} \times \begin{Bmatrix}0\\0\\(I_{G3G3}+m0.455^2)\dot{\theta}\end{Bmatrix}$$

$$= \begin{Bmatrix}0\\0\\(I_{G3G3}+m0.455^2)\ddot{\theta}\end{Bmatrix}$$

DOOR EXERCISE – solution

Dynamic analysis:



$$\begin{cases}\sum \vec{F} = m\vec{a}_G \\ \sum \vec{M}_O = \dot{\vec{H}}_O\end{cases} \Leftrightarrow \begin{cases}\dots \\ \dots \\ -c\dot{\theta} - k\theta = \left(I_{G3G3} + m0.455^2\right)\ddot{\theta}\end{cases}$$

$$\Leftrightarrow \begin{cases}\dots \\ \dots \\ (7.2 + 60 \times 0.455^2)\ddot{\theta} + c\dot{\theta} + k\theta = 0\end{cases}$$

$$\Leftrightarrow \begin{cases}\dots \\ \dots \\ 19.62\ddot{\theta} + c\dot{\theta} + 25\theta = 0\end{cases}$$

Natural frequencies:

$$W_n = \sqrt{\frac{k_{eq}}{m_{eq}}} = \sqrt{\frac{25}{19.62}} = 1.13(rad/s)$$

DOOR EXERCISE – solution

Damping coefficient:

$$\beta = \frac{c_{eq}}{c_{critico}} = c_{eq} / 2\sqrt{k_{eq} m_{eq}} = 1 \Leftrightarrow$$

$$\Leftrightarrow c_{eq} = 44.3 (N.s/m)$$

Movement equation:

$$19.62\ddot{\theta} + c\dot{\theta} + 25\theta = 0$$

Possible solutions for the movement equation:

$$\theta = Ae^{st} \qquad \dot{\theta} = (As)e^{st} \qquad \ddot{\theta} = (As^2)e^{st}$$

Substituting the solution into the eqaution:

$$\begin{Bmatrix} s1 \\ s2 \end{Bmatrix} = \frac{-c}{39.24} \pm \frac{\sqrt{c^2 - 1962}}{39.24}$$

DOOR EXERCISE – solution

For vibrations critically damping, the solution presents two real double solutions

$$\theta(t) = e^{-W_n t}(A_1 + tA_2)$$

Differentiating one more time

$$\dot{\theta}(t) = -W_n e^{-W_n t}(A_1 + tA_2) + e^{-W_n t}(A_2)$$

Using the necessary boundary conditions

$$\begin{cases} t = 0 \Rightarrow \theta = \theta_0 \\ t = 0 \Rightarrow \dot{\theta} = \dot{\theta}_0 \end{cases} \Leftrightarrow \begin{cases} A_1 + 0 \times A_2 = \theta_0 \\ -W_n A_1 + A_2 = \dot{\theta}_0 \end{cases} \Leftrightarrow \begin{cases} A_1 = \theta_0 \\ A_2 = \dot{\theta}_0 + W_n \theta_0 \end{cases}$$

Substituting the constants into the movement equation:

$$\theta(t) = e^{-W_n t}(\theta_0 + t(\dot{\theta}_0 + W_n \theta_0))$$

DOOR EXERCISE – solution

Mathematically, to obtain the maximum of a function, its is necessary to differentiate and equal to zero the function.

$$\frac{d\theta(t)}{t} = 0 \Leftrightarrow$$

$$\Leftrightarrow 0 = e^{-W_n t}(\dot{\theta}_0 + W_n\theta_0) + \left(-W_n\right)e^{-W_n t}(\theta_0 + t(\dot{\theta}_0 + W_n\theta_0)) \Leftrightarrow$$

$$\text{Any function } e^{-W_n t} \neq 0, \text{ it can be verify:}$$

$$\Leftrightarrow t = \frac{1}{W_n}$$

The maximum value for the function is: $\theta = 70^{\circ} = 1.22(rad)$

The intial angular velocity: $\theta(t = \frac{1}{W_n}) = 1.22 = e^{-W_n t}(0 + \frac{1}{W_n}(\dot{\theta}_0 + \frac{1}{1.13}0)) \Leftrightarrow$

$$\Leftrightarrow 1.22 = \frac{1}{e}\frac{1}{1.13}\dot{\theta}_0 \Leftrightarrow$$

$$\Leftrightarrow \dot{\theta}_0 = 3.74(rad/s)$$

DOOR EXERCISE – solution

Time to get 5º in rotation and in a critical damping system:

$$\theta = 5^{o} = 0.0872(rad) = e^{-W_n t}\left(0 + t(3.74 + 0)\right) \Leftrightarrow$$

$$\Leftrightarrow 0.0872 = e^{-1.13t} t(3.74)$$

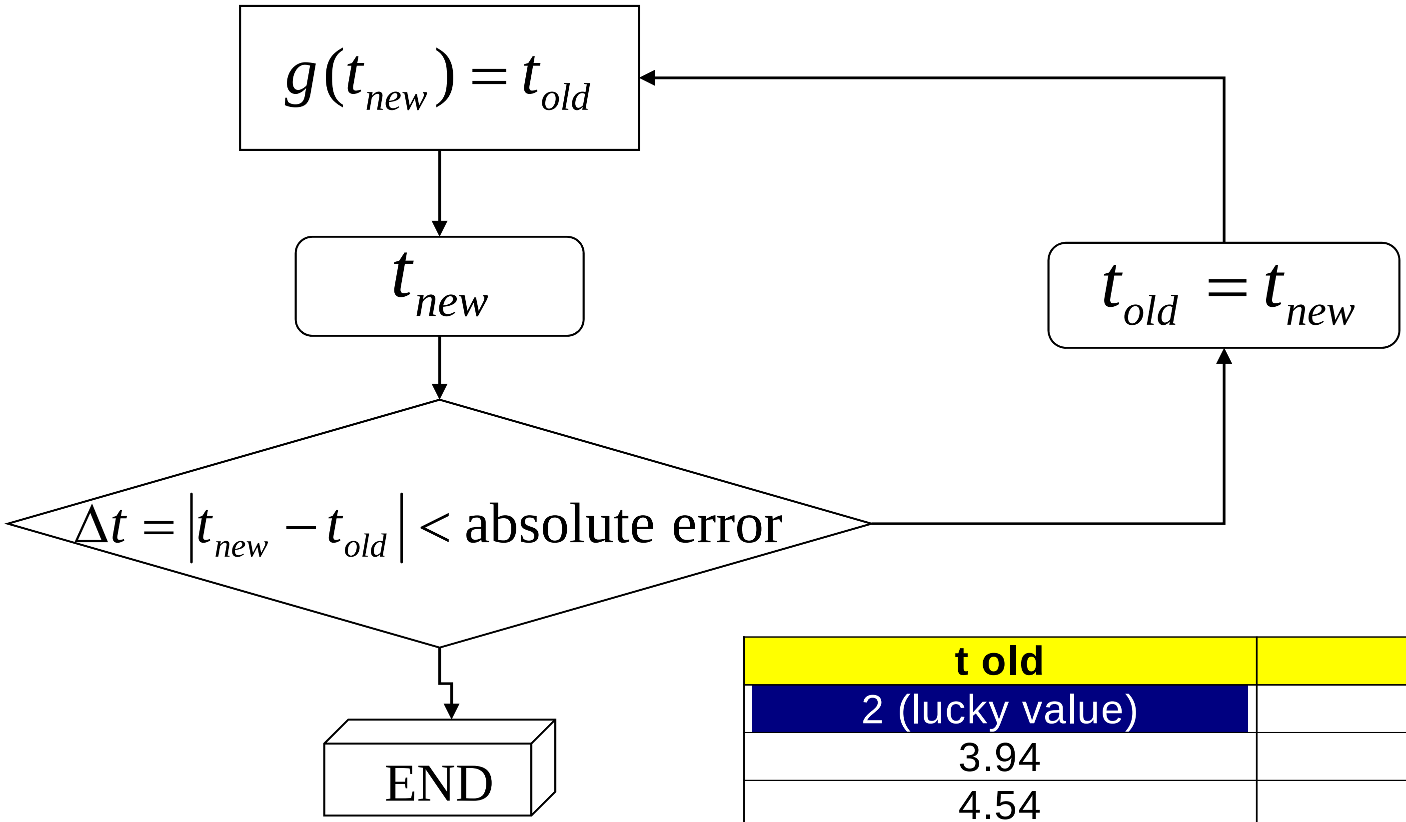
Being a non linear equation type, numerical methods for solution are required

$$0.0872 = e^{-1.13t} t(3.74) \Leftrightarrow 0.0233 e^{1.13t} = t \Leftrightarrow$$

$$\Leftrightarrow g(t) = t$$

DOOR EXERCISE – solution

Numerical solution method



t old	t new
2 (lucky value)	3.94
3.94	4.54
4.54	4.66
4.66	4.69
4.69	4.69

Note: Absolute error defined by user and equal to 0.01(s)

DOOR EXERCISE – solution

For a over damping system, with 1.3 coefficient

$$\beta = \frac{c_{eq}}{c_{critico}} = c_{eq} / 2\sqrt{k_{eq} m_{eq}} = 1.3 \Leftrightarrow$$

$$\Leftrightarrow c_{eq} = 57.6 (N.s/m)$$

Possible solution type:

$$\theta(t) = e^{-\beta W_n t}\left[\left(A_1 + A_2\right) ch\left(W_d t\right) + \left(A_1 - A_2\right) sh\left(W_d t\right)\right]$$

Differentiating one time

$$\dot{\theta}(t) = (-\beta W_n) e^{-\beta W_n t}\left[\left(A_1 + A_2\right) ch(W_d t) + \left(A_1 - A_2\right) sh(W_d t)\right] +$$
$$+ e^{-\beta W_n t}\left[-W_d\left(A_1 + A_2\right) sh(W_d t) + W_d\left(A_1 - A_2\right) ch(W_d t)\right]$$

Applying boundary conditions

$$\begin{cases} t = 0 \Rightarrow \theta = \theta_0 \\ t = 0 \Rightarrow \dot{\theta} = \dot{\theta}_0 \end{cases} \Leftrightarrow \begin{cases} A_1 + A_2 = \theta_0 \\ -1.13 * 1.3 (A_1 + A_2) + W_d (A_1 - A_2) = \dot{\theta}_0 \end{cases}$$

$$\Leftrightarrow \begin{cases} A_1 = \theta_0 \\ A_2 = \dot{\theta}_0 + W_n \theta_0 \end{cases}$$

DOOR EXERCISE – solution

Substituting into motion equation

$$\theta(t) = e^{-\beta W_n t}\left[\left(A_1 + A_2\right)ch\left(W_d t\right) + \left(A_1 - A_2\right)sh\left(W_d t\right)\right]$$

$$\theta(t) = e^{-W_n t}(\theta_0 + t(\dot{\theta}_0 + W_n\theta_0))$$

Mathematically, the maximum of a function is the result of a time function derivative.

$$\dot{\theta}(t) = \frac{d\theta}{dt} = 0 \Leftrightarrow$$

$$\Leftrightarrow 0 = -\beta W_n e^{-\beta W_n t}\left[\left(A_1 + A_2\right)ch(W_d t) + \left(A_1 - A_2\right)sh(W_d t)\right] +$$

$$+ e^{-\beta W_n t}\left[-\left(A_1 + A_2\right)W_d sh(W_d t) + \left(A_1 - A_2\right)W_d ch(W_d t)\right] \Leftrightarrow$$

For any value of $e^{-\beta W_n t}$

$$\Leftrightarrow 2W_d t = 1.51 \Leftrightarrow t = 0.805(s)$$

DOOR EXERCISE – solution

$$\theta(t=0.805)=\theta_{\text{maximum}}=70^{o}=1.22(rad)\Leftrightarrow$$

$$\Leftrightarrow 1.22=e^{-\beta W_n 0.805}\left[\frac{\dot{\theta}}{0.938}\left(\frac{e^{W_d 0.805}-e^{-W_d 0.805}}{2}\right)\right]\Leftrightarrow$$

$$\Leftrightarrow \dot{\theta}_0=4.5(rad/s)$$

Substituting 5º or 0.0872(rad), we obtain:

$$0.0872=e^{-1.469t}\left[\frac{4.5}{0.938}\left(\frac{e^{W_d t}-e^{-W_d t}}{2}\right)\right]\Leftrightarrow$$

$$\Leftrightarrow 0.03635=e^{-0.531t}-e^{-2.407t}$$

By a numerical similarly procedure

$$0.03635+e^{-2.407t_{old}}=e^{-0.531t_{new}}$$

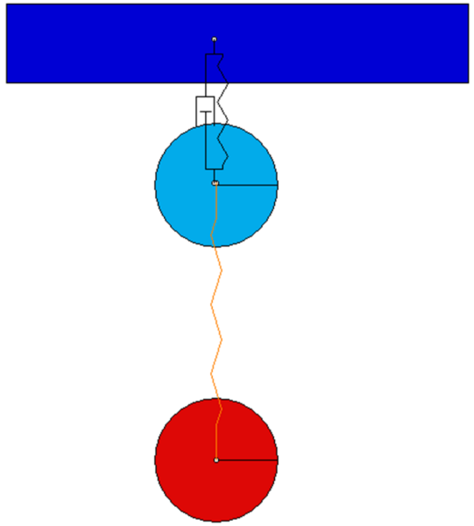
Table of convergence	
t old	**t new**
2 (lucky value)	5.86
5.86	6.24
6.24	**6.24**

A stronger kick must be used in a over damping system, even the time to open the door 70º is almost the same. The initial angular velocity is grater [3.74 to 4.5 (rad/s)], reflecting the increasing resistant viscous moment.

SYSTEMS WITH TWO DEGREES OF FREEDOM

- Instead of one equilibrium equation, it will take two equations.

-The motion equations may be obtained by the dynamic laws, by other virtual working principle or by other methods (numerical, etc.).

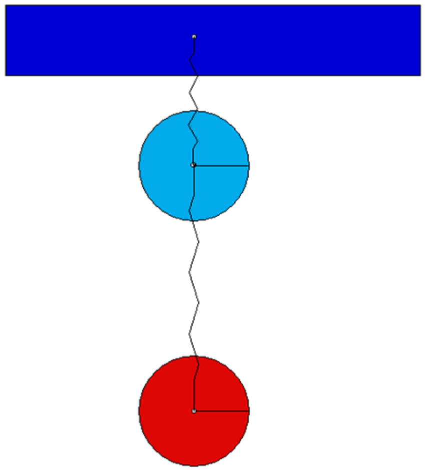


For two degrees of freedom:

$$[M]\{\ddot{x}\}+[C]\{\dot{x}\}+[K]\{x\}=\{f(t)\}$$

Mass matrix — Damping matrix — Stifness matrix — Load Vector

DIFFERENTIAL SOLUTION



-Integration of the motion equation:

1- Modal superposition method

2- Direct numerical integration

For mode shapes and natural frequencies determination, load vector is not used.

$$[M]\{\ddot{x}\}+[K]\{x\}=\{0\}$$

Possible solution:

$$\{x\}=\{X\}\cos(wt)$$

Time derivatives should verify also:

$$\{\dot{x}\}=-\{X\}\,w\sin(wt)$$

$$\{\ddot{x}\}=-\{X\}\,w^2\cos(wt)$$

DIFFERENTIAL EQUATION

$$[M]\left(-\{X\}\, w^2 \cos(wt)\right) + [K]\left(\{X\}\cos(wt)\right) = \{0\}$$

Simplifying:

$$\left(-w^2\,[M] + [K]\right)\left(\{X\}\cos(wt)\right) = \{0\}$$

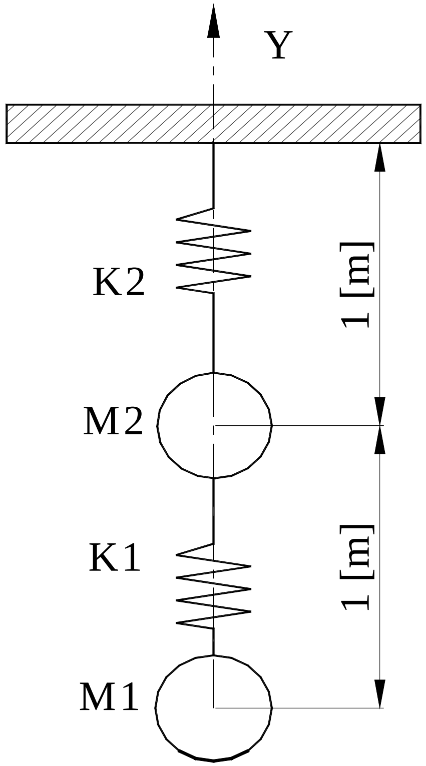
Homogeneous system with null solution ?

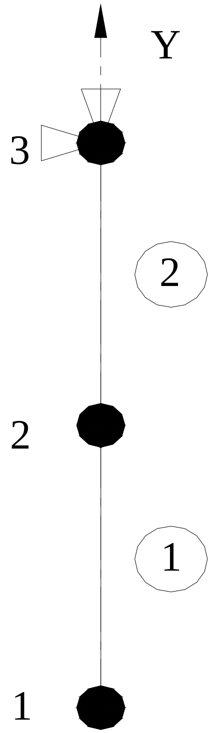
Searching for other solution rather then null, implies mathematically:

$$\det\left(-w^2\,[M] + [K]\right) = 0$$

Frequency determination w1 and w2: Multiple solutions, mass matrix normalization.
For each frequency determination, mode shapes can also be calculated.

VIBRATION ANALYSIS EXERCISE

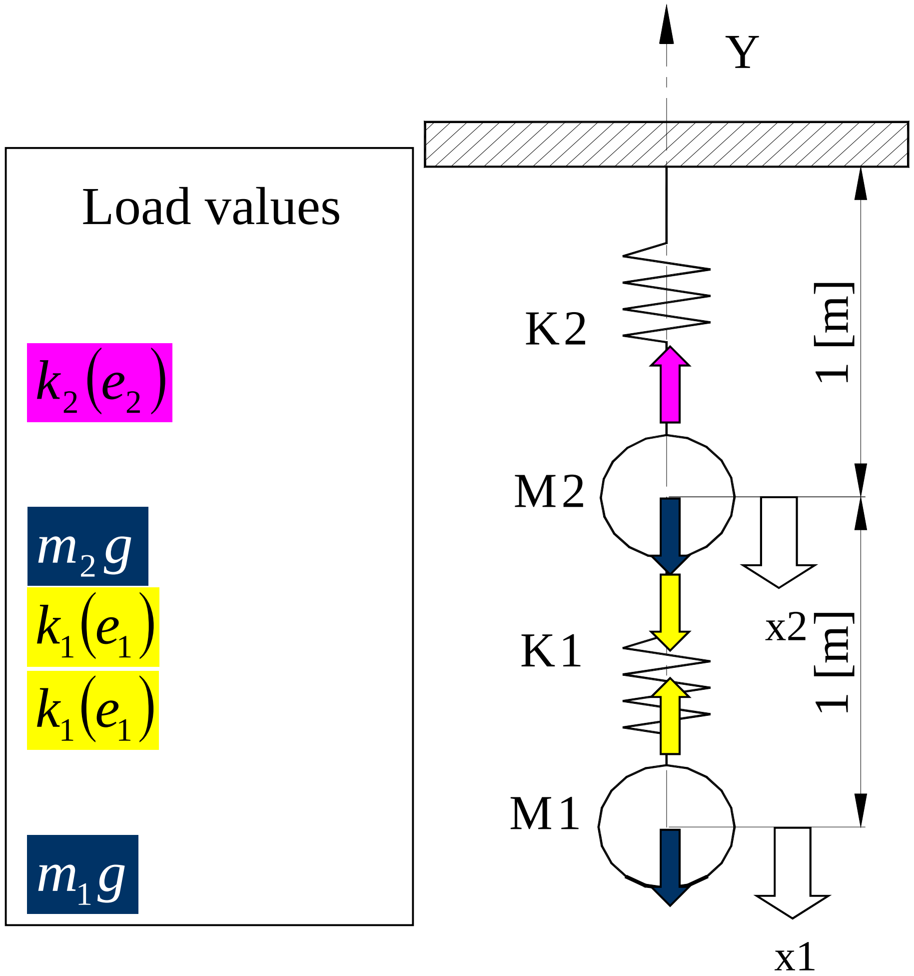




- Analytical solution for determining the natural frequencies and natural modes of vibration.

- Numerical solution by ANSYS and Interactive Physics 2000 (frequency and time domain)

SOLUTION



Findig the position from static equilibrium: e1, e2

$$\sum \vec{F} = \vec{0} \qquad k_2 e_2 - m_2 g - k_1 e_1 = 0$$

$$\sum \vec{F} = \vec{0} \qquad -m_1 g + k_1 e_1 = 0$$

SOLUTION

Findig the motion equilibrium equation

$$\sum \vec{F} = m\vec{a}_G \qquad k_2(e_2+x_2)+m_2\ddot{x}_2-m_2g-k_1(e_1+x_1-x_2)=0$$

$$\sum \vec{F} = m\vec{a}_G \qquad k_1(e_1+x_1-x_2)+m_1\ddot{x}_1-m_1g=0$$

Applying the information from static equilibrium:

$$k_2(x_2)+m_2\ddot{x}_2-k_1(x_1-x_2)=0$$

$$k_1(x_1-x_2)+m_1\ddot{x}_1=0$$

Introducing matrix formulation:

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}\begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix}+\begin{bmatrix} k_1 & -k_1 \\ -k_1 & (k_1+k_2) \end{bmatrix}\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix}=\begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

Load values

$k_2(e_2+x_2)$

$m_2\ddot{x}_2$

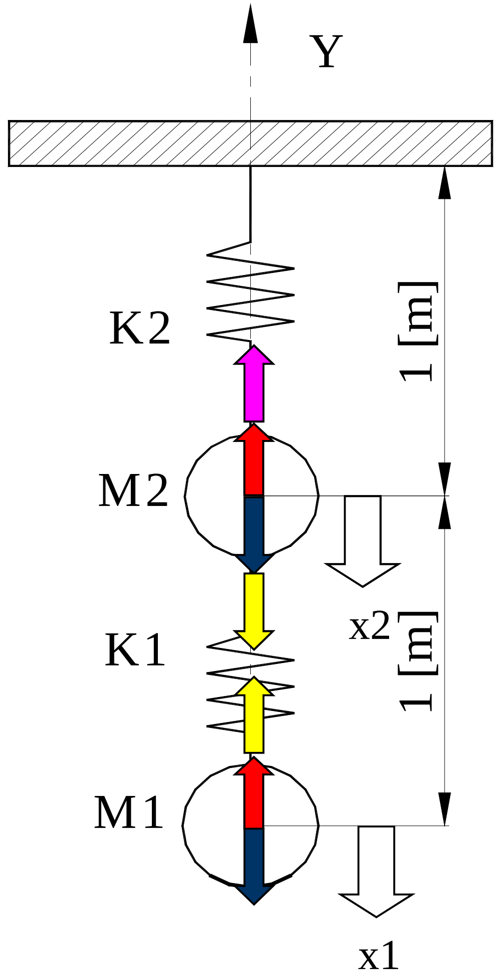
$m_2 g$

$k_1(e_1+x_1-x_2)$

$k_1(e_1+x_1-x_2)$

$m_1\ddot{x}_1$

$m_1 g$



Note: the degrees of freedom are displacements, producing mass matrix with units of mass. If the degrees of freedom were rotations, this should produce units of inertial moments.

SOLUTION

For the homogeneous system:

$$\left(-w^2\,[M]+[K]\right)\left(\{X\}\cos(wt)\right)=\{0\}$$

The non zero solution lead to:

$$\det\left(-w^2\,[M]+[K]\right)=0 \Leftrightarrow \det\begin{bmatrix}-w^2 m_1+k_1 & -k_1\\ -k_1 & -w^2 m_2+(k_1+k_2)\end{bmatrix}=0$$

$$\Leftrightarrow w^4 m_1 m_2 - w^2\left[m_1(k_1+k_2)+k_1 m_2\right]+(k_1.k_2)=0$$

By substitution, w^2=a, a quadratic equation is obtained:

$$a^2 m_1 m_2 - a\left[m_2 k_1 + m_1(k_1+k_2)\right]+(k_1.k_2)=0$$

$$a=\frac{+\left[m_2 k_1+m_1(k_1+k_2)\right]\pm\sqrt{\left[m_2 k_1+m_1(k_1+k_2)\right]^2-4\times(m_1 m_2)\times(k_1.k_2)}}{2(m_1 m_2)}$$

SPECIFIC SOLUTION

Being: $m_1 = m_2 = 2\,[kg]$

$k_1 = 6\,[N/m] \quad k_2 = 16\,[N/m]$

Solution for each real value:

$$a_1 = 2.00 \quad a_2 = 12.00 \quad \Rightarrow \quad w_1 = 1.4142 \quad w_2 = 3.4641 \quad \Rightarrow \quad f_1 = 0.22508\,[hz] \quad f_2 = 0.5513\,[hz]$$

For the first frequency, the first mode of vibration lead to a indetermined system:

$$\left(-w_1^{\,2}\,[M] + [K]\right)\begin{Bmatrix} x_{11} \\ x_{12} \end{Bmatrix} = \{0\}$$

Choose x_{12}=1 and take the solution for x_{11}=3.0

Choose x_{22}=1 and take the solution for x_{21}=-0.333

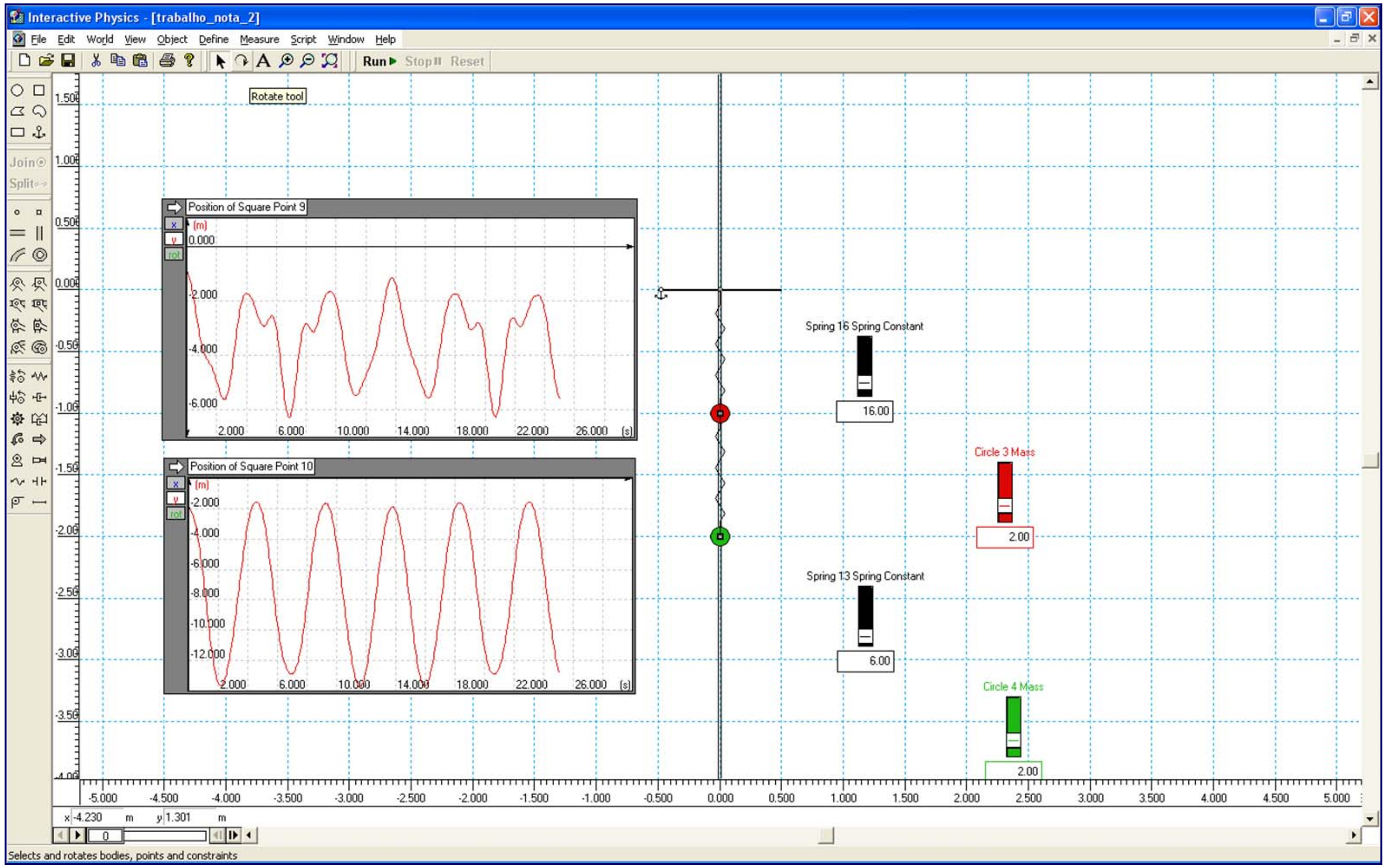
Normalize both solution to mass matrix

$$\begin{Bmatrix} \psi_{11} \\ \psi_{12} \end{Bmatrix} = \frac{1}{\sqrt{\langle 3.0 \quad 1 \rangle \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{Bmatrix} 3.0 \\ 1 \end{Bmatrix}}} \begin{Bmatrix} 3.0 \\ 1 \end{Bmatrix} = \begin{Bmatrix} 0.6708 \\ 0.2236 \end{Bmatrix}$$

$$\begin{Bmatrix} \psi_{21} \\ \psi_{22} \end{Bmatrix} = \frac{1}{\sqrt{\langle -0.333 \quad 1 \rangle \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{Bmatrix} -0.333 \\ 1 \end{Bmatrix}}} \begin{Bmatrix} -0.333 \\ 1 \end{Bmatrix} = \begin{Bmatrix} -0.2236 \\ 0.6708 \end{Bmatrix}$$

Numerical solution using
INTERACTIVE PHYSICS 2000

The student should be able to model two degrees of freedom, as presented, and obtain the time domain solution for each particular problem (m1, m2, k1 and k2).

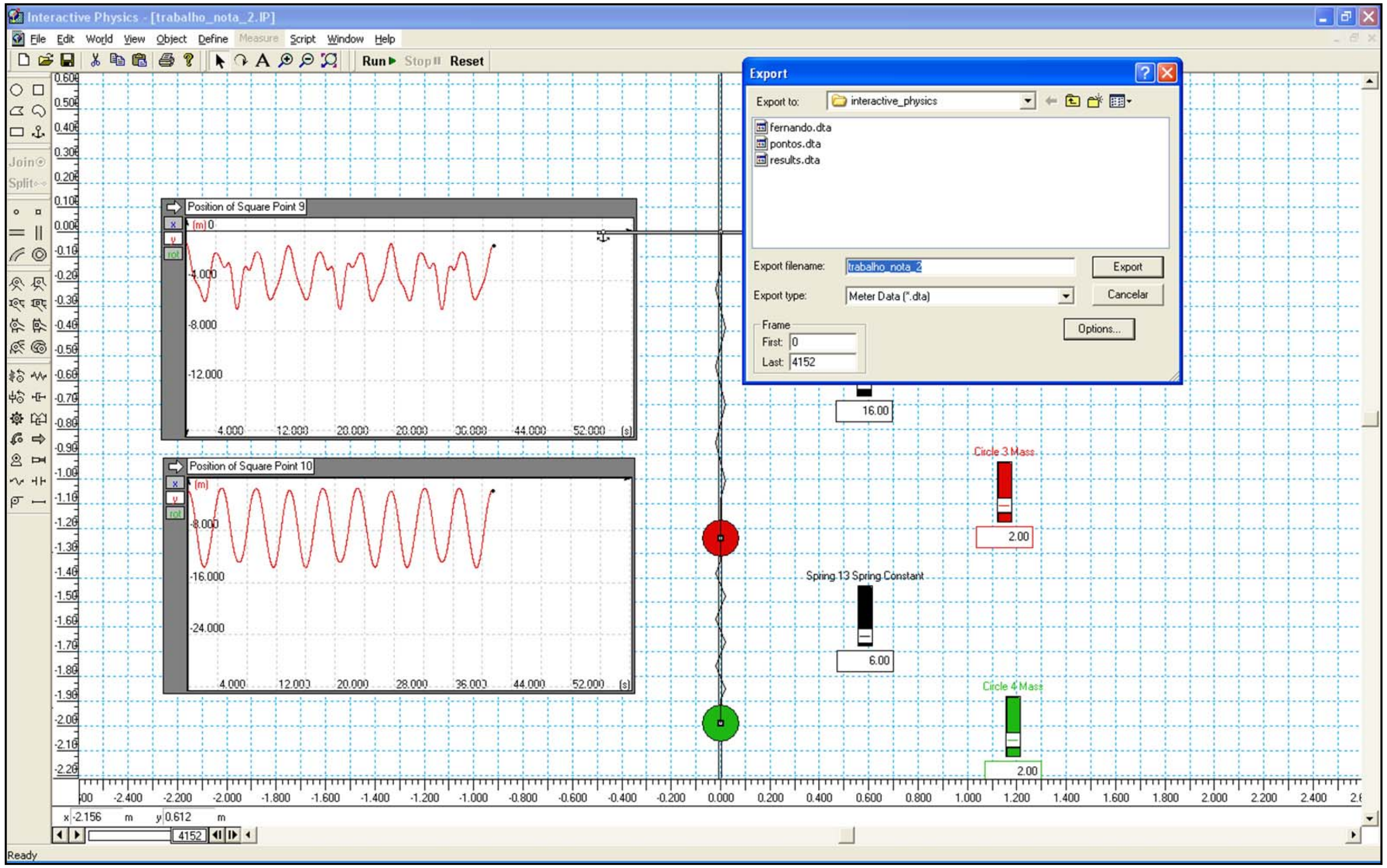


The student should compare the frequency value with the analytical, ANSYS and interactive physics solution (for this last one, please follow the Fourier Analysis.

FFT – Fourier Analysis 1/2

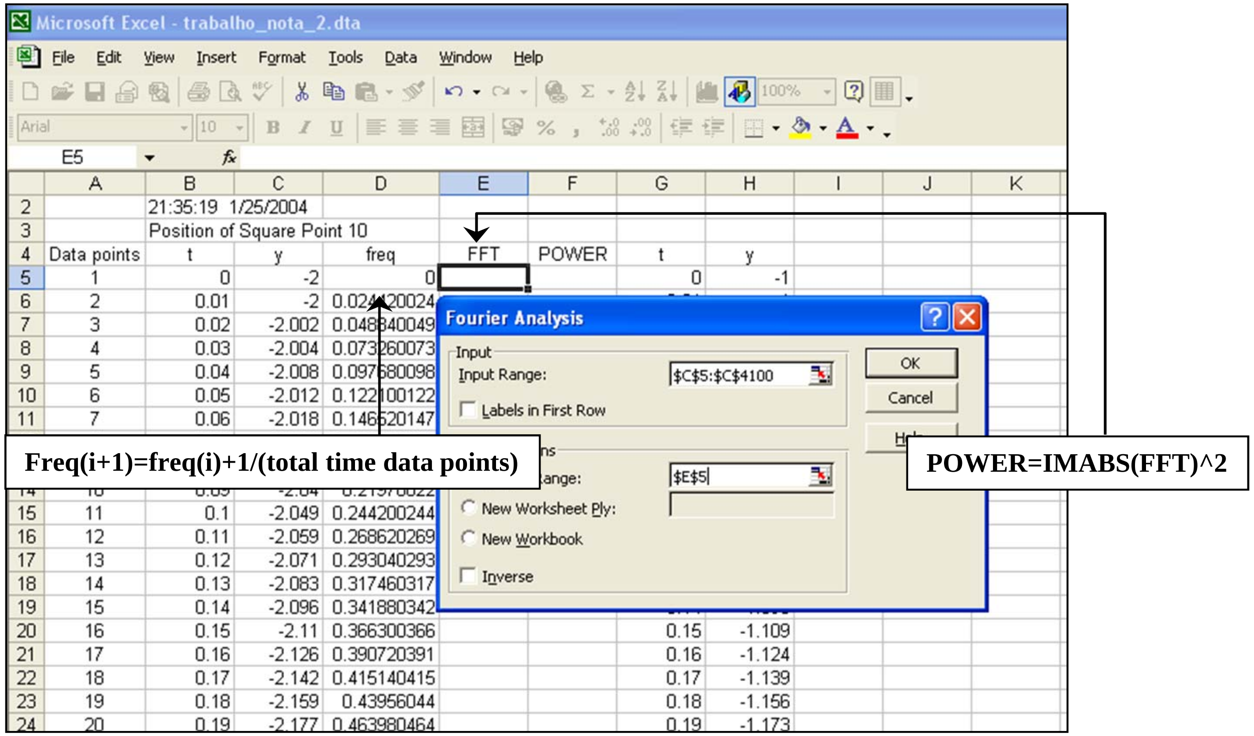
•Time domain transformation in frequency domain, using Excel data analysis capability.

•Interactive Physics can export “any number” of data points and these should be carefully worked at excel. Excel has a limited number of data points for analysis (4096).



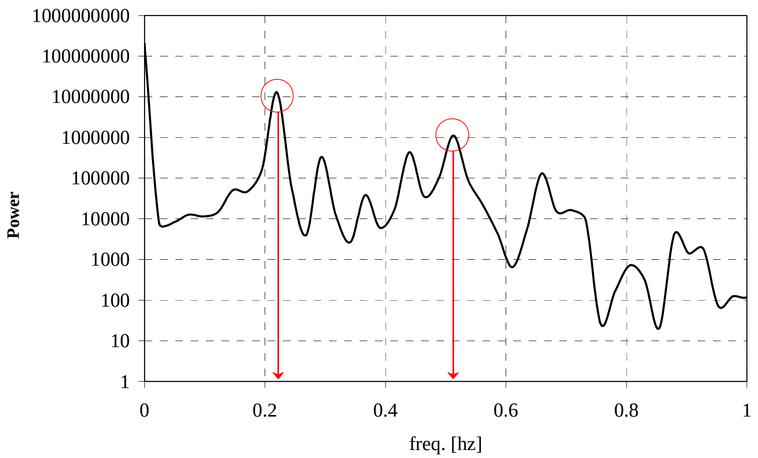
FFT – Fourier Analysis – 2/2

•Excel needs to apply for “Add Ins”, new tool, called “Data Analysis”. For this new capability, the student should work with Fourier Analysis, over a maximum of 4096 data points.

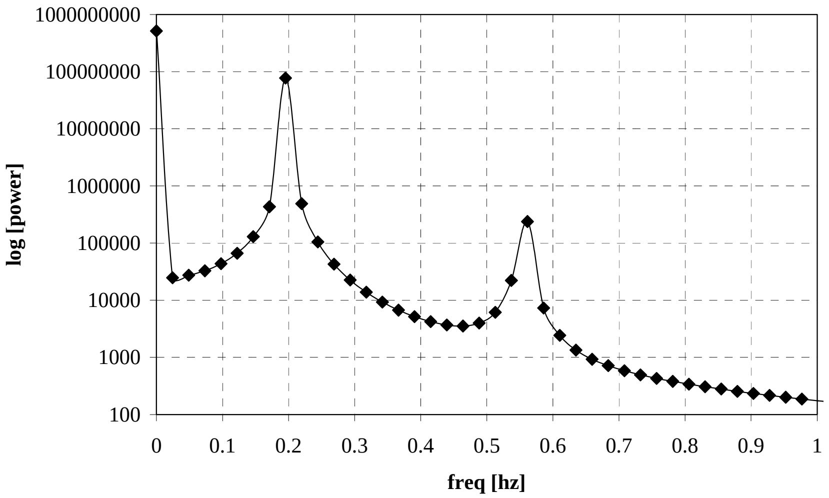


POWER GRAPH REPRESENTATION

•Excel give the possibility for graphing frequency versus power, for proper frequencies determination. For the specific solution presented (k1=6, m1=2, k2=16, m2=2).

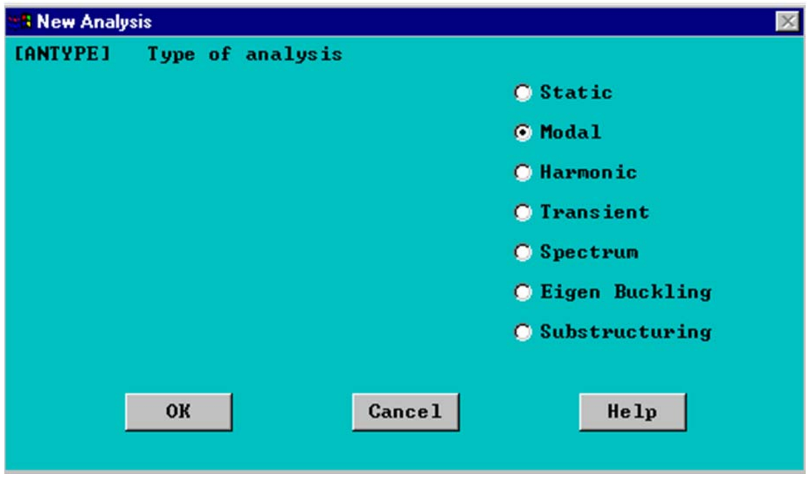


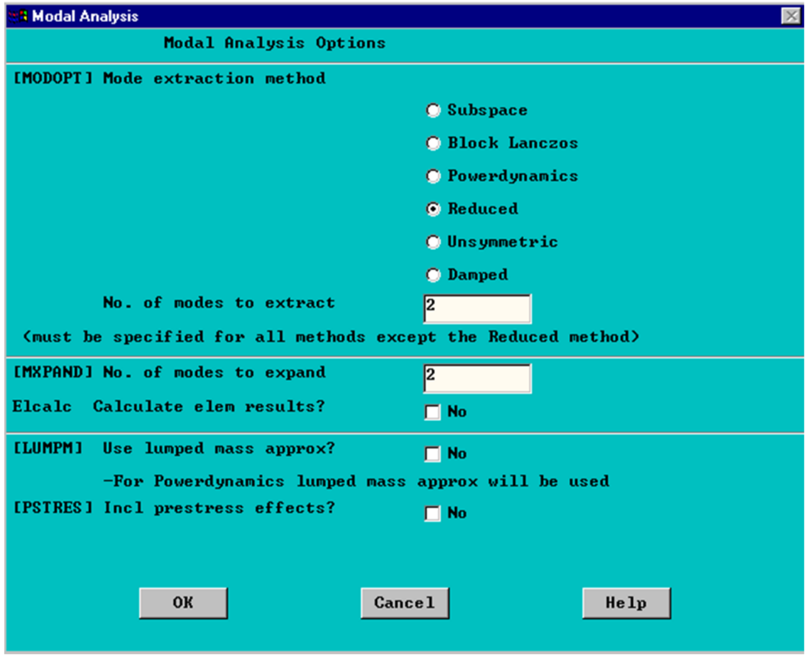
•For the above configuration results, the first two frequencies are not well identified. For other specific case like (k1=15, m1=4, k2=15, m2=3), the results presents much more well suitable interpretation (see next figure).



Numerical solution using
ANSYS FOR MODEL ANALYSIS

The student should be able to model two degrees of freedom as presented and obtain the frequency domain solution for each particular problem (m1, m2, k1 and k2).





The student should use the modal analysis capability with reduce mode shape extraction.

Note: See User Manual from LPAC.